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Embodied design: constructing means for constructing meaning

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Abstract Design-based research studies are conducted as iterative implementation-analysis-modification cycles, in which emerging theoretical models and pedagogically plausible activities are reciprocally tuned toward each other as a means of investigating conjectures pertaining to mechanisms underlying content teaching and learning. Yet this approach, even when resulting in empirically effective educational products, remains under-conceptualized as long as researchers cannot be explicit about their craft and specifically how data analyses inform design decisions. Consequentially, design decisions may appear arbitrary, design methodology is insufficiently documented for broad dissemination, and design practice is inadequately conversant with learning-sciences perspectives. One reason for this apparent under-theorizing, I propose, is that designers do not have appropriate constructs to formulate and reflect on their own intuitive responses to students' observed interactions with the media under development. Recent socio-cultural explication of epistemic artifacts as semiotic means for mathematical learners to objectify presymbolic notions (e.g., Radford, Mathematical Thinking and Learning 5(1): 37–70, 2003) may offer design-based researchers intellectual perspectives and analytic tools for theorizing design improvements as responses to participants' compromised attempts to build and communicate meaning with available media. By explaining these media as potential semiotic means for students to objectify their emerging understandings of mathematical ideas, designers, reciprocally, create semiotic means to objectify their own intuitive design decisions, as they build and improve these media. Examining three case studies of undergraduate students reasoning about a simple probability situation (binomial), I demonstrate how the semiotic approach illuminates the process and content of student reasoning and, so doing, explicates and possibly enhances design-based research methodology.

Keywords Cognition · Reasoning · Problem solving · Semiotics · Gesture · Multimodality · Probability · Binomial · Combinatorial analysis · Outcome distribution · ProbLab · Media · Computer · Model · Simulation · College

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1 Background and objectives

A not-too-well-kept secret among designers of educational mathematical artifacts is that our practice is as much an art as it is a science. Granted, we develop and employ design frameworks to create mathematical objects (e.g., Abrahamson & Wilensky 2007; Freudenthal 1986; Fuson 1998; Gravemeijer 1994; Schoenfeld 2005; Wilensky 1997). However, much of the subsequent interactive process of tuning these objects toward supporting productive engagement is still under-conceptualized (Schön 1990, 1992).

In general, opaqueness of creative practice need not denigrate its products, because these can be evaluated by consumers or tested empirically. Yet opaqueness becomes problematic for educational designers, whose practice is both situated in academic contexts of complex intellectual structures and accountable to manifold stakeholders such as sponsors, policy makers, colleagues, practitioners, and target audiences. Designers therefore wish to render their processes transparent by sharing their rationales for specific decisions and, more broadly, documenting their methodology, so that these can be studied, critiqued, and improved upon.

In an attempt to render their design practice transparent, design-based researchers appropriate constructs from industrial engineering and the cognitive sciences – constructs that enable the researchers to plan, coordinate, execute, reflect on, and communicate their processes (Barab, Zuiker, Warren, Hickey, Ingram-Goble, Kwon et al. 2007; Brown 1992; Collins 1992; Confrey 1998, 2005; Edelson 2002; Kelly 2003; Sandoval & Bell 2004). In turn, implementing and analyzing these designs result in the articulation of 'ontological innovation' (diSessa & Cobb 2004) or 'humble theory' (Cobb, Confrey, diSessa, Lehrer, & Schauble 2003), some of which pertain to the design practice itself.

Such scientification of discourse around design-based research practice at the macro level of activities and constructs might connote that a pellucid methodology undergirds the design process at the micro level of nuanced design decisions, such as decisions based on noticing and responding to aspects of the implementation. Schön (1990), however, argues that design is essentially inventive work and therefore cannot be captured as a formulaic problem-solving process (cf. Newell & Simon 1972). Schön thus suggests that any explication of design decisions is perforce post-facto rationalization of the designer's intuitive reasoning, which is cognitively impenetrable and therefore problematically reconstructed. Whereas I agree with Schön's view of design as explorative, reflexive, and emergent, I propose that some aspects of design remain opaque – even post facto – not because they are intuitive per se but because designers lack constructs to conceptualize those tacit aspects of their practice. Specifically, I conjecture, designers notice and respond to, yet do not commonly explicate, the non-verbal behaviors that students manifest as they struggle to use available media to express emergent understandings (whereas other researchers, e.g., Church & Goldin-Meadow 1986, have been attending closely to students' non-verbal behavior in the context of mathematical learning).

There is a non-coincidental analogy here between, on the one hand, students' attempts to use various media at their disposal to express their emerging mathematical notions and, on the other hand, researchers' attempts to use various theoretical constructs at their disposal to communicate their insight into students' learning. Namely, whereas the researchers and students draw on different mathematical *knowledge, goals, and beliefs* (Schoenfeld 1985) each are ultimately constrained in their respective sense-making by the nature or availability of expressive means, whether epistemic or substantive.[1] I am thus conceptualizing students' mathematical problem-solving processes as well as, reflexively, design-based researchers' data-analyses of these processes through the same theoretical lens, a semiotic approach. By elucidating non-verbal as well as verbal aspects of students' interactions with learning tools, I conjecture, design-based researchers will be in a better position to elucidate tacit aspects of their creative responses to students' observed behaviors.

By conceptualizing individual expression as contingent on available media, I am espousing a semiotic account of reasoning as the (re-)invention of meaning. The objective of this paper is to explore the potential utility of this semiotics approach for the practice of design-based researchers by examining whether this approach illuminates students' mathematical reasoning – its nature, struggles, and trajectories – as they engage with proposed materials.

A priori, the semiotic approach appears to be well suited to the practice of design-based research. To begin with, a focus on media – the objects that formulate, store, convey, and mirror student expression – is closely aligned with design-based researchers' analytical and pragmatic objectives (Bakker & Hoffmann 2005): to better theorize the roles of mathematical objects in creating opportunities for mediated content learning and, so doing, to delineate principles for effective design. Methodologically, the approach improves calibration between, on the one hand, the sampling density of microgenetic analysis (Schoenfeld, Smith, & Arcavi 1991; Siegler & Crowley 1991) and, on the other hand, the apparent micro-intentions of the participant students. Namely, framing participants' actions as 'problem solving activity' is too broad a focus, yet framing these actions as 'sense making' does not appear to capture equitably what the students are in fact attempting to do. Instead, I have been framing students' micro-actions as attempts to express and elaborate aspects of situations pertinent to a problem-solving goal – students are noticing, 'seeing as,' and articulating patterns and interactive potentialities in the materials (Schön 1992); they are objectifying and inscribing presymbolic ideas (Radford 2003). The semiotic approach also relaxes tensions in the clinical-interview paradigm (e.g., diSessa 2007; Ginsburg 1997), because the approach explicitly acknowledges the nature and implications of the inherently discursive interaction through which the data under investigation are elicited.

