Abstract Meditations on the Concrete and Concrete Implications for Mathematics Education

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This paper was published in I. Harel & S. Papert (Eds.) Constructionism. Norwood N.J.: Ablex Publishing Corp. 1991.

No ideas but in things - **William Carlos Williams**

INTRODUCTION

Seymour Papert has recently called for a "revaluation of the concrete": a revolution in education and cognitive science that will overthrow logic from "on top and put it on tap." Turkle and Papert (1991) situate the concrete thinking paradigm in a new "epistemological pluralism" - an acceptance and valuation of multiple thinking styles, as opposed to their stratification into heirarchically valued stages. As evidence of an emerging trend towards the concrete, they cite feminist critics such as Gilligan's (1982) work on the contextual or relational mode of moral reasoning favored by most women, and Fox Keller's (1983) analysis of the Nobel-prize-winning biologist Barbar McClintock's proximal relationship with her maize plants, her "feeling for the organism." They cite hermeneutic critics such as Lave (1988), whose studies of situated cognition suggest that all learning is highly specific and should be studied in real world contexts.

For generations now we have viewed children's intellectual growth as proceeding from the concrete to the abstract, from Piaget's concrete operations stage to the more advanced stage of formal operations (e.g., Piaget, 1952). What is meant then by this call for revaluation of the concrete?

And what are the implications of this revaluation for education? Are we being asked to restrict children's intellectual horizons, to limit the domain of inquiry in which we encourage the child to engage? Are we to give up on teaching general strategies and limit ourselves to very context specific practices? And what about mathematics education? Even if we were prepared to answer in the affirmative to all the above questions for education in general, surely in mathematics education we would want to make an exception?

If there is any area of human knowledge that is abstract and formal, surely mathematics is. Are we to banish objects in the head from the study of mathematics? Should we confine ourselves to manipulatives such as Lego blocks and Cuisinaire Rods? Still more provocatively, shall we all go back to counting on our fingers?

We often use phrases such as "concrete thinking", "concrete-example", "make it concrete" when thinking about our own thinking as well as in our educational practice. In this paper I will show that such phrases, although often used, are not well understood -- indeed the standard definitions we have of "concrete" are flawed and inadequate. I present a new characterization of the concrete that addresses these inadequacies, expands our notion of the concrete, draws implications for our educational practice and, in the sense I will develop, concretizes it.

To begin our investigation we will need to take a philosophical detour and examine the meaning of the word concrete. What do we mean when we say that something - a concept, idea, piece of knowledge (henceforward an *object*) - is concrete?

STANDARD DEFINITIONS OF CONCRETE

Our first associations with the word concrete often suggest something tangible, solid; you can touch it, smell it, kick it; [1] it is real. A closer look reveals some confusion in this intuitive notion. Among those objects we refer to as concrete there are words, ideas, feelings, stories, descriptions. None of those can actually be "kicked." So what are these putative tangible objects we are referring to?

One reply to the above objection is to say: No no, you misunderstand us, what we mean is that the object referred to by a concrete description has these tangible properties, not the description itself. The more the description allows us to visualize (or, if you will, *sensorize*) an object, to pick out, say, a particular scene or situation, the more concrete it is. The more specific the more concrete, the more general the less concrete. In line with this, Random House says *concrete* is "particular, relating to an instance of an object" not its class.

Thus, my pillowcase is concrete; it is a unique instance with its particular color (faded milky white), texture (abraded cotton, but soft, like a well-worn jean) and elasticity (less than it was). But a mathematical triangle is not; it is described purely by its formal properties and has no color, thickness, nor any richness of detail apart from its defining properties.

A common opposition or dichotomy is to oppose abstract to concrete. Webster's says "a poem is concrete, poetry is abstract." We thus have an implied continuum that, as it moves from pillowcases to triangles, gets less concrete and more abstract. This particular pen that I am currently using, which is made by Papermate, is black, has a cap roughly one sixth as long as the stem, which has some chew marks on it, is much more concrete than just plain "pen" or even "Papermate pen". These descriptions of my pen ascend[2] in levels of abstraction and can be further abstracted by making the move to "writing implement" or "communication tool". Note that, in the last case, objects which are not at all similar to pens,[3] objects such as language itself, are subsumed together with plain Papermates under the heading of "communication tool". Under this view, an attempt to operationalize a criterion for deciding if a concept or description is concrete might look like: "determine how many objects in the world could fit this description; the lower the number the more concrete."[4]

Let us call the notion of concrete specified by the above the standard view. If we adopt the *standard view*, then it is natural for us to want our children to move away from the confining world of the concrete, where they can only learn things about relatively few objects, to the more expansive world of the abstract, where what they learn will apply widely and generally.

