Learning Axes and Bridging Tools in a Technology-Based Design for Statistics[1]

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Abstract

S.A.M.P.L.E.R. (Abrahamson & Wilensky, 2004a, 2004b) is a computer-based classroom learning-environment built in the NetLogo (Wilensky, 1999a) and HubNet (Wilensky & Stroup, 1999) modeling environments. S.A.M.P.L.E.R. is the statistics component of 'ProbLab' (Abrahamson & Wilensky, 2002) a middle-school curricular unit built at The Center for Connected Learning and Computer-Based Modeling as part of the 'Connected Probability' project (Wilensky, 1993, 1995, 1997a). We report results from an implementation of S.A.M.P.L.E.R. (two Grade 6 classrooms, total $n = 38$), and frame our analysis of student discussion in terms of two novel design-research constructs: 'learning axes' and 'bridging tools.' A *bridging tool* (Abrahamson, 2004; Fuson & Abrahamson, 2005) is a classroom artifact designed to tap students' previous mathematical knowledge and situational intuitions and help students build understanding of new mathematical concepts that are linked to symbols, procedures, and vocabulary. A *learning axis* is a space of potential learning extending between two competing perceptual interpretations of a bridging tool—students construct new concepts through reconciling the tension created by this dual interpretation. We discuss some tradeoffs inherent in the design of learning environments that use this approach.

> "Such problems [—especially problems like that of composing a poem, inventing a machine, or making a scientific discovery—] are intimations of the potential coherence of hitherto unrelated things, and their solution establishes a new comprehensive entity, be it a new power, a new kind of machine, or a new knowledge of nature" (Polayni, 1967, p. 44).

> "Not that I mean as sufficing for invention the bringing together of objects as disparate as possible; most combinations so formed would be entirely sterile. But certain among them, very rare, are the most fruitful of all" (Poincaré, 1903/2003, p. 51).

1 Introduction

1.1 Objective

This paper is about a specific learning environment for mathematics and about a theory of learning that evolved through analysis of data from iterative implementations of the design. The

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design, S.A.M.P.L.E.R. (Abrahamson & Wilensky, 2002, 2004a, 2004b), Statistics As Multi-Participant Learning-Environment Resource, is a computer-based probability-and-statistics learning environment created at The Center for Connected Learning and Computer-Based Modeling. S.A.M.P.L.E.R. is embedded in the NetLogo (Wilensky, 1999a) agent-based modeling environment and makes use of the NetLogo extension HubNet (Wilensky & Stroup, 1999) architecture.[3] S.A.M.P.L.E.R. is part of ProbLab (Abrahamson & Wilensky, 2002, 2004c), a suite of probability-and-statistics interactive models and construction activities that extend Wilensky's 'Connected Probability' project (1993, 1995, 1997a). The theory, "learning axes and bridging tools," is a theoretical–pragmatic lens that we are using both to frame our design rationales and to interpret student interaction with our design. This theory extends Abrahamson's (2004) work on bridging tools (see also Fuson & Abrahamson, 2005). We will first explain the design of S.A.M.P.L.E.R. and then demonstrate student interaction with this design through the lenses of the proposed theoretical model.

1.2 Design Problem: Student Difficulty With Probability and Statistics

'Connected Probability' is an attempt to respond to a century of theoretical and empirical studies reporting and analyzing student difficulty in the domain of probability and statistics (von Mises 1928/1957; Piaget, 1952; Hacking, 1975, 2001; Simon & Bruce, 1991; Shaughnessy, 1992; Konold, 1994; Wilensky, 1993, 1995a, 1997a; Biehler; 1995; Papert, 1996; Gigerenzer, 1998; Maher, Speiser, Friel, & Konold, 1998; Henry, 2001). Authors have critiqued as detrimental to student learning the symbolical notation of the domain (Gigerenzer, 1998), embedded assumptions in learning environments regarding randomness (e.g., Henry, 2001; Maher, Speiser, Friel, & Konold, 1998), and a general disconnect in most mathematics curricula between student real-world experiences and formal mathematical expressions (Wilensky, 1997a; Pratt, 2000). Our reading of this body of literature is that there are challenging dualities inherent in the domain—pairs of juxtaposed subconstructs such as theoretical- vs. empirical probability (Abrahamson & Wilensky, 2003; Hacking 2001), dependent- vs. independent events (Abrahamson, Berland, Shapiro, Unterman, & Wilensky, 2004), exploratory data analysis vs. probability (Biehler, 1995), single events vs. expected values (von Mises, 1928/1957; Hacking, 2001), and the ultimate tenuousness of statistical measurement vs. "true" population properties (Abrahamson & Wilensky, 2004a).

We regard these dualities as defining the core learning issues that students must face if they are to master the domain of probability and statistics. Further, we propose that student difficulty with these learning issues could be treated not as 'confusions' indicating poor learning but as 'tensions' that could be generative of inquiry and deep understanding. We call the learning potential defined by a conceptual duality a *learning axis*, to emphasize the conceptual space extending between two juxtaposed subconstructs, e.g., between 'theoretical probability' and 'empirical probability.' Moreover, we regard each pair of juxtaposed subconstructs as inter-defining in a dialectical semiosis: 'theoretical probability' has little if any meaning without some understanding of 'empirical probability,' and vice versa. The new ideas that students have to construct, e.g., 'distribution,' are conceptual capstones that bridge this semiosis. We propose to leverage the potential inherent in learning axes by building *bridging tools* that evoke the tension along learning axes and stimulate students to construct new understandings that resolve this

[3] CCL material including all models is available for free download at http://ccl.northwestern.edu/

tension. In this paper, we describe some of the bridging tools we designed for the domain and interpret classroom episodes in terms of students' conceptual constructions that are grounded in these tools.

1.3 Previous Learning Designs for Probability and Statistics

The promise of the computer as a medium for learning environments for mathematics, in general, and for probability and statistics, in particular, has long been discussed by Papert (1980, 1996) and others (e.g., Wilensky, 1993; Konold, 1994; Finzer, 2000). Oft-cited advantages of the computer environment are high-speed errorless data processing, dynamic-visualization capabilities, and interactive facilities that can support exploration and the testing of conjecture (e.g., Pratt, 2004).

In order to contextualize the learning-environment designs we discuss in this paper, we will now look at two of probability and statistics learning environments, TinkerPlots (Konold, 2004) and Fathom (Finzer, 2000), both products of multi-year development process that incorporate state-of-the-art technology and are achieving penetration into the education market.