The semiotic approach I propose to apply to the analysis of student interaction with new materials is situated within diverse yet converging literatures, wherein reasoning is theorized as:

- necessarily situated, dynamic, and distributed over artifacts, people, and time (Clancey 2008; Greeno 1998; Heidegger 1962; Hutchins 1995; Merleau-Ponty 1992; Norman 1991; Polanyi 1967); and therefore
- intrinsically contingent on and mediated through personally available expressive forms, whether epistemic or material, endosomatic or exosomatic, predetermined or emergent, actually perceived, mentally simulated, or blended (Barsalou 2008; Collins & Ferguson 1993; Fauconnier & Turner 2002; Goldin 1987; Hutchins 2005; Kosslyn 2005; Presmeg 2006; Saxe 1981; Slobin 1996; Stetsenko 2002; Vygotsky 1978/1930);

[1] It seems only a matter of difference in discursive norms of two communities of practice – mathematicians, designers – that mathematical problem solving is currently held to higher standards of accountability than design-based problem-solving. Yet in a climate of public accountability, I am concerned, under-conceptualization might lead a designer or reviewer to devalue or even reject a potentially sound decision because it apparently cannot be rationalized.

- mediated by "embodied mathematical tools" (endosomatic) – ecologically adapted innate/early cognitive mechanisms that support reasoning by privileging the encoding of certain quantitative relations in perceptual stimuli (e.g., 'enabling constraints,' Gelman & Williams 1998; 'natural frequencies,' Gigerenzer 1998; aspect ratio, Suzuki & Cavanagh 1998; 'intuitive statistics,' Xu & Vashti 2008); and therefore
- facilitated by *cognitively ergonomic* artifacts (cf. Artigue 2002) – mechanical, virtual, inscriptional, or procedural systems – historically or recently evolved – that are tuned to the affordances and constraints of humans' bio-mechanical and epistemic systems;
- generating, interpreting, and coordinating signs in a range of semiotic systems embodied in multiple modalities/media such as verbal utterance, gesture, inscription of text, tables, or diagrams, and manipulation of objects (Alibali, Bassok, Olseth, Syc, & Goldin-Meadow 1999; Arzarello, Robutti, & Bazzini 2005; Bartolini Bussi & Boni 2003; Bartolini Bussi & Mariotti 1999; Becvar, Hollan, & Hutchins 2005; Lemke 1998; Radford 2003, 2006; Rotman 2000; Schegloff 1984);
- negotiating between personal image schemas of mathematical constructs and mediated ways of seeing mathematical objects (Abrahamson 2004a; Stevens & Hall 1998);
- transpiring over a sequence of expressing-for-thinking micro-actions, each objectifying an emerging idea, only to reflexively constitute contextual input toward the formulation of a subsequent idea produced by the same or another agent (McNeill & Duncan 2000);
- contingent on tacit social norms of discourse, both general and discipline-specific (Barnes, Henry, & Bloor 1996; Bloor 1976; Borovcnik & Bentz 1991; Cobb & Bauersfeld 1995; Ernest 1988; Grice 1989; Schegloff 1996); and
- potentially conducive of individuals' gradual appropriation of artifacts that they instrumentalize as problem-solving tools and ultimately utilize as content-bound mental schemas (Trouche 2004; Vérillon & Rabardel 1995; Vygotsky 1978/1930).

Applied to design-based research, these resources underlie the practice of *embodied design*.[2] Here, I focus on the semiotic aspect of embodied design so as to evaluate its potential for deepening our understanding of design-based research practice. I do so by presenting and analyzing case studies of students attempting to express their insights as they are working with learning materials under development. As often occurs in the scrutiny of human practice, moments of expectation breakdown are particularly revealing of tacit structure and function of otherwise opaque mechanisms (Garfinkel 1967), and the same applies to design-based research (Abrahamson & White 2008; White 2008). I therefore present three case studies of students whose situated mathematical reasoning I interpret as modulated by available means that either enable or hinder expression, and thus elaboration, of emerging ideas relevant to constructing meaning for the mathematical content. So doing, I interpret designers' creative activity as responding to multimodal aspects of students' hindered semiotic actions.

[2] When the Embodied Design Research Laboratory (EDRL) was established in 2005, an Internet search located only one prior use of 'embodied design' (Van Rompay & Hekkert 2001). Since then, the phrase has been used idiosyncratically by several scholars (e.g., Schiphorst 2007).

The larger research program that frames this paper is the development of a principled methodology for implementing realistic/constructivist/constructionist pedagogical philosophy (Freudenthal 1986; Papert 1980; von Glasersfeld 1990) in the form of content-targeted objects, activities, and facilitation guidelines, so as to contribute to the theory and practice of mathematics education. We have outlined a treatment of mathematical representations as conceptual composites so as to frame the *initial* design of objects (‘bridging tools’) supporting students’ recomposition of the target concepts (Abrahamson 2006b; Abrahamson & Wilensky 2007) and have demonstrated how an embodied-cognition approach elucidates students’ *in situ* negotiation and appropriation of these objects (Abrahamson 2004a; Fuson & Abrahamson 2005). Here, I focus on how data analysis informs *iterative* design, specifically on how analysis of videotaped interactions with participant students guides the selection, creation, and/or improvement of the learning materials under development, for subsequent studies.

I begin, below, by explaining the design. Next, I present and discuss the cases. Then, following some conclusions, I offer implications for learning theory and design practice.

2 Context: a design-based research study of an experimental unit on probability[3]

The data discussed in this paper come from a design-based research project investigating students’ intuitive understanding of random processes. We first worked with middle-school students to examine their intuitions for outcome distributions in “urn”-type sampling experiments and how these intuitions could potentially be kindled toward supporting grounded appropriation of normative mathematical procedures for the binomial function (Abrahamson 2008b; Abrahamson & Cendak 2006). Then, to understand trajectories and interactions of intuition and learning, we replicated the study with university students (Abrahamson 2007c).