Yet somehow our attempts at teaching abstractly leave our expectations unfulfilled. The more abstract our teaching in the school, the more alienated and bored are our students, and far from being able to apply their knowledge generally across domains, their knowledge displays a "brittle" character, usable only in the exact contexts in which it was learned.[5] Numerous studies have shown that students are unable to solve standard math and physics problems when these problems are given without the textbook chapter context. Yet they are easily able to solve them when they are assigned as homework for a particular chapter of the textbook (e.g., DiSessa, 1983; Schoenfeld, 1985).

CRITIQUES OF THE STANDARD VIEW

Upon closer examination there are serious problems with the standard view. What does it mean for something to be specific as opposed to general? Surely, we know what we mean by that? Well, let's see: specific descriptions can be satisfied by only a few objects, while general ones can be satisfied by many. Let's take an example, say the word "snow".[6] Is snow concrete? Your first reaction would probably be: Of course, snow, that fluffy stuff that falls from the sky each winter, that stuff that covers the ground and for a moment makes the land a virgin untouched by human imprint, that stuff that fell on Robert Frost's horse while they were stopped one evening. Surely, if anything is specific and concrete, snow is!? Unless, of course, you are an Eskimo (and my apologies for presuming otherwise). For Eskimos, as we all know, have many words for snow (twenty-two according to my Funk and Wagnalls)[7], and each of them describes a particular kind of snow with its particular sensory qualities. Snow, for an Eskimo, is a vast generalization, combining together twenty-two different substances, some of which may be as different to an Eskimo as "pens" and "languages" are to us.

So we see here one faulty assumption that underlies the standard view: the assumption that there are a fixed number of objects in the world, i.e., that people's ontologies are identical [8], or that there is one universal ontology in an objective world [9]. But as was first noted by Quine (1960), this is not the case: There are a multitude of ways to slice up our world. Depending on what kind and how many distinctions you make, your ontology can be entirely different. Objects like snow which are particulars in one ontology can be generalizations in another. Indeed for any concrete particular that we choose, there is a world view from which this particular looks like a generalization.

An even more radical critique of the notion of a specific object or individual comes out of recent research in artificial intelligence (AI). In a branch of AI called emergent AI, objects that are typically perceived as wholes are explained as emergent effects of large numbers of interacting smaller elements. Even the human mind, our once archetypical example of an individual, is now said to be made up of a society of agents (Minsky, 1987). Research in brain physiology as well as in machine vision indicate that the translation of light patterns on the retina into a "parsing" of the world into objects in a scene is an extremely complex task. It is also underdetermined; by no means is there just one unique parsing of the inputs. Objects that seem like single entities to us could just as easily be multiple and perceived as complex scenes, while apparently complex scenes could be grouped into single entities. In effect the brain constructs a theory of the world from the hints it receives from the information in the retina. Thus it is entirely possible to imagine a visiting alien "seeing" what you call this concrete chair as a random collection of variegated particles designated by some strange abstract label. The alien might not even perceive the area of space you call chair to be filled at all, or to be filled partially by one object and partially by another. To make this alien more concrete, just imagine for instance a virus's "eye" - view of say a wicker chair. It is because children share a common set of sensing apparatus (or a common way of obtaining feedback from the world, see Brandes & Wilensky, 1991) and a common set of experiences such as touching, grasping, banging, ingesting [10], that children come as close as they do to "concretizing" the same objects in the world [11].

This critique of the standard view is also beholden to Piaget. But instead of focusing on the progression of the child through stages, this view takes as its focus Piaget's emphasis on construction; that the child actively constructs his/her world. Each object constructed is added to the personal ontology of the child. The phenomenon of conservation indicates the creation of a new stable entity that is added to the ontology. Before the conservation of number, there is no number object in the child's world [12]. One consequence of this view is that we can no longer maintain a simple sensory criterion for concreteness, since virtually all objects, all concepts which we understand, are constructed, by an individual, assembled in that particular individual's way, from more primitive elements [13]. Objects are not simply given to the senses; they are actively constructed [14].

We have seen that when we talk about objects we can't leave out the person who constructs the object. (To paraphrase Papert: You can't think about *something* without thinking about *someone* thinking about *something*.). It thus follows that it is futile to search for concreteness in the object -- we must look at a person's construction of the object, at the relationship between the person and the object.