TinkerPlots , designed by Cliff Konold and Craig Miller, is a dynamic data exploration environment for Grades 4 through 8. The developers introduce their software package as follows (Konold, 2004):

> Students can begin using TinkerPlots without knowledge of conventional graphs or different data types, without thinking in terms of variables or axes. By ordering, stacking, and separating data icons, students gradually organize data to answer their questions. Students can analyze data that come with the program, that they download from the Internet, or that they enter themselves. Using the construction set of basic operations, students create a wide variety of graphs, including standards like pie charts, histograms, and scatterplots, and novel graphs of their own invention. Because plot are built up in stages, students can deconstruct unfamiliar plot to learn how to interpret them. Students can save the current plot configuration as a new command ("skyline graph") to later recreate that plot type in one step. To perceive variability in data, TinkerPlots offers more than position along axes; it also offers differences in icon size, color, and sound. These additional modalities allow students to detect covariation in powerful and intuitive ways.

Fathom Dynamic Statistics (Finzer, 2000) is a powerful and versatile environment for studying statistics from a data-driven perspective. Fathom integrates data analysis with mathematical modeling and is targeted at secondary-education students. The environment enables users to start from data or from scratch. Features of this software include: direct manipulation of mathematical objects (axes, sliders, lines, and data) and synchronous update off all dependent objects during dragging operations; parameterization with sliders; omnipresence of algebraic formulas; and ease of data import (Finzer, 2000). A typical use of Fathom might begin with importing a data set and follow with the testing of hypotheses through manipulation-based generation of various graphs, histograms, scatterplots, etc. as well as tables, all interlinked with equations. The user can also build and run simulations of probability experiments.

Both TinkerPlots and Fathom incorporate dynamic visualization elements that support student learning of mathematical representations of probability and statistics. We will now explain the rationale of using NetLogo, as a software platform for the study discussed in this paper. As we will explain, the parallel-processing architecture of the NetLogo environment enables an 'embodied' probability, thus helping students to ground the concepts of probability and statistics in personal experiences of chance. That is, our choice of NetLogo, an agent-based environment, was informed by a reconceptualization of the domain of probability and statistics—a reconceptualization that foregrounds the micro-level of probabilistic events as well as the macro-level of summative mathematical representations. We now explain the agent-based conceptualization of the domain.

1.4 Connected Probability: Agent-Based Computer Models for Learning Probability and Statistics

In this section we discuss the 'Connected Probability' project, and, specifically, how agent-based computer models may contribute to student learning of probability and statistics. We then briefly overview the ProbLab curricular suite of models and classroom activities, and situate ProbLab within the Connected Probability project. This design will be explained in further detail in a later section of this paper.

1.4.1 What is Agent-Based Modeling?

Computer-based modeling has become a standard research method in academe and industry—it enables researchers to think through complicated phenomena and visualize innovative ideas. One form of modeling is *agent*-based modeling (e.g., NetLogo, Wilensky, 1999; *Repast*, Collier & Sallach, 2001; *Swarm*, Langton, & Burkhardt, 1997). An "agent" is a computational entity that can represent an object that is being studied, e.g., a "coin." Agents are often associated with an icon that appears on the computer interface and possibly moves there, e.g., the coin could "flip." The modeler "creates" agents, assigns properties to them, and defines rules that the agents follow when the computer procedures are activated. These rules govern the agents' interactions with each other and with elements in their environment. Papert (1980) argued that modeling from the agent's perspective enables students to draw on their personal resources to make sense of mathematical concepts—students embody the agent by tapping their kinesthesia and sense of spatial orientation/navigation. A sense of chance, too, we will see later, can be embodied in agent-based modeling.

Following educational reformers call to harness the computer in education, education researchers have collaborated with computer developers to build computer-based learning environments that are accessible to students with little or no background in programming (e.g., *LOGO*, Papert, 1980; *StarLogoT*, Wilensky, 1997b). Students use the modeling environments to express hypotheses and then run simulated experiments to examine these hypotheses (Hoyles & Noss, 1992; Wilensky & Reisman, 2005). For instance, a young student using a computer to simulate the trajectory of a missile may learn about ballistics when her rocket descends in straight lines instead of along an arc.

Modeling from the perspective of the agent, e.g., an ant, rather than from the perspective of the aggregate, e.g., a colony, is conducive to understanding mechanisms underlying phenomena involving multiple interacting agents. For instance, an entomologist studying ant colonies might wish to assign rules to "ants" and then observe whether their interactions emulate what she has observed in the field. To support such research, *multi-agent* parallel-processing modeling environments have been developed (e.g., NetLogo, Wilensky, 1999a; Swarm, Langton & Burkhardt, 1997; Repast, Collier & Sallach, 2001).

Note that when a multi-agent simulation is activated, each agent follows its rules independently. Often, the simulation consists of repeated iterations of procedures, with each iteration's output feeding in as input into the subsequent iteration. Whereas the agents follow the same rules, the agents often behave differently one from another, for several reasons: (1) Each agent may interact with objects in its immediate spatial environment that possibly differs from other agents' immediate environment; (2) The modeler may incorporate randomness to modify agents' behaviors, and agents' random values are independent and (3) Agents may repeatedly update their properties based on their experience in each iteration of the procedure. The aggregate result of all these behaviors can be quantified and represented in mathematical notation on the screen. For instance, if 1,000 coins flip, a monitor might show that 529 coins "landed" on 'heads.' A subsequent run of this simulated experiment may yield different outcomes, e.g., 478 'heads,' and a graph could track the outcomes of repeated iterations. We will return to agent rules and aggregate quantification later when we discuss our design.

1.4.1.1 What is NetLogo?

The learning environments described in this paper are embedded in the NetLogo (Wilensky, 1999a) multi-agent modeling-and-simulation environment. NetLogo is a more mature successor to StarLogoT (Wilensky, 1997b). NetLogo is designed to suit the needs both of young learners and practicing researchers (the "low threshold—no ceiling" principle). Middle-school students can build NetLogo simulations that enhance their understanding of the natural sciences, and an increasing number of researchers uses NetLogo to simulate experiments in a wide range of subject matter. For both young and mature practitioners, the iterative process of building a model, running experiments, and adjusting the model has helped achieve deeper understanding of the phenomena being modeled (e.g., Abrahamson & Wilensky, 2003; Abrahamson, Berland, Shapiro, Unterman, & Wilensky, 2004; Wilensky, 1999b; Wilensky & Reisman, 2005).