Twenty-five self-selected undergraduate and graduate students, all enrolled in mathematics or mathematics-oriented programs, participated in individual semi-clinical interviews, lasting about 70 minutes each. Students worked with materials from *ProbLab* (Abrahamson & Wilensky 2002), an under-development unit initially created under the umbrella of the *Connected Probability* project (Wilensky 1997). ProbLab is designed in light of current agreement over the importance of developing means for helping students connect theoretical and empirical activities pertaining to probability (Jones, Langrall, & Mooney 2007). That is, students should learn how analysis of a randomness generator (its combinatorics and independent probabilities) relates to frequency distributions from actual experimentation with this generator.

The interview began with students analyzing an experimental procedure in which four marbles were drawn out randomly from a box containing hundreds of marbles, of which there were equal amounts of green and blue marbles. To draw out these samples, we used a dedicated utensil, the *marbles scooper*, consisting of four concavities arranged in a 2-by-2 array (a 4-block; see Fig. 1a). Thus a scoop with 1 green marble and 3 blue marbles could have four different appearances (permutations), depending on the specific location of the green marble relative to the scooper handle. According to probability theory, the most likely outcome has exactly 2 green marbles and 2 blue marbles in any order. Specifically, the distribution of expected outcomes by number of green marbles is 1:4:6:4:1, corresponding

[3] See other publications for literature reviews and details of the design’s rationale, evolution, and implementations that herein are necessarily condensed.

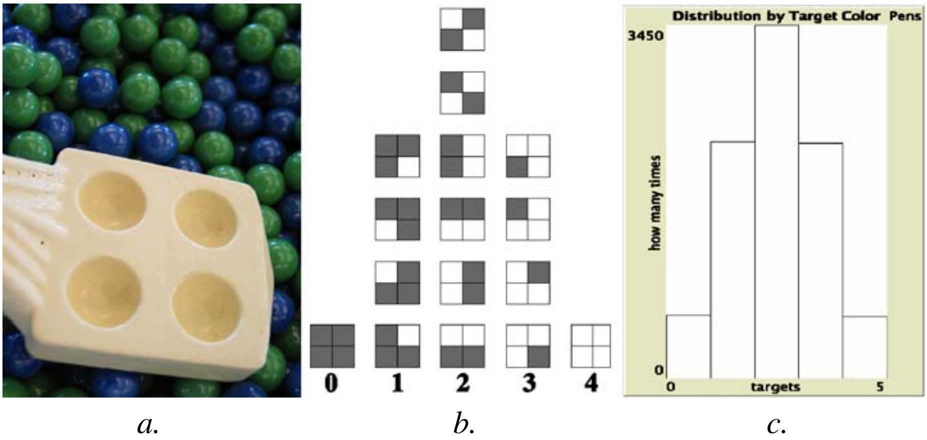

Fig. 1 ProbLab materials used in the study – theoretical and empirical embodiments of the 4-Block mathematical object: (a) The marbles scooper; (b) the combinations tower; and (c) an actual experimental outcome distribution produced by a computer-based simulation of the marbles-scooper probability experiment

to 0 green, 1 green, 2 green, 3, green, and 4 green, respectively.[4] We asked students, "What will we get when we scoop?," which is an ambiguous question since it specifies neither whether we are referring to a single scoop or the long run, nor whether we mean an event (combination) or a specific compound outcome (permutation). By and large, students correctly intuited the ranking of likelihood, i.e., that a 2-green–2-blue scoop would be the most likely, the 4-green or 4-blue the least likely, etc.[5]

Next, students were guided to use a set of stock-paper cards, each depicting an empty 4-block (a blank 2-by-2 grid), and green and blue crayons so as to create all the different 4-block patterns one could possibly scoop out of the box (thus performing what is called in probability theory '*combinatorial analysis*'). Once students built the 16 unique green/blue configurations (2^4), they were guided to arrange these cards in columns by number of green cells in the 4-block. This resulted in a *combinations tower* (see Fig. 1b), the sample space of the 4-block stochastic device that is distributed such that it anticipates the shape of things to come for a .5 p value (see Fig. 1c and see Abrahamson 2006c). Note that the combinations tower shares figurative properties with two semiotic tools complementary to the construction of meaning for the targeted content – a sample space and a likelihood distribution (compare Fig. 1b & c). This structural hybridity is a hallmark of *bridging tools* (Abrahamson 2004b; Abrahamson & Wilensky 2007).

We have thus been attempting to create contexts for students to synthesize intuitive and mediated resources (Case & Okamoto 1996; Schön 1981): event-based inductive inference grounded in judging properties of the random generator (Abrahamson & Cendak 2006; Pratt 2000; Tversky & Kahneman 1974; Xu & Vashti 2008) and outcome-based deductive

[4] The ratio of sample size, four marbles, to population, hundreds of marbles, renders the 'without-returns' issue negligible.

[5] In expressing their judgments, students alternatively used figures of speech such as 'more/less likely,' 'greater/smaller chance,' and 'you'll get more/less of that.' Whether students' judgments could be called 'intuitive *probability*' or 'intuitive *frequency*' remains a moot question (Gigerenzer 1998, calls this 'natural frequency'; Xu and Vashti 2008, call it 'intuitive statistics').

inference grounded in constructing and analyzing the sample space (Abrahamson 2008b). The combinations tower enabled triangulation of intuitive and analytic processes, because its vertical trajectories served as *semiotic means of objectification* (Radford 2003) of the presymbolic sense of distribution. That is, the tower functioned as a *material anchor* for the *conceptual blending* (Fauconnier & Turner 2002; Hutchins 2005) of intuitive expectation into the sample space – through guided noticing, students came to see the columns of discrete icons as indexing the tacit sense of relative frequencies (Abrahamson, Bryant, Howison, & Relaford-Doyle 2008).

After working with the combinations tower, students were asked to describe outcome distributions they expected to receive in a computer-based probability experiment that simulates the operation of the 4-block random generator and tallies outcomes according to the number of green in each scoop (see Applet 1).[6] We asked, "What will the histogram look like?"; "Now, what if we change the probability of getting green from .5 to a greater number?" As we will see in the data excerpts, students spontaneously used the combinations tower on their desk so as to describe the outcome distribution they anticipated in the computer-based simulation. Thus, in accord with the design's objective, the activity sequence fostered opportunities for students to construct meaning through juxtaposing theoretical and empirical artifacts in a single semiotic action.

The interview ended with computer-based activities that offered a conceptualization of the empirical distribution as a non-uniform multiplicative scale-up of the sample space, using a simulation that stacks the actual outcomes in "stalagmite" columns (Abrahamson & Cendak 2006 and see Applet 2). This conceptualization blends the empiricism of probability experiments, in which numerous random trials are accumulated randomly, into the sample space, a fixed structure. The conceptualization, too, is thus aligned with the design's initial premise to create contexts for students to coordinate theoretical and empirical aspects of the binomial.