As an example, let us return to play in the "snow" one more time. As we have seen, "snow" which for a New Englander is concrete, is an abstract generalization for an Eskimo. The particular varieties of snow which for an Eskimo are concrete are not even objects in the New Englander ontology. The distinctions, unimportant to most New Englanders, have not been made, leaving the specific snow objects unconstructed. We search in vain if we seek a property of snow that will determine its concreteness. We must look at our construction of a snow description, our relationship with snow, in order to find out if it is concrete for us or not.

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TOWARDS A NEW DEFINITION OF CONCRETE

"*Only Connect*" - E.M. Forster

The above discussion leads us to see that concreteness is not a property of an object but rather a *property of a person's relationship to an object* [15]. Concepts that were hopelessly abstract at one time can become concrete for us if we get into the "right relationship" with them.

I now offer a new perspective from which to expand our understanding of the concrete. The more connections we make between an object and other objects, the more concrete it becomes for us. The richer the set of representations of the object, the more ways we have of interacting with it, the more concrete it is for us. Concreteness, then, is that property which measures the degree of our relatedness to the object, (the richness of our representations, interactions, connections with the object), how close we are to it, or, if you will, *the quality of our relationship with the object*.

Once we see this, it is not difficult to go further and see that any object/concept can be become concrete for someone. The pivotal point on which the determination of concreteness turns is not some intensive examination of the object, but rather an examination of the modes of interaction and the models which the person uses to understand the object. This view will lead us to allow objects not mediated by the senses, objects which are usually considered abstract - such as mathematical objects - to be concrete; provided that we have multiple modes of engagement with them and a sufficiently rich collection of models to represent them.

When our relationship with an object is poor, our representations of it limited in number, and our modes of interacting with it few, the object becomes inaccessible to us. So, metaphorically, the abstract object is high above, as opposed to the concrete objects, which are down and hence reachable, "graspable." We can dimly see it, touch it only with removed instruments, we have remote access, as opposed to the object in our hands that we can operate on in so many different modalities. Objects of thought which are given solely by definition, and operations given only by simple rules, are abstract in this sense. Like the word learned only by dictionary definition, it is accessible through the narrowest of channels and tenuously apprehended. It is only through use and acquaintance in multiple contexts, through coming into relationship with other words/concepts/experiences, that the word has meaning for the learner and in our sense becomes concrete for him or her. As Minsky says in his *Society of Mind:*

> The secret of what anything means to us depends on how we've connected it to all the other things we know. That's why it's almost always wrong to seek the "real meaning" of anything. A thing with just one meaning has scarcely any meaning at all (Minsky, 1987 p. 64).

This new definition of concrete as a relational property turns the old definition on its head. Now, thinking concretely is seen not to be a narrowing of the domain of intellectual discourse, but rather as opening it up to the whole world of relationship. What we strive for is a new kind of knowledge, not brittle and susceptible to breakage like the old, but in the words of Mary Belenky, "connected knowing" (Belenky, Clinchy, Goldberger, & Tarule, 1986).

Keeping in mind that the adjective concrete applies not to things, not to concepts or ideas or physical objects, but rather to relationships between people and these things, it follows that what we would like to achieve in the schools by revaluing the concrete is not a restriction of children's knowledge to a smaller but more "concrete" domain, but rather an enrichment of the child's relationship to the whole panorama of human intellectual endeavor. The lesson we take from Piaget is not that the child develops by leaving behind the primitive world of concrete operations and leaping into the enlightened world of adult formal operations. Rather what we desire is that the child concretize his or her world by engaging in multiple and complex relationships with it.

CONSEQUENCES OF THE NEW VIEW

Let us return to the classroom and try to gain insight by seeing how these ideas work out in a school setting. We will take as our example the teaching of fractions, a subject thought to be difficult for most children to apprehend because the material is "so abstract." [16] Indeed fractions are an appropriate example for study, since one of the primary difficulties in understanding fractions is in grasping that the fraction expresses a *relationship* between a part and a whole (e.g., Harel, 1988). The difficulty lies in the child's confusion about what the whole is, the very same difficulty we encountered when trying to define *concrete*. The traditional approach to teaching the manipulation of fractions is to give rules for each operation, rules such as "to add fractions, make a common denominator," "to divide fractions, invert and multiply." [17] These rules are given as if they were definitions: they are supposed to serve as the meaning of their corresponding operations. They are not connected to each other, nor to previous knowledge about fractions. Indeed, studies have shown that, in the case of dividing fractions, no connection is made between the notion of division in fractions and familiar division of whole numbers (e.g., Ball, 1990; Wilensky, 1989). These practices lead to a disconnected knowing, a knowledge of fractions that can only bear up if one is given problems that just call for application of these rules [18].