1.4.2 Rationale

NetLogo was authored with the vision of enabling students to participate actively in rigorous and engaging inquiry. The pedagogical motivation of these environments was articulated in Wilensky's (1993) doctoral thesis, in which he critiqued the inadequacy of prevalent mathematics curricula to support students' 'connecting' to knowledge. Wilensky (1993, 1997a) diagnosed mathematics curricula as both ahistorical and "acognitve"—they do not provide students with opportunities to decipher mathematicians' historical progress from intuition to formal structures; that therefore students' use of formal structures is not grounded in their *own* intuition; that therefore students do not engage in learning mathematics; and that therefore students suffer from *epistemological anxiety* (Wilensky, 1997a)—they have little if any meaning for the solution procedures they are taught to use. This diagnosis has led Wilensky to design environments for students to connect to mathematical constructs by building from their intuitions towards mathematical articulation of these intuitions.

Arguably, the historical development of probability has been constrained by the availability of suitable computational media and devices. For example, an abacus is not a suitable or powerful enough tool to generate and monitor large numbers of random outcomes. At the same time, even young students have intuitions about chance (Piaget & Inhelder, 1975; Fischbein, 1975). Wilensky's project, 'Connected Probability,' was to use the computer environment so as to nurture students' intuition of chance towards fluency with the fundamental concepts of the domain—randomness, sampling, and distributions.

1.4.3 Previous Connected-Probability Work

Phenomena involving probabilistic behaviors can be viewed both from "micro" (agent) and "macro" (aggregate) perspectives, and connecting these perspectives is a key to understanding probability (Wilensky, 1993, 1995, 1997a). Therefore, multi-agent computational environments are conducive to the study of probability: each agent "rolls its own die" (micro perspective), and these random outcomes can be viewed as a single distribution (macro perspective). Students who explore simulations of probability experiments have opportunities to study, discuss, and understand the relation of micro and macro perspectives and to thus deepen their understanding of probability (Wilensky, 1995; Papert, 1996; Pratt, 2000; Abrahamson & Wilensky, 2005a). The Connected Probability project has recently been extended through the *ProbLab* (Abrahamson & Wilensky, 2002a) curriculum under development.

It is worthwhile to dwell here on a difference between agent-based and other approaches to simulations of probability experiments. To do so, we will consider a generic case of flipping 100 "coins" on a computer screen. The agent-based approach is to have each coin land on 'heads' with a probability of, say, .5. The aggregate approach is to have a normally-distributed-around-50 number of the coins land on 'heads.' For the naïve viewer, there is no difference between these approaches—in each case roughly 50 of the coins will land on 'heads.' Moreover, over repeated samples of 100 coins, either approach will eventually lead to an accumulation of sample means that is normally distributed around 50 coins, or .5 of the sample. Yet, the agent-based approach, we argue, is more loyal to student intuition of chance, whereas the aggregate approach is couched in terms of a mature understanding of the concept (Wilensky, 1997a). That is, a student who is building a simulation of a probability experiment could only author such an "aggregate" procedure if the student already understood distributions. Student-authentic procedures are crucial if one believes, as we do, that the procedures underlying the simulations should be "glass boxes" rather than "black boxes."(Wilensky, 2001). In this sense, agent-based modeling of probability experiments nurtures a simple notion of chance so as to sustain the credibility of the simulation. Moreover, the agency of the coins supports a tension between the agent-based phenomenology of chance (a group of individual coins is flipping) and its aggregate quantification (e.g., .54 of the coins fell on 'heads'). This tension, we believe, can engender student *self*-construction of probability-and-statistics constructs, e.g., 'distribution,' that are grounded both in the agent and the aggregate meanings and bridge these meanings. Therefore, as compared to strictly aggregate modeling of probability, agent-based modeling is possibly more student driven rather than concept-driven.

1.4.4 ProbLab: Connected-Probability Curricular Suite of Models

Earlier Connected-Probability computer-based models were authored in both the StarLogoT and NetLogo environments. For instance, in the library of sample models that comes with NetLogo, models simulate Galton's Box and the "Monty Hall paradox." ProbLab (Abrahamson & Wilensky, 2002a) extends this approach by packaging a suite of NetLogo models and classroom construction activities designed to help students experience probability and statistics from micro and macro perspectives and to relate these perspectives. ProbLab has been implemented in a sequence of design-research studies with elementary and middle-school students who worked as individuals, in focus groups, and in urban classrooms. In these studies, student participation in ProbLab activities stimulated classroom discussions of combinatorial analysis and its relation to sampling and distribution (Abrahamson & Wilensky, 2004a-c, 2005a-e).

1.4.4.1 NetLogo Models

The ProbLab interactive simulations are authored and run in NetLogo. NetLogo models typically include a "graphics window"—the visualization space where the agents are embodied as icons, various sliders, switches, and choice buttons for setting the simulation parameters and activating the simulation, and monitors, graphs, and histograms that show current-state values from the simulation, e.g., the total number of coins that fell on 'heads.' Students can study the code—the program procedures that underlies the agents' behavior, they can modify this code to extend the model, or author their own model. NetLogo models also include "info windows" with texts that explain the content of the model and how to operate it as well as suggested activities that scaffold students' learning paths.

1.4.4.2 Participatory Simulations and HubNet

Participatory simulation activities ("PSA," Wilensky & Stroup, 1999a) are group activities in which learners take part in simulating a phenomenon they are studying. For instance, students embody gas molecules and move around in the classroom space to simulate Brownian motion. Wilensky and Stroup (1999b) developed the HubNet technological infrastructure to enable computer-based PSA. In HubNet, instead of acting the role of agents *physically*, students each operate a *computer-based* agent. The shared space of agent activity is not the classroom physical space but the interface of the facilitator's computer—the "server." The server interface is projected onto a large classroom overhead screen, so that students can view the shared space.

Using a "client" that can be a laptop personal computer, a calculator, or some other handheld device, students control agents that share a common space. For instance, in the "Disease" PSA, the teacher "infects" one of the student–agents, and that student–agent chases and infects other student–agents, who, in turn, infect yet more student–agents. A graph records the total number of infected agents over time (the aggregate perspective). Typically, this graph is S-curved, because the overall rate of infection rises gradually at first, accelerates in the middle and then tapers off as all students are infected. Students initiate experiments to explore these aggregate mathematizations. To simulate these experiments, students use interface features to change the parameter values that modify the output of the program's procedures. Students can also download back to their client information from the shared interface.

1.4.4.3 S.A.M.P.L.E.R.

S.A.M.P.L.E.R. (Abrahamson & Wilensky, 2002b), Statistics As Multi-Participant Learning-Environment Resource, is ProbLab's statistics component. S.A.M.P.L.E.R. is a PSA that runs in HubNet. In the current version, the clients are laptop computers. In S.A.M.P.L.E.R., students take samples from a hidden "population." Each student take his/her own sample, uses that sample data to estimate the population statistics, and inputs this estimated statistic into the common histogram that accumulates all students' guesses. This PSA has been implemented with

10-, 12-, and 14-year-old students, who had opportunities to ground rich discussion of mathematical content in their collaborative sampling activities (Abrahamson & Wilensky, 2004a, 2004b).