Note that the activities began with a situation in which $p=.5$ (equal numbers of green and blue marbles in the box). This design decision introduced a tradeoff, because it enabled initial entries into the mathematical content, yet it made for later difficulty in assimilating other p values. Moreover, as the data will demonstrate, a pivotal conceptual challenge in this design lies within the theoretical, per se – even before the expectations have been tested empirically – as students attempt to coordinate intuitive and analytic expectations for cases of $p=.5$ and beyond.

3 Data and analysis

Each of the following three episodes from a design-based research study demonstrates an aspect of object-mediated mathematical reasoning, as illuminated through a semiotic analytic approach. The episodes were selected to represent the three main phases of the design. The first excerpt shows a student who, to convey a mathematical property she notices in an object, uses a serendipitously available medium that is part of the object itself: she overlays upon the marbles box her imaged re-configuration of its content. In the second excerpt, a student's spontaneous gesture adumbrates structural elements of a physically absent artifact, an outcome distribution, without alluding to it verbally; this artifact is then constructed physically, enabling the student to anchor and elaborate his initial reasoning. In the third excerpt, a student struggles in vain with a malleable medium, the combinations-

[6] All ProbLab computer-based simulations are built in *NetLogo* (Wilensky 1999).

tower cards, to express an image in another, unmalleable, medium, a histogram on a computer screen. Analysis of the semiotic affordances/constraints in this latter case framed a set of design specifications for a new mathematical object, a computer-based module, that would enable the desired expressivity in subsequent interviews. Thus, the cases collectively demonstrate how students' expressivity may be either facilitated or limited by the adequacy of available media to encode emerging images they wish to communicate and how the semiotic analytic approach both elucidates students' actions and, reflexively, explicates the designer's insight and process.

3.1 Reflexive artifacts: mathematical objects as means to objectify their own properties

3.1.1 Setting the scene

When students were initially asked to estimate outcomes, they spoke of equal proportions of green and blue in the marbles box. By and large, this warrant included a gesture to one side of the box and then to the other, as though the hundreds of marbles were separated by color. Some of the participants indicated – even touched – the middle point of the box immediately prior to gesturing to the "blue half" and the "green half." What is the nature of this gesture and how did it serve in constructing mathematical meaning? We will focus on Rose, a recently graduated statistics major, who was typical of students who gestured "half-half" toward the marbles box (see Abrahamson 2007b, for a case of a Grade 6 student).

Three minutes into the interview, the interviewer asks Rose what she would expect to "get" when she scoops. Rose has scooped twice and both yielded a '2 green, 2 blue' sample. Now the marbles scooper is on the desk, where Rose has just placed it, with two blue marbles on the left and two green marbles on the right. Rose responds that she expects to receive a '2 green, 2 blue' sample, "because that's what I've gotten this far, and it looks like it's about half-and-half in there." In saying "half and half," Rose flapped her hands up and down above the marbles box, with the left hand over the left side of the box and the right hand over the right side. The interviewer responds that the ratio in the box is indeed half-and-half.[7] Rose then says (see Movie 1):

> Now that I know that it's 50 percent blues [gestures to the left half of box surface] and 50 percent greens [gestures to the right half of box surface], I would guess that [gaze shifts to the marbles scooper] I would get two blues [fingers touch two blue marbles on left side of scooper] and two greens... [fingers touch two green marbles on right side of scooper] is the most likely combination of marbles to come out when I do the scooping [right hand gyrates rapidly in mimed scooping motions].

Note the spatial analogy Rose has built between the marbles box and the scooper – she maps the left and right sides of the marbles box, respectively, onto the left and right sides of the marbles scooper. More significantly, note that it is not the case that there are 50 percent blue marbles on the left of the box and 50 percent on the right. Rose's multimodal assertion appears to be counterfactual. Why would Rose offer a statement incompatible with the distal stimuli?

[7] This response changes the situation for Rose from one of statistical investigation – attempting to determine the green-to-blue distribution in the marbles "population" – to a probability experiment, where one can apply the Law of Large Numbers and/or compute expected values (see Abrahamson 2006a, on nuanced relations between statistics and probability).

3.1.2 Analysis

The epigenesis of a specific gesture is in actual physical manipulation (Vygotsky 1978/1930). If there were fewer marbles, Rose might have sorted them physically to compare color groups. Yet it is precisely because the actual marbles (the 'factual,' Radford 2003) are not readily given to physical grouping that Rose expresses the global image that would result from physical sorting, tacitly channeling this image to the gestural modality yet mapping it deictically upon the source object it qualifies (thus maintaining the 'contextual,' Radford 2003).

Roth and Welzel (2001) submit that gestures can provide the material that "glues" layers of phenomenally accessible and abstract concepts. Indeed, Rose appears to "glue" the half–half structure upon the collection of marbles; the marbles constitute a *material anchor* into which Rose *blends* the half–half image (Fauconnier & Turner 2002; Hutchins 2005). Yet the semiotic perspective is geared to expand on this cognitive analysis by revealing the *motivation* for this complex cognitive action, thus relating the action to its framing activity practice, the discursive didactical-mathematical interaction.

Rose perceives the pragmatics of the interaction as demanding a mathematical argumentation genre, so she intends to articulate her intuitive anticipation of a '2 green, 2 blue' sample mode as warranted by properties of the marbles collective. Gazing at the box with the intention of determining the color ratios, she experiences a presymbolic notion of equivalence between the green and blue intensities. She identifies the linear continuity of the plastic container as a means of objectifying this notion, which is consonant with the scooper structure.[8]

In designing the activity, I had not anticipated that the marbles box would constitute a semiotic tool for reflexively objectifying its own properties of color distribution. Yet this affordance attracted the students spontaneously. Appreciating the significance of this unexpected gesture, in turn, honed my attention to nuances of expected appropriation of semiotic artifacts.

3.2 Cognitively ergonomic semiotic mathematical artifacts designed for tacit appropriation

Mark is a senior economics major who has taken many courses in probability and statistics. Below, I compare two data excerpts from his interview, before and after he has constructed the combinations tower (see Movies 2 & 3). This case study demonstrates how imagery evoked by a mathematical artifact is subsequently blended into another artifact.