The solution to this problem, however, is not to avoid abstract objects like fractions, or even to replace rules for manipulating them with situated practices such as suggested by Lave (1988). These solutions use the old mistaken notion of concrete, a notion of concrete as a property of certain objects but not others, in order to restrict the domain of learning. Rather, we must present multiple representations of fractions, both sensory (pies, blocks, clocks) and non-sensory (ratios, equivalence classes, binary relations), and give opportunities for the child to interact with all of these and establish connections between them. This kind of enrichment of the relationship between the child and the fraction will make the fraction concrete for the child and provide a robust and meaningful knowledge of fractions.

By establishing this kind of complex and multifaceted relationship with the fraction, the child may still not fall in love with fractions as Papert did with the gears of his childhood (Papert, 1980), but at least fractions will be brought into the "family" thus enabling a lifelong relationship with them.

Most of us who have participated in mathematics classes have had the experience of myriad definitions and theorems swirling about you, in the air, out of reach, any attempt to grab hold of one sends the others speeding away. Okay, so you can do the homework, but what is really going on here?

If you were one of the fortunate ones, at some point in the class something clicked and it all came together [19]. But the fact that it all came together for you, though doubtless due in part to your own native talent, is largely a matter of happy accident. Almost nothing is done in our math classrooms to facilitate this clicking into understanding.

Those of us who click are rewarded, and often pursue the study of mathematics. Those of us who do not, learn that they "aren't good at math" and rarely continue on in it. I argue here that this sudden click of understanding, this dawn of early light, is nothing other than our old friend the concretizing process (henceforward *concretion*) at work. Concretion is the process of the new knowledge coming into relationship with itself and with prior knowledge, and thus becoming concrete.

It would thus appear again that the standard Piagetian view of stage is turned on its head. In the school setting, rather than moving from the concrete to the formal, we often begin our understanding of new concepts (just as we often do with new people) by having a formal introduction. Gradually, as the relationship develops it becomes more intimate and concrete. Outside of school, in the world, our nascent understanding of a new concept, while not usually formal is often abstract because we haven't yet constructed the connections that will concretize it. The reason we mistakenly believed we were moving from the concrete to the abstract is that the more advanced objects of knowledge (e.g., permutations, probabilities) which children gain in the formal operations stage are not concretized by most adults. Since these concepts/operations are not concretized by most of us, they remain abstract and thus it seems as if the most advanced knowledge we have is abstract. It follows that the actual process of knowledge development moves from the abstract to the concrete. Only those pieces of knowledge that we have not yet concretized remain abstract.

HOW SHOULD EDUCATORS RESPOND?

Translated into practical advice for educators, this perspective gives a few answers and raises many questions. How do we foster the concretion process? What kind of learning environment nurtures it and promotes its growth? Clearly, much more research is needed to explore the many facets of this question. Here we point to only one: the *constructionist paradigm* for learning (see Harel & Papert, 1990). When we construct objects in the world, we come into engaged relationship with them and the knowledge needed for their construction. It is especially likely then that we will make this knowledge concrete. When Harel's fourth and fifth graders (Harel, 1988) construct a computer program for representing and teaching fractions, they have the opportunity to meet and connect multiple representations of fractions and to construct their own idiosyncratic relationships with and between them.

When people construct objects in the world external to them, they are forced to make explicit decisions about how to connect different pieces of their knowledge. How does one representation fit with another? Which pieces of knowledge are the most basic? Which are important enough to incorporate into the construction and which can be safely left out? Which really matter to them and which don't engage them at all? The constructionist paradigm, by encouraging the externalization of knowledge, promotes seeing it as a distinct other with which we can come into meaningful relationship.

I leave you with a thought experiment: What kinds of relationships between people would be fostered by a society which stipulated that people be introduced to each other formally and thereafter relate only in prescribed, rule-driven ways? If you shudder at this prospect, consider the analogy between this scenario and the *instructionist paradigm* for learning (see Harel & Papert, 1990). It is through people's own idiosyncratically personal ways of connecting to other people that meaningful relationships are established. In a similar way, when learners are in an environment in which they construct their own relationships with the objects of knowledge, these relationships can become deeply meaningful and profound.