Having overviewed the rationale of the Connected-Probability project and specifically of the ProbLab experimental unit, we will now overview the theoretical component of this paper. In later sections, we will explain the ProbLab activities and models in further detail so as to contextualize a subsequent elaboration of the theory in terms of student work in ProbLab.

1.5 Learning Axes and Bridging Tools: A Theoretical–Pragmatic Approach to Design for Learning

Learning axes and *bridging tools* are theoretical–pragmatic constructs for design-research in mathematics education. The two constructs are inter-defining, with a learning axis being more about the cognition of mathematics, and a bridging tool—more about the design of learning environments. These two constructs that have emerged through our studies of student learning in diverse mathematical domains enable us to articulate our design rationales in terms of our understanding of how students learn mathematics, and, in turn, to couch students' mathematical learning in terms of their interactions with our designs.

X X X

X X X

Figure 1. A bridging tool anchoring student construction of the commutative property of multiplication. Attending alternately to 2 rows of 3 X's or 3 columns of 2 X's—two competing disambiguations of this ambiguous figure—may stimulate an understanding of commutativity.

Figure 1 is an example of a picture that could serve as a bridging tool in a mathematics-education design. The picture can be attended to in many ways—it *affords* (Gibson, 1977) different perceptions. Two of the possible interpretations of this picture—the picture's meanings—are as 2 rows each made up of 3 X's or as 3 columns each made up of 2 X's. For the learner to build new understanding with these contrasting meanings, s/he would need to be aware of some quality of the picture that is conserved or remains constant across the competing perceptions—in this case the constant is the cardinality of the group (6 X's) or the 'object permanence' of the picture (knowing that the picture does not *really* change).

Between two alternate meanings for a single object extends the *learning axis*, a space of potential learning. The potential learning is evoked when the coexistence of two competing perceptions is foregrounded and problematized, stimulating the learner to construct a logical reconciliation (a "bridging") of these competing perceptions. In this case, one might construct the commutative property of multiplication ($a*b = b*a$) as a rule that relaxes the tension inherent in the object's ambiguity. In later sections, we elaborate on these constructs, situate them within the broader literature, and further demonstrate them in the analysis of classroom data.

1.6 Summary of Introduction

We began this section by discussing students' difficulty in studying probability and statistics. We analyzed the domain as configured by dualities of subconstructs. After an overview of some previous designs for probability and statistics, we discussed agent-based modeling-and-simulation environments, focusing on the Connected Probability project and the NetLogo software, and we suggested that these environments hold potential for students to bridge the challenging dualities of the domain. Specifically, we explained the structure of ProbLab, the experimental unit for probability and statistics that we are currently developing. Next, we introduced two theoretical constructs, learning axes and bridging tools, that help us design classroom learning artifacts in light of our domain analysis and inform our analysis of classroom data.

In Section 2, Design, we will elaborate on ProbLab's NetLogo models and construction activities. Specifically, we will explain one activity design {?}that culminates with the S.A.M.P.L.E.R. PSA. Section 3, Theory, builds on Section 2 to ground 'learning axes' and 'bridging tools' in the context of our design. Section 4 details the methodology of an implementation of ProbLab in two Grade 6 classrooms, and Section 5 presents results from that implementation. Section 6 is an analysis of students learning through the learning-axes and bridging-tools theoretical lenses. We conclude the paper in Section 7 with an evaluation of our design-research perspective, implications to mathematics education, and pointers to future work,

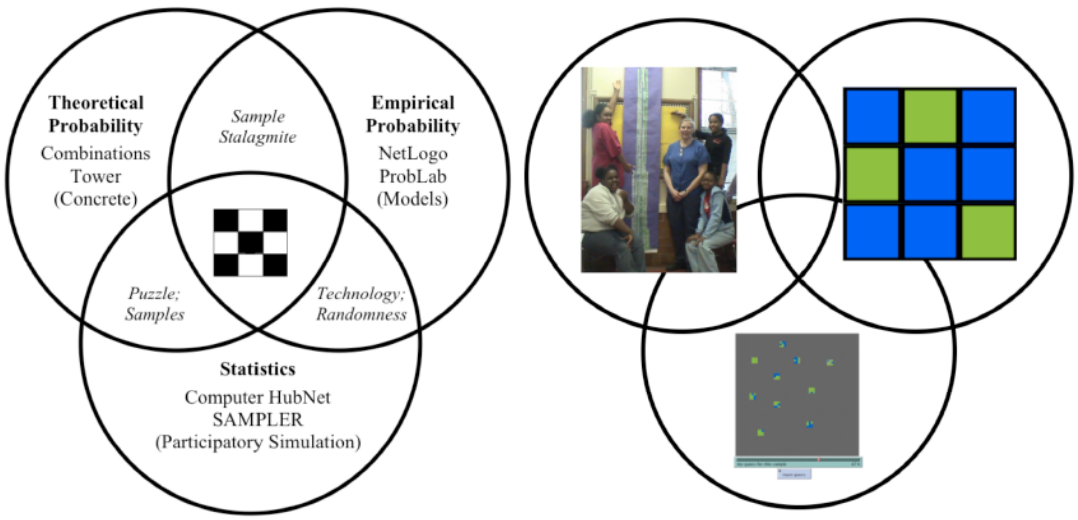
2 Design

In this section we will elaborate on the design rationale and activities of ProbLab (Abrahamson & Wilensky, 2002a), a computer-based experimental unit that extends the Connected Probability project (Wilensky, 1997a). Specifically, we will overview the activities that, in the current version of the unit, lead up to the S.A.M.P.L.E.R. participatory simulation activity. Explaining the earlier activities will contextualize both an explanation of the design of S.A.M.P.L.E.R., in this section, and discussions of our theory and of classroom data, in later sections.

2.1 ProbLab: Overview and Design Rationale

ProbLab was designed as an attempt to untangle and restructure the domain of probability and statistics so as to make it accessible to middle-school students. We view the domain as a set of three interleaved sub-constructs, or 'pillars,' that students need to coordinate: theoretical probability, empirical probability, and statistics (see Figure 2, below, on left). These pillars are not target cognitive constructs. Rather, this tripartite structure is a theoretical–pragmatic framework organizing our domain analysis, design space of learning tools and classroom activities, communication with teachers, and data analysis. The semiotic interdependency between these pillars is one reason probability and statistics can be hard to teach and to learn.