3.2.1 Working with the marbles box

The first data excerpt begins 8 minutes into the interview (see Fig. 2a). By that point, it has been established that there are equal numbers of green and blue marbles in the box. Mark had scooped only once and received a sample with 3 blue balls and 1 green ball. Now, he is building an argument for his expectation that in an experiment with numerous trials we will receive equal numbers of green and blue balls:

> So when the times that you have 3 blue balls [left hand gestures to the left] will be weighted against when the times you have 3 green balls [left hand gestures to the right, lightly touching desk], and that will make it a combinations that... weighted to be 2 green balls and 2 blue balls [hands parallel on desk].... And... these two

[8] A future study is necessary to determine whether students would use similar gestures for other-than-half-half proportions. Also, if an actual number line, running from 0 to 100, were attached to the rim of the marbles box, students' spontaneous part–part gesture could index a part-to-whole numerical value, thus bridging from the preverbal to the numerical.

Fig. 2 Mark, a senior economics major, explaining why he expects a '2 green and 2 blue' sample as the central tendency of a hypothetical experiment with the marbles scooper: (a) using hands as event categories; (b) using columns of the combinations tower; (c) elaborating on expected variance by squeezing columns toward each other

> combinations in expected value, it's the same....as the scenario that you got 2 green balls and 2 blue balls.... /8 sec/ And, for the same reasons... the 4 blue balls will also be weighted against the 4 green balls, and in expected value they should be in the same ratios. So that would.... eventually give you the same expected values, when you calculate it that way.

Mark's manner of speech is hesitant and his gestures often precede his verbalizations, suggesting that his argument emerges as he develops it.[9] Whereas each of Mark's gesture constructions is intact with his verbalized ideas of balance and compensation that are central to his reasoning – 3 blue against 3 green, 2 blue for 2 green, and 4 blue against 4 green – these gesture constructions also relate *to each other* spatially in terms of location, including order and distance. Specifically, Mark aligns the outcome categories equispatially on the desk, abiding with certain structural properties of a number line or, perhaps, a type of interval scale running to the left and right of his body center. This is not a mathematically normative '0-1-2-3-4' scale, which could index the single dimension "number of green," but rather a palindrome scale, '4-3-2-2-3-4' – '4 blue'; '3 blue'; '2 blue'; '2 green'; '3 green'; '4 green.' This spontaneous form indexes two juxtaposed dimensions, "number of blue" and "number of green," and is thus appropriate for the embodied argument of balance. However, note the duplication of the '2 blue' and '2 green' categories that are in fact mathematically equivalent (they refer to the same event category).

The gestured edifice emerges dynamically as an attraction of available media (body, desk) and conceptual topology (semantic categories in a proto-bar-chart). Note how Mark's initial gesture ("3 blue balls") is in the air, the second ("3 green balls") touches the desk, and thereon he places the categories upon the desk. As in the case of the marbles box, so this typical desk, with its flat surface and straight edge, is fortuitously recruited as an auspicious medium for accommodating the progressively elaborated complex construction that includes categories of events and relations among them. Note also how the first gestures are conducted single-handed and sequentially, yet the latter are ambidextrous and simultaneous, as though once the notion of symmetry has been evoked through the left–

[9] Mark uses the mathematical term 'expected value,' yet his explanation does not abide with the formal procedure for determining an expected value but rather it is a qualitative argument for why the mean sample should have 2 green balls and 2 blue balls. (To calculate the expected value, Mark should add up the four independent .5 probabilities of getting green, one for each concavity, to receive the sum of 2 green balls as the expected value of a single scoop.)

right placement of the initial categories, this embodied notion is promoted as skeletal to the mathematical argument. Thus, the desk and the body are recruited and aligned as media of expression, each with its unique affordances.[10]

Mark, similarly to Rose in the earlier data excerpt, negotiates and blends topological properties of his epistemic resources with those of the artifacts in question. Namely, just as Rose constructed the marbles box as a bipartite object resonating with the 2-and-2 structure of the 4-block array, so Mark coordinates his emergent outcome categories with the 2-and-2 concavities of the 4-block. In particular, over a span of 8 seconds of silence, Mark laboriously negotiates the coordination of available embodied structures – the gestured structure of *two* instantiated outcome categories, '2 blue' and '2 green' (that are mathematically co-referreing) and the material structure of a *single* scenario of a '2 blue, 2 green' 4-block. The coordination is facilitated, perhaps as a result of a serendipitous bio-mechanical constraint, when Mark draws his palms against each other to embody equivalence, whilst using his thumbs – which perforce become adjacent – to indicate the expected value upon the material scooper. That Mark has "only" two hands serves as an enabling constraint, because this limitation on his expressivity impelled a compact amalgam. That is, the material and conceptual resources become reconciled, bound in a new co-located and co-expressive blend.

Note that Mark's hands-as-columns are of equal height, as in a histogram representing a flat distribution. Indeed, Mark's focus has been on the symmetry of compensation between juxtaposed outcome categories, not on relative frequency of all categories. In the following excerpt, when Mark works with the combinations tower, we will see how this flat distribution takes on *y*-axis verticality. This verticality will be expressive both of the cardinalities of the five category sets qua sample space and emergently implicative of expected frequencies in projected experiments.

3.2.2 Working with the combinations tower

Ten minutes later, Mark has constructed the combinations tower and is interpreting it (see Fig. 2b, Movie 3). He has said that the relative height of the middle column – the six 2-green permutations – denotes an expected plurality of outcomes of that category in experiments with the scooper. Now, focusing on the "added value" of the combinations tower relative to an unstructured sample space, Mark reiterates his earlier argument about expected value, only now he attends to the distribution's verticality:

> And also I... *As I say before* [left hand on the 4-blue card, right hand on the 4-green card], the 2 columns, in having.... 3 greens and 3 blues [left and right forearms on these columns, respectively], will be weighted against each other in expected value terms.

Mark is conscious of this utterance being a reiteration, repeating the symmetry/balance metaphor of categories weighted against each other in a hypothetical experiment. Yet, Mark's gestures indicate that he is attuned to features of the artifact before him and incorporating them into his mathematical argumentation: he fits his *forearms* onto the 3-blue and 3-green columns, respectively, highlighting variable verticality in the distribution.

[10] Note how the palindrome scale is co-centered with the marbles box. This spatial co-positioning of artifact and gestured construction may appear epiphenomenal to normative bio-mechanics of tool use and, thus, barely pertinent to that which we may wish to call mathematical reasoning. However, a cognitive-ergonomics approach to artifact-based mathematical learning and design should tend to this spatial relationship and monitor its emergent consequences.

Importantly for the conceptual construction of binomial distribution, Mark implicitly establishes semiotic equivalence between his earlier 'hand' categories (events) and the current 'forearm categories' (event *sets*).