ACKNOWLEDGMENTS

I wish to thank Seymour Papert and Ellie Baker for stimulating discussions which gave birth to this paper. Donna Woods, Mitchel Resnick, David Rosenthal, Idit Harel, David Frydman, and Aaron Brandes provided helpful and insightful comments on earlier drafts.

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[1] As in Samuel Johnson's famous refutation of idealism by kicking a stone.

[2] Our language uses height as a metaphoric scale to measure concreteness. Thus the very concrete is down and the abstract up (where presumably it is hard to reach and to "grasp" and to "hold on to").

[3] That is, if we use a sensory metric as a measure of similarity.

[4] Provided that there is at least one such object. If there are no objects satisfying the description, the concreteness of the object is not specified. An amusing probably apocryphal anecdote concerns a mathematician giving a guest lecture at a research university. He defines a Rotman-Herstein group and then proceeds to spend the next hour proving all kinds of marvelous theorems about groups of this kind. Towards the end of the lecture a graduate student gets up and asks: "Esteemed professor, I am very impressed by all these amazing theorems that you have proved about Rotman-Herstein groups, but I have been trying to come up with specific (concrete) examples of such groups, and I can't find any except the trivial cases of which your theorems are manifestly true." It took the professor and the student only a little more time to show that no other examples of Rotman-Herstein groups exist.

[5] A fruitful analogy can be made here between this kind of brittleness and the brittle representation of knowledge that so called expert systems in AI exhibit. In effect this kind of abstract teaching is akin to programming our children to be rule-driven computer programs. A similar kind of brittleness can be found in simple animals such as the sphex wasp. For a discussion of "sphexish" behavior and how it differs from human behavior, see Hofstadter (1982) Dennett (1984).

[6] Famous for its inclusion in the defining sentence of the the logicist's account of the correspondence theory of truth: "'Snow is white' is true if and only if snow is white" (see Tarski, 1956).

[7] Recent research has disputed this claim(e.g., Pullum, 1989; Woodbury, 1991). There seems to be no present consensus as to precisely how many words or lexemes exist in Eskimo languages, such as Inuit or Yup'ik, which refer to a type of snow. The argument here though does not depend on the accuracy of this specific example. Experts in any domain develop distinctions that break up former unitary concepts into multiple sub-concepts, thus transforming a "concrete" object into a generalization.

[8] For if this were not the case, then each person who applied the definition would get a different result and moreover, the same person would get a different result at different periods of his or her development.

[9] A tenet of the empiricist world view. For an illuminating comparison of the empiricist, rationalist, and hermeneutic stances, see Packer and Addison (1989).

[10] I ignore here the social experiences which play a large role in determining which objects are useful to construct.

[11] Alternatively, we can say that children construct a model of the world through feedback they receive

from their active engagement with the world (again, see Wilensky, 1991).

[12] Or alternatively, there is a number concept, but it is incommensurable with the adult concept (see Carey, 1985).

[13] In other words, whether something is an object or not is not an observer-independent fact; there is no universal objective (sic) way to define a given composition as an object. It thus follows that when we call an object concrete, we are not referring to an object "out there" but rather to an object "in here", to our personal constructions of the object.

[14] Even the recognition of the most "concrete" particular object as an object requires the construction of the notion of object permanence.

[15] Or as we shall say later, "concretion" is the process by which "stuff", (i.e. sense data, more primitive objects) become objects for an individual -- in other words the process of an individual coming into relationship with an object.

[16] For those of us who think that fractions are not abstract, substitute imaginary numbers for fractions. I recall that, in grade school, when I first encountered imaginary numbers, they were very mysterious. What, I wondered, made some numbers imaginary and others real? Reflection on this question helped me see that despite the suggestive language real numbers weren't so real and imaginary numbers weren't so imaginary. Later, a high school student told me: "a mathematician is someone for whom imaginary numbers are just as real as real numbers." Not a bad definition.

[17] A rhyme gathered from one classroom goes, "Ours is not to reason why, just invert and multiply."

[18] Though one can go remarkably far with such limited knowledge. When a class of MIT graduate students was asked, "What does it mean to divide two fractions?" almost no one could muster any kind of answer. Of course all of them knew how to perform the calculation, yet each student, when made aware of the question, expressed a lack of understanding of what's going on. Though they could all state the flip and multiply rule, no one felt that this was a sufficient explanation of what division of fractions meant.

[19] Like a self-organizing system reaching a stable state?