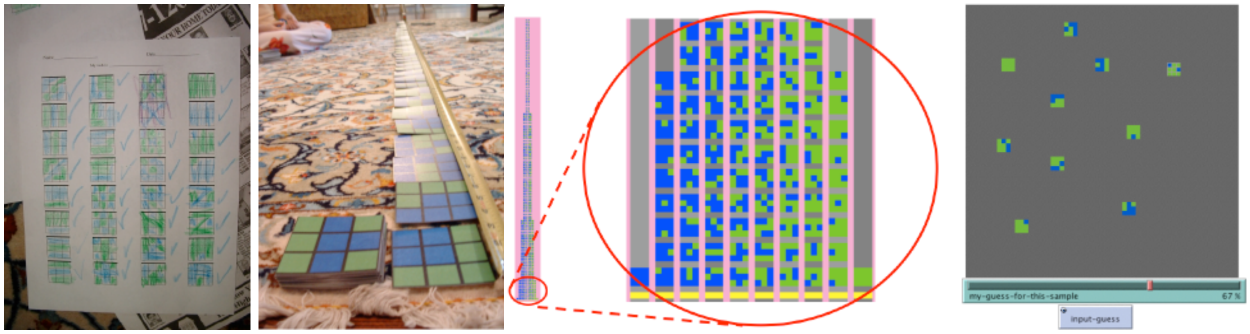
In teaching probability and statistics, it is tempting for educators to focus on one pillar or another and then move sequentially to the next pillar. A teacher may plan a unit as an orderly sequence of concepts, e.g., "theoretical probability" and then "empirical probability," but the students might remain with disconnected pockets of procedural knowledge or even not understand these pockets at all. Yet, it is also intractable to teach these pillars all at once, because we would then foster understanding that is muddled and undifferentiated. Our response to this dilemma is to design *bridging tools* (Abrahamson, 2004; Fuson & Abrahamson, 2005). A bridging tool is a mathematical object that can be viewed from different activity-driven perspectives (in Figure 2, below, the names of some of these bridging tools are in the overlapping areas of the circles on the left, and some of these tools are pictured on the right). Each classroom-activity context, e.g., theoretical or empirical probability, lends pillar-specific meaning to that object, and, moving between these frames, a student is stimulated to construct around the object a conceptual structure that coordinates the competing frames coherently. In this sense, bridging tools are what Papert has called "an object to think with."



Design Structure Classroom Tools

Figure 2. The ProbLab experimental curricular unit in probability and statistics.

ProbLab activities cohere around the '9-block' (see Figure 2, above, and see Figure 3, below). The *9-block* is what we call the *math-thematical* object of the ProbLab unit—it is the pivotal bridging tool of the design that features in each of ProbLab's classroom activities. The 9-block is a 3-by-3 array of little squares, each of which can be either of some 'target color,' such as green, or some 'other color,' such as blue. Framed from the theoretical-probability pillar, a given 9-block is one of all 512 (2^9) green/blue permutations in its combinatorial sample space (see Figure 3a, for a student's worksheet in which she created different 9-blocks; see Figure 3b, for a "9-Block Deck" of cards). Framed from the empirical-probability pillar, if you "roll" a 9-block in a computer model, you get different green/blue permutations. So the 9-block functions as ProbLab's computer-based counterpart of a coin or a die that are traditional mechanical probability objects serving in traditional probability curricula (see Figure 3c, for a fragment of a NetLogo model, in which 9-blocks are generated randomly and accumulated into a "histogram"). Framed from the statistics pillar, a 9-block is a sample out of a population of thousands of green/blue squares (see Figure 3d). Yet as a computational artifact, the 9-Block is a more powerful 'object to think with' than traditional stochastic devices such as a coin or a pair of dice.



a *b* *c* *d*

Figure 3. The 9-block: bridging theoretical probability, empirical probability, and statistics; bridging concrete and virtual media

The following sections explain further the classroom activities of ProbLab. Each section highlights how the activities help students to bridge between the domain's pillars. The unit begins with students building the combinatorial sample space of all green/blue 9-blocks (the *combinations tower*). Next, students work with computer microworlds in which they conduct simulated experiments: they randomly generate 9-blocks and then analyze patterns in the distribution of these random samples. Finally, students work in S.A.M.P.L.E.R., in which they take and analyze 9-block samples from a hidden population of thousands of green and blue squares. (For other objects, models, and activities and for further references, see http://ccl.northwestern.edu/curriculum/ProbLab/). Note that whereas we work with teachers to structure the ProbLab experimental unit according to lesson plans, the activities could be used in a more exploratory fashion.