The combinations-tower columns thus act as "gloves" into which Mark's forearms fit smoothly – it is a cognitively ergonomic semiotic means of objectification that accommodates-yet-elaborates his reasoning: In its materiality, the tower anchors and offloads key imagistic aspects of Mark's argument, so that he needn't commit cognitive resources (working memory) or embodied semiotic resources (forearms) just to make the columns intersubjectively present. Moreover, the combinations tower enables Mark to *elaborate* his thoughts (to "run the blend," see Fig. 2c). Namely, the malleability of this discrete set of cards allows Mark to condense the columns horizontally so as to demonstrate gradual diminution in the variance of the outcome distribution in projected experiments, as the number of samples increases. So doing, Mark ignores the histogram's anticipated vertical growth, focusing on the overall shape of the tower-as-histogram. Thus, the combinations tower, essentially a sample space, enables the student to coordinate aspects of theoretical and empirical probability, in line with the design's rationale.

Mark, like Rose, used available media as semiotic means of objectifying elements of emergent mathematical argumentation, and both participants used the media in ways that the designer had not foreseen. We now turn to discuss the case of Mary, whose argumentation was hindered by the unavailability of appropriate semiotic means for anchoring a complex blend.

3.3 Obduracy of the world: stretching the blend, breaking the blend

Mary, a senior statistics major, has been working with a computer-based simulation of the 4-block marbles-scooper experiment, with the p value set at .5. An on-screen histogram, which tracks the accumulation of actual experimental outcomes (Fig. 1c), has been converging on a 1-4-6-4-1 distribution. The interviewer asks what the distribution might be for higher p values. Mary replies (see Movie 4):

> If it was more likely to be green, it would be skewed. This [touches left-side histogram columns] would get sma[ller]... This [touches right-side; see Fig. 3a] would get bigger.

Yet, recognizing that she cannot *directly* manipulate the graphical elements of the virtual histogram to express the distribution she anticipates, Mary tries alternative semiotic techniques, beginning with "remote-manipulating" the histogram through gesture.

Framing her view of the histogram, Mary peers at the screen from between her hands and "tilts" the histogram to the right (see Fig. 3b). Yet still unsatisfied that she has communicated the image adequately, Mary searches further for suitable semiotic means. Her right hand hovers momentarily over a pen (see Fig. 3c), as though she considers drawing the expected histogram on an available sheet of paper. But she abandons that medium, too, and turns to the combinations tower on the desk, saying: "...but it would shift, like..." Sliding the cards in opposite vertical directions – left hand down, right hand up – she makes the 1-green-column shorter and the 3-green-column taller (see Fig. 3d). Now she is working with the single card on the right of the tower. She wishes to show that this 4-green column, too, would become taller. She lays her right hand on this single card and slides the card up to the desired height (see Fig. 3e). Yet, once raised, this card is no longer aligned with the bottom of the tower, and so Mary returns the card down to its original location (Fig. 3f), stating that the card medium ultimately limits her expression: "You can't really do it on these cool things [cards], but it would be more like that."

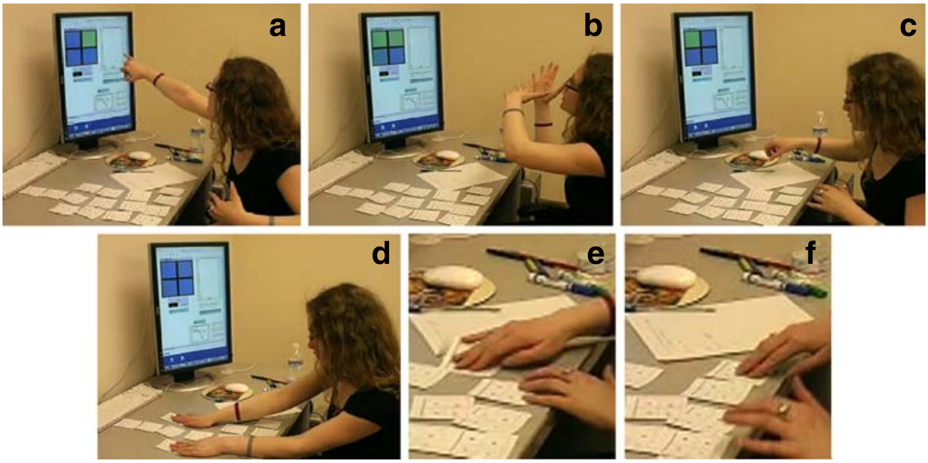

Fig. 3 Negotiating media constraints on image expressivity, Mary: (a) manipulates the on-screen histogram "hands on"; (b) manipulates the on-screen histogram "hands off"; (c) considers pen and paper, but declines; (d) remote-manipulates the on-screen histogram "hands on"; (e) shifts a card up to show the expected shape; but (f) returns the card because the shift violated constraints of the representational form

3.3.1 Running the blend

Mary spontaneously instrumentalizes the combinations tower, a theoretical-probability construction, as the best available semiotic means for expressing an anticipated transformation in the structure of a frequency-distribution histogram, an empirical-probability construction. Like Mark, Mary adjusts the combinations tower, a set of discrete iconic objects (the cards standing in for possible scoops) so as to blend in the anticipated contours of a set of continuous indexical inscriptions (the columns signifying frequencies). Both Mark and Mary manipulate the properties of an object at hand so as to communicate change in missing or obdurate objects. Yet the object at hand (the tower) is not an arbitrary semiotic tool, such as paper and pen – the object had been designed for this didactical situation specifically as bearing complementary properties of the mathematical situation. *By using a bridging tool to remote-act on the mathematical situation, Mary inadvertently runs the theoretical–empirical blend, so that a spontaneous semiotic act engenders an opportunity for linking core conceptual elements, per the design objective.*

3.3.2 Breaking the blend

Manipulating the tower had worked well for the initial case of $p=.5$, but Mary encounters technical difficulty for a different p value. Walking a semiotic tightrope between a medium's affordances and constraints, Mary runs the conceptual blend, stretches it to its limits, and ultimately breaks it. Imagistically, Mary appears to have run the blend unfettered by conceptual disparity between the sample space and the expected frequencies, but in attempting to objectify this blend physically, she faces the semiotic obduracy of the available means of construction. Yet this very obduracy embodies the conceptual disparity emerging from the design – the very disparity that is to be bridged – so that the semiotic breakdown reveals the conceptual disparity. Namely, Mary's disavowal, "You can't really do it on these cool things," suggests that she recognizes the problematic affordance of the combinations tower ("cool things") as a means for objectifying expected frequency distribution ("it"). Thus, an aborted semiotic act discloses the equipmentality of the semiotic means – "the environment announces itself afresh" (Heidegger 1962, p. 105) – thus affording post-facto reflection on the mathematical meaning embodied in the media.

In sum, the combinations tower serves as a means for spontaneously blending 'sample space' and 'anticipated outcome distribution,' yet this blend is liable to break when run for *p* values other than .5. The blend breaks due to mechanical incompatibility between available semiotic means for objectifying two distinct yet complementary mathematical ideas: discrete cards iconizing all possible outcomes (the sample space) and continuous columns indexing probabilities or frequencies. As a designer, I ask how we might sustain this blend. Namely, what mathematical object would satisfy the criteria that it both enables a user to run the blend for all *p* values and does not lose the semiotic grounding in the sample space?

3.3.3 Fixing the blend

Consider the histogram. Note that histogram columns are themselves ambiguous figures in that they are semiotic means for articulating either theoretical or empirical frequencies. That is, a histogram *per se* can be construed as signifying either a probability distribution ('what we will get') or a frequency distribution ('what we got').[11] The contextual contingency of the histogram suggests that the desired blend could migrate from the combinations tower, the grounding blend, to the histogram's columns that, being continuous, would not break for non .5 cases. Yet, whereas the histogram is unconstrained by the discreteness of the sample space, how could a histogram nevertheless incorporate vestiges of the 16 discrete sample-space outcomes so that the sample space and independent probabilities could play a role as the blend's logical warrant. These questions guided the design of Histo-Blocks (see below), an interactive module in which an expectation distribution is stratified into 16 units dynamically controlled by *p*.[12]

3.4 Histo-blocks: a virtual semiotic means for running a conceptual blend

Histo-Blocks (see Fig. 4), a computer-based module, takes advantage of the medium's affordances to enhance the expressivity, vividness, and precision of the physical manipulations observed in our interviews. Specifically, students can perform "electronic gestures," in which the blend of 'sample space' and 'expected outcome distribution' does not break down for *p* values other than .5. The simulation includes interlinked "click-able" objects: a slider controlling the *p* value, a virtual combinations tower, a dynamic histogram, and output monitors (see Applet 3). With Histo-Blocks we thus complete the laying out of a

[11] At the limit, the histogram signifies both, and explicating this theoretical-empirical homomorph in terms of properties of the random generator – its combinatorics and probabilities – could be regarded as the apex of coordinating theoretical and empirical aspects of the binomial. The Law of Large Numbers predicts that the empirical outcome distribution will converge on the theoretical expectation. After several thousand trials in a simulated marbles-scooping experiment (see Applet 1), the frequency distribution stabilizes at the expected shape.

[12] Searching for a new material anchor that would enable us to run the blend for all *p* values, I attempted various design variants on the combinations tower, such as a stretchable combinations tower, and I have examined the affordances of different media for implementing this and related design solutions (see 'Sample Stalagmite,' Abrahamson 2006c and Applet 2, for another solution, which is only glossed over in this article).

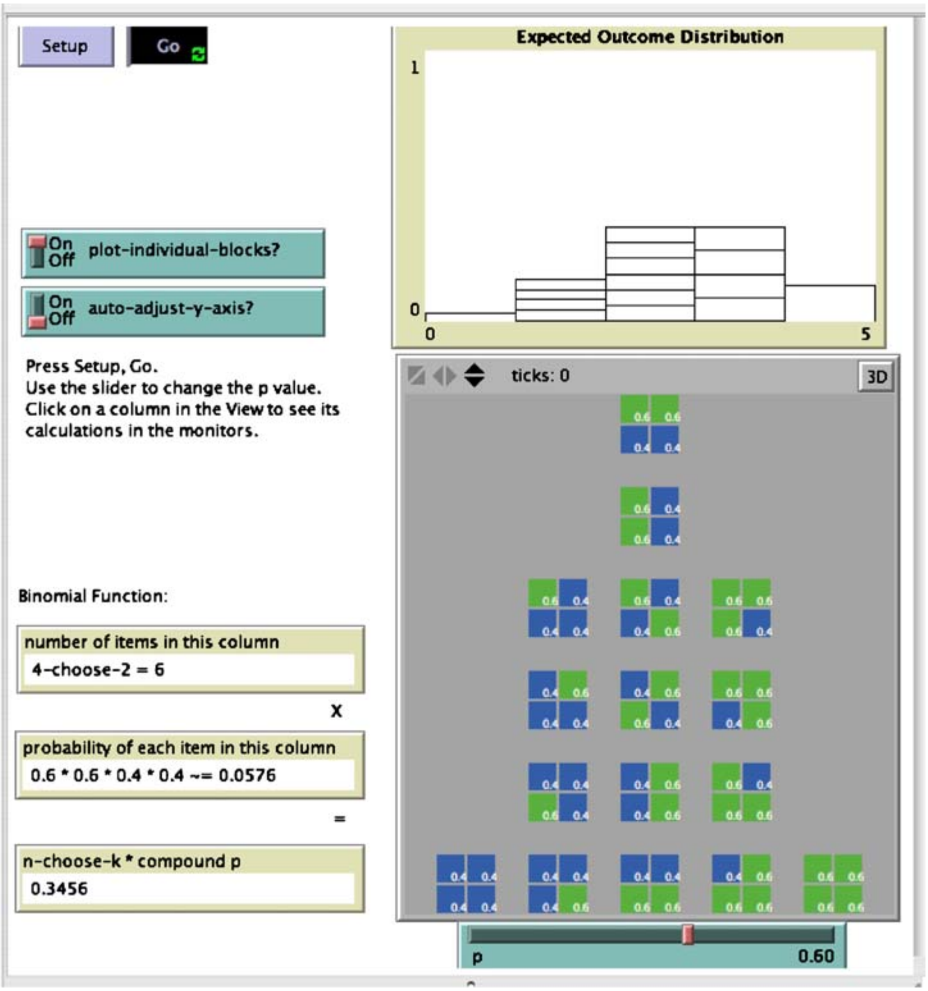

Fig. 4 Interface of the ProbLab model Histo-Blocks, built in NetLogo. In this screen-shot, the probabilities, histogram, and monitors all express properties of the middle column of the combinations tower for a *p* value of .6. The corresponding middle column in the histogram, directly above the combinations tower, is equal in height to the column immediately adjacent to the right, because the properties of that column are 4 * 0.6 * 0.6 * 0.6 * 0.4=~.3456

semiotic trajectory from the marbles box (actual) to the combinations tower (iconic), to the histogram columns (indexical), and finally to the binomial function (symbolical), the latter three all integrated within a single model.