2.2 The Combinations Tower

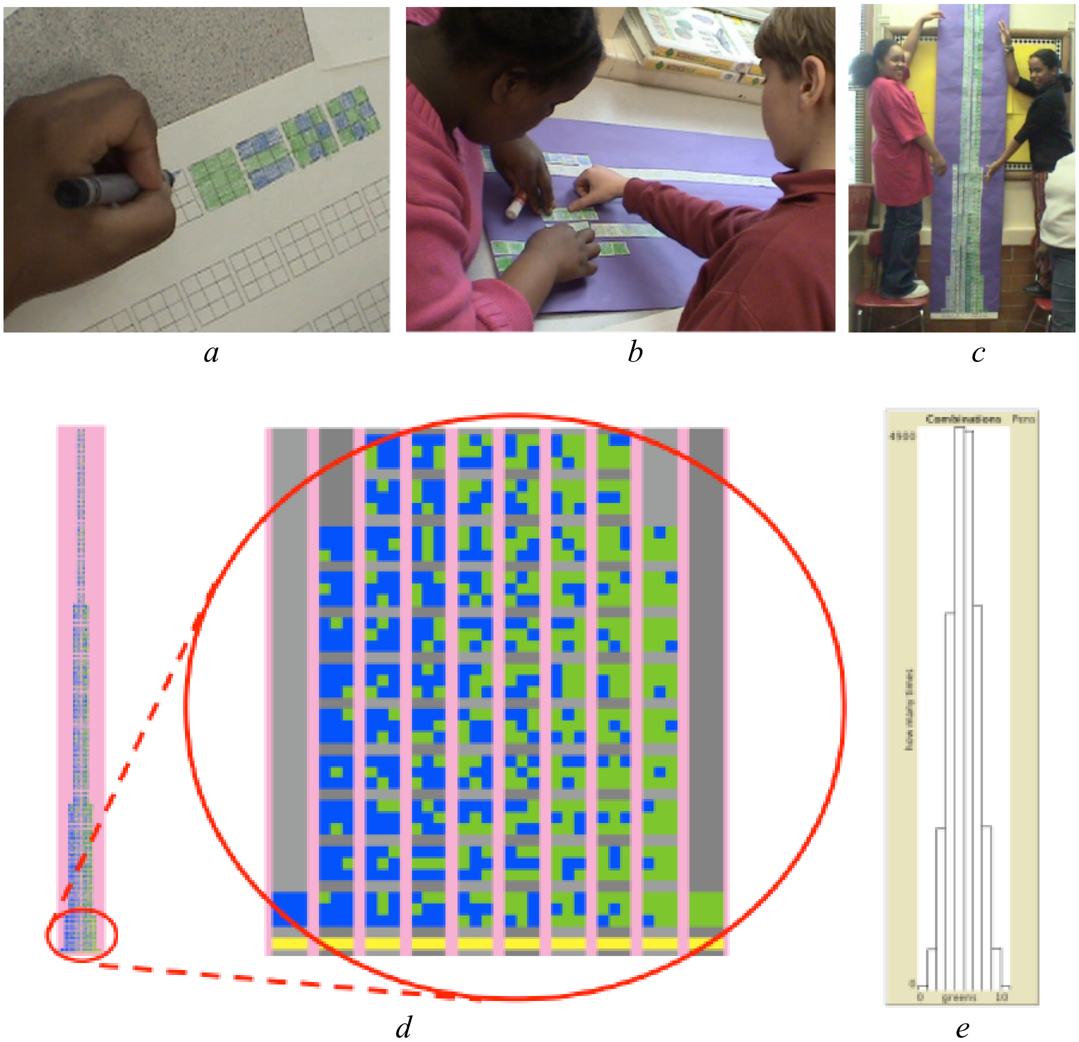


Figure 4. The 'combinations tower' ProbLab activity

ProbLab activities use traditional as well as computer-based media.[4] In the ProbLab activity *combinations tower*, students use paper, crayons, scissors, and glue to build the combinatorial sample space of all possible green/blue 9-blocks (see Figure 4a, b, and c, above). Abrahamson & Wilensky (2005b, 2005c, 2005d) report on how a 6th-grade classroom self organized to engineer, strategize, and produce this challenging mathematical object. Sixth-grade students create collections of 9-blocks (Figure 4a, above), with particular attention to avoiding duplicates. Students are then guided to construct this sample space in the shape of a histogram according to the number of green squares in the 9-blocks (see Figure 4d, above, for a clear view of the histogram). The heights of the histogram columns correspond to the coefficients of the binomial

[4] see Abrahamson & Wilensky, 2005e, for the *marble scooper*, a device, shaped as a 9-block, for sampling sets of nine marbles out of a box of marbles with known or unknown ratio between marbles of different color

distribution: 1, 9, 36, 84, 126, 126, 84, 36, 9, and 1. For example, there is only a single (1) combination with no green squares, there are 9 different combinations with exactly one green square, 36 with exactly two green squares, and so on. Although students do not yet understand binomial distribution functions, they do notice the symmetry and general emerging shape of the distribution and use this knowledge to inform their search for new combinations. Students paste the 9-blocks they have created onto a poster, grouped according to the number of green squares in each column and without duplicates (Fig. 4b). This histogram—the *combinations tower*—grows into a narrow and very tall poster that begins near the floor and soars up to the ceiling (Figure 4c, above). The students and teacher refer to this display in subsequent activities, as we will discuss in a later section.

Note the resemblance in shape of the pictures in Figure 4d and 4e (see above). Figures 4d and 4e are fragments from two NetLogo models: Figure 4d is the combinatorial sample space of 9-blocks (a combinations tower), created in the ProbLab model '9-Block Stalagmite' (Abrahamson & Wilensky, 2002c). Figure 4e shows the frequency distribution of randomly generated 9-blocks, created in the ProbLab model '9-Blocks' (Abrahamson & Wilensky, 2002d). So in content, the combinations tower is a combinatorial sample space, and, in shape, it shows the frequency distribution one would get if one were to generate large numbers of 9-blocks randomly. This hybrid or ambiguous property of the combinations tower is a hallmark of pedagogical bridging tools, designed artifacts that elicit student engagement in and negotiation between two or more action models that are complementary in understanding a domain. Thus, the combinations tower is poised to stimulate student thinking along the learning axis 'theoretical probability ←→ empirical probability' and students' construction of 'probability' as the mathematical concept that bridges this axis.

We find the juxtaposition of theoretical and empirical probability a fruitful pairing, generative of the domain's most 'powerful idea,' if to use Papert's (1980) term. We wish to emphasize that it is in student negotiating *between* two subconstructs of a domain that the most profound ideas can potentially be grasped (see Abrahamson, 2004, for evidence from the domain of ratio and proportion; see Abrahamson & Wilensky, 2004b, for evidence of students using this design to connect theoretical and empirical probability; see Episode Three in section 6.3 in this paper).

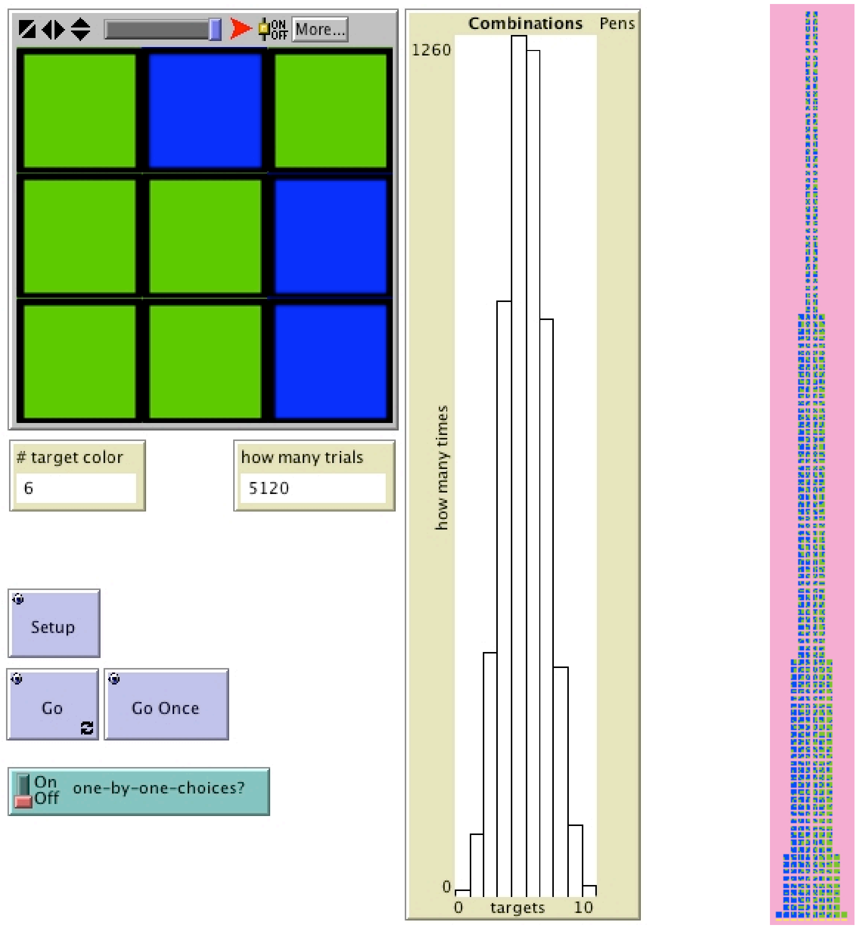
2.3 NetLogo Models

ProbLab interactive computer-based simulations are authored and embedded in NetLogo (Wilensky, 1999a). Each model constitutes a small microworld, an environment with built-in constraints that foregrounds target aspects of phenomena and invites users to engage in exploratory discovery-through-construction activities; the user constructs new knowledge through understanding the environment's constraints and adapting to them (for a discussion of constructivism, learning, and computers, see Forman & Pufall, 1988). Learners' exploration is facilitated through a set of modifiable parameters that shape both the environment and the behavior of elements within it. The model's parameters and their range of values define the parameter space of the model. Working within this space, students examine connections between the parameter space and experimental results. Users set the values of these parameters either through the interface or by accessing and altering the computer procedures. Given appropriate support, the users also write new procedures, such as assigning new properties to elements in the environment or initiating extensions of the model.

ProbLab models range along several dimensions, including specific content, the number of experimental parameters that the user can modify from the interface, and whether there currently are activities outside of the computer environment that relate to aspects of the simulation. To characterize the content of the models, we apply the learning-axes and bridging-tools perspective. For instance, the model 9-Block Stalagmite (Abrahamson & Wilensky, 2002c) bridges the theoretical probability and empirical probability spaces, and the model 9-Blocks (Abrahamson & Wilensky, 2002d) bridges between independent events and dependent events. Other models bridge out of the domain of probability and statistics, e.g., Dice (Abrahamson & Wilensky, 2002e) bridges concrete and computer-based random-outcome generators, Equidistant Probability (Abrahamson & Wilensky, 2002f, 2003) bridges geometry and empirical probability, and ProbLab Genetics (Abrahamson & Wilensky, 2005f) bridges Mendelian genetics and probability, and so these models are beyond the scope of this paper (see http://ccl.northwestern.edu/curriculum/ProbLab/index_models.html). We will now demonstrate with one of these models, 9-Blocks, that was used in the study reported in this paper.

9-Blocks (see Figure 5, below) was designed to help students understand outcome distributions obtained in empirical-probability experiments. The procedure generates a random 9-block by having each of the nine squares choose *independently* between green or blue (in the default version of the model, there is an equal chance of getting each of these colors). The user watches as, one by one or all at once, the squares each become green or blue: the one-by-one option enhances the independence of each of the nine outcomes, whereas the concurrency option suggests visually as though the procedure were selecting from the sample space of all 512 different 9-blocks. By the latter interpretation, each 9-block event is not a collection of independent outcomes but a single outcome.

Once the 9-block is all colored, the procedure treats the 9-block as a *compound* event: It counts up how many green squares are in the 9-block and records this number in a histogram. For instance, if the 9-block has 6 green squares in it (see Figure 5a, below), the sixth column from the left will rise up one unit. The histogram grows as the experiment runs and gradually takes on a stable shape—a normal (discrete) distribution (see, in Figure 5a, below, a histogram after 5,120 trials). The shape of the stable distribution resembles the combinations tower (compare Figure 5a, below, to Figure 5b). Why is this so? Recall that in the combinations-tower activity, the totality of 512 different 9-blocks was arranged in columns according to the number of green squares in each 9-block (see Figure 5b, below). In the 9-Blocks model, the histogram is, in form and function, the traditional mathematical representations—it represents diagrammatically a count of random outcomes by type. So how does a random procedure produce a distribution identical in shape to a form derived through combinatorial analysis? The answer lies in considering independent probabilities as well as numbers of combinations.



a. *b.*

Figure 5. Interface of ProbLab interactive simulation "9-Blocks"

There is an equal chance, 1/(2^9), that the program will produce any *specific* 9-block (e.g., the 9-block in Figure 5a, above), because each square has a 1/2 chance of being green and a 1/2 chance of being blue. However, the procedure does not tally the occurrence of specific 9-blocks but of classes of possible outcomes—a total of ten classes defined by the number of green squares in a 9-block. These classes (sets) of outcomes differ in probability because they differ in the number of members in each class. For example, there is a higher chance of getting a 9-block with a single green square as compared to the chance of getting a 9-block with no green squares, because there are 9 different 9-blocks with a single green square (it can be in 9 different locations), but only a single 9-block with no green squares. Similarly, there is a higher chance of getting a 9-block with 2 green squares as compared to getting a 9-block with 1 green square, because there are more combinations with 2 green squares (36 combinations) than with just 1 (9 combinations). Of course, this is the logic that led historically to the development of traditional formulas for computing binomial distributions, e.g., for determining the chance of getting *n* green squares in 9 independent trials. However, students in primary, secondary, and tertiary school and even experts often cannot explain the logic underlying these formulas (Wilensky, 1997a). The pedagogical objective of ProbLab is to help students build understandings so that eventually they can connect to the mathematical knowledge expressed in these formulas.

By allowing the user to view the 9-blocks appearing either square by square or all at once, the 9-Blocks interactive model provides opportunities for students to discuss differences between independent and dependent outcomes. By incorporating a histogram that resembles the *theoretical*-probability structure, the 9-Block *empirical*-probability activity is designed to help students bridge theoretical and empirical probability. Specifically, students who have participated in combinations-tower activities should have an experiential basis for understanding outcome distributions in terms of combinatorial analyses and random selection.

2.4 S.A.M.P.L.E.R.: ProbLab's HubNet Computer Participatory-Simulation Activity

The ProbLab experimental unit, which begins with combinatorial-analysis activities and continues with empirical-probability experiments, culminates with S.A.M.P.L.E.R., a statistics activity. S.A.M.P.L.E.R., Statistics As Multi-Participant Learning-Environment Resource (Abrahamson & Wilensky, 2002), is a participatory-simulation activity (Wilensky & Stroup, 1999) built in NetLogo (Wilensky, 1999a) and is extended through the HubNet technological infrastructure (Wilensky & Stroup, 1999a) to include a classroom of students who share aspects of the same data, act upon it simultaneously, and inform each other's actions (see the methodology section for further explanation of the classroom setting and the technology supporting S.A.M.P.L.E.R.).

We will begin by explaining the S.A.M.P.L.E.R. interface and then discuss students' work with S.A.M.P.L.E.R. Whereas this activity could function as a standalone, we will contextualize the activity design in light of students' earlier work in ProbLab. Such contextualization will allow us to highlight the opportunities we are designing for students possibly to achieve deeper understanding by bridging between the pillars of the domain, theoretical probability, empirical probability, and statistics. In a later section, we will discuss classroom data so as to focus on these connections between the pillars, which students may be building.

2.4.1 S.A.M.P.L.E.R. Interface: Population, Samples, and Distributions

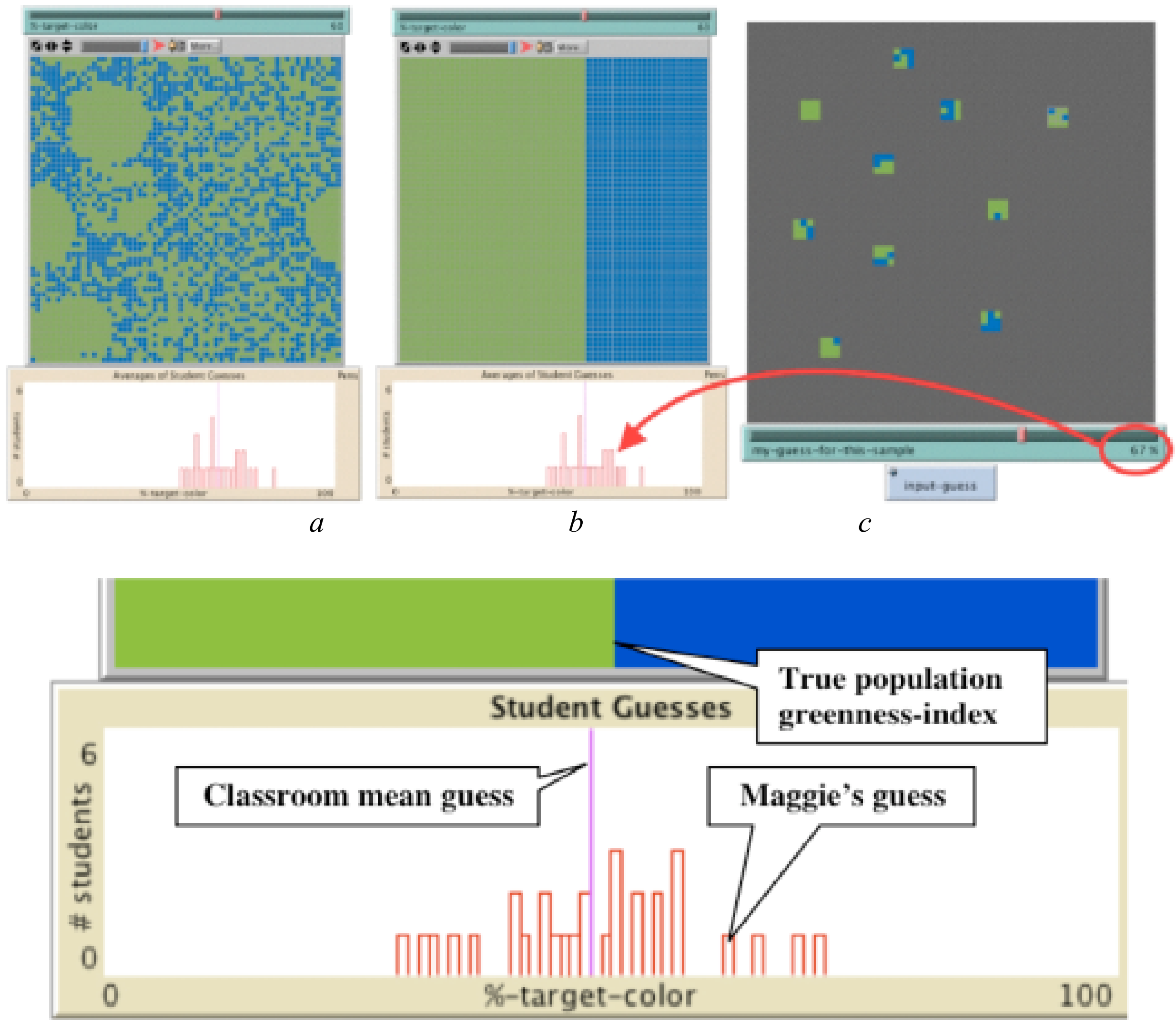


Figure 6. Selected features of the S.A.M.P.L.E.R. computer-based learning environment.

In S.A.M.P.L.E.R. (see Figure 6, above), students take individual samples from a population so as to determine a target property of this population. The "population" is a matrix of thousands of green or blue squares (Figure 6a) and the target property being measured is the population's greenness, i.e., the proportion of green in the population. A feature of the activity is that population squares can be "organized"—all green to the left, all blue to the right (Figure 6b). This "organizing" indexes the proportion of greenness as a part-to-whole linear extension that maps onto scales both in a slider (above it) and in a histogram of students' collective guesses (below it). Students participate through *clients* (in the current version of S.A.M.P.L.E.R., these clients run on students' personal computers). These clients are hooked up to the facilitator's *server*. Students take individual samples from the population (Figure 6c), and analyze these samples so as to establish their best guess for the population's target property. (Note that whereas all students sample from the same population, by default each student only sees their own samples, unless these are "pooled" on the server.) Students input their individual guesses and these guesses are processed through the central server and displayed as a histogram on the server's interface that is projected upon the classroom overhead screen (Figure 6d).

The histogram shows all student guesses and the classroom mean guess and interfaces with the self-indexing green–blue population. Note the small gap (Figure 6d, middle) between the classroom mean guess and the true population index. Because a classroom-full of students takes different samples from the same population, the histogram of collective student input typically approximates a normal distribution and the mean approximates the true value of the target property being measured. The students themselves constitute data points on the plot ("I am the 37"… "So am I!"… "Oh no… who is the 81?!"). So students can reflect both on their individual guesses as compared to their classmates' guesses and on the classroom guess as compared to the population's true value of greenness. Such reflection and the discussion it stimulates may be conducive to understanding typical distributions of sample means.

S.A.M.P.L.E.R. can constitute a standalone set of activities, yet the general framework of ProbLab is that students will participate in activities that interleave and juxtapose the statistics component, the theoretical-probability component, and the empirical-probability component. The 9-block plays a pivotal