Histo-Blocks was also designed to foster deep understanding of the binomial function $P(X = k) = n\text{-}choose\text{-}k * p^k(1-p)^{n-k}$– the product of combinatorial and probabilistic factors – as follows. Consider the case of flipping four coins (n=4), each with a *p* value of .6 ("unfair" coins). What is the chance of getting exactly 3 Heads (k=3)? The solution is a product of 4-choose-3 (= 4), i.e., the number of different coin orders for the combination of 3 Head and 1 Tail, and the compound probability of the four coins in this combination, i.e., .6*.6*.6*.4. Thus, the solution is 4*.6*.6*.6*.4=.3456. Analogously, the binomial function can express the probability of sampling a 4-block with exactly three green marbles, when 60% of the marbles are green.

In order to help students build an understanding of the binomial, its two factors have been distributed over separate properties of the virtual combinations tower: the combinatorial factor is represented by the 1-4-6-4-1 sample space of 16 discrete objects, and the probabilistic factor is embedded as probability labels upon each of the independent green/blue squares. The specific cases of these two factors appear numerically in output windows (see bottom-left side of Fig. 4), below which a third window shows their product, the solution.

In later interviews, we successfully engaged several students in pilot activities involving the Histo-Blocks simulation. These students were able to explicate spatial properties of the stratified histogram – e.g., the equivalent heights of the 2-green and 3-green columns at $p=.6$ (see Fig. 4) – as grounded in quantitative properties of the combinations tower and manifest in symbolical signs in the monitors (see Movies 5 and 6).

4 Concluding remarks

A foundational socio-cultural conceptualization of learning (Vygotsky 1978/1930) is that teachers, or adults in general, intuitively attune their didactic practice in real time in response to how students *see* mathematical artifacts (Alibali, Flevares, & Goldin-Meadow 1997; Stevens & Hall 1998). In the luxury of video-based investigative hindsight, education researchers practice a methodical elaboration of this same intuitive pedagogical attuning, only that this post facto process is aimed at improving theoretical models. Design-based researchers supplement these investigations by examining how artifacts in the learning environment either facilitated or hindered students' reasoning and expressivity and how these artifacts might be modified in light of these observations. It is such attention to the affordances of learning tools that this paper has attempted to foreground. The argument is not that design-based researchers *should* be attending to how their design is being used in context, because this is common practice. Rather, the objective of this paper has been to expose and frame *tacit* aspects of designers' practice from an embodied-cognition semiotic perspective, as a means of offering routes to systematizing this practice in the form of recommended methodology. To do so, I have presented several data episodes in which problem solvers appropriated available media in their attempts to materialize, communicate, and elaborate their reasoning. Specifically, I have investigated the nature of the available media vis-à-vis the task and the design rationale and whether the media afforded student reasoning evaluated as conducive to conceptual development along desired trajectories. My examples demonstrated the utility of methodological attention to multimodal multimedia semiotic activity as a means of eliciting the processes of students' embodied reasoning. Once students' semiotic intentionality is interpreted, I proposed, the designer is in a better position to develop improved tools that enhance students' expressivity and thus, to facilitate student appropriation of mathematical ideas embodied in the designed artifacts.

This study bolsters a prevalent notion that students' mathematical reasoning is not either with or without objects, perceptual or conceptual, situated or symbolic, concrete or abstract (Barsalou 2008; Bartolini Bussi & Boni 2003; Hutchins 1995; Radford 2003). In fact, these pairs of constructs assume an ontology that does not capture the phenomenology of mathematical reasoning. Rather, mathematical reasoning is enacted coordination of multiple multimodal resources, including semiotic means distributed over space, time, and participants.

It has not been my intention, in this paper, to prescribe design processes, just as I would not wish to prescribe mathematical creativity – if either of these were completely proceduralizable, we would relegate these practices to software (Boden 1994; Schoenfeld 2005; Schön 1992) – yet by foregrounding for designers tacit aspects of their own practice, I hope to have made available for them discursive means of reflecting on and communicating this practice.

5 Implications for design and future work

Mathematical learning has been described as guided reinvention (Freudenthal 1986). Casting mathematical learning as invention may suggest that students' construction of ideas is necessarily deliberate, laborious, and protracted, culminating in a 'eureka!,' when the blend is born (Arieti 1976; Fauconnier & Turner 2002; Poincaré 2003/1897; Steiner 2001). And yet a conjecture rising from this study is that blending occurs spontaneously in semiotic activity, as a person scrambles to use any available means of objectification – be it substantive or epistemic – to communicate an emerging idea. In fact, when the selected medium fits the idea like a glove, a student may appropriate the medium without reflecting on the mathematical implications of the blend. Only when the blend breaks down, i.e., when ready-to-hand semiotic tools are conspicuous, obtrusive, obstinate, or missing, are its sources "disclosed for circumspection" (Heidegger 1962, p. 106; see also Koschmann, Kuuti, & Hickman 1998).

> Perception is consummated when one *addresses* oneself to something as something and *discusses* it as such. This amounts to *interpretation* in the broadest sense; and on the basis of such interpretation, perception becomes an act of *making determinate*. What is thus perceived and made determinate can be expressed in propositions, and can be retained and preserved as what has thus been asserted. (Heidegger 1962, p. 89, original italics)

Thus, usability per se, such as through avoiding ambiguity, need not necessarily be the golden standard in educational design as it is in industrial design (cf. Norman 2002). Rather, educational design is to some extent a craft of tradeoffs between usability and struggle.

I am proposing a semiotic approach as an intellectual foundation for a principled design framework (Abrahamson 2003, 2004b; Abrahamson & Wilensky 2007; Bakker & Hoffmann 2005; Fuson & Abrahamson 2005). Students perform semiotic acts with available media *apropos* of warranting intuitive inferences. So doing, they construct new meaning by instrumentalizing the semiotic tools so as to describe properties they notice in other objects, themselves potential semiotic tools. Learning environments should thus include a constellation of semiotic objects strategically selected/created on the basis of the desired compositions their linking affords (Abrahamson 2006b). The teacher problematizes the blends so that they break, and students reflect on the mathematical implications of this communication breakdown.

The embodied-design approach demonstrated herein may be helpful in guiding this craft of constructing tools for student construction of meaning. To do so effectively, communities of mathematics-education scholars operating from the complementary perspectives of cognition, socio-cultural theory, and semiotics should pool their resources and enter in dialogue.[13]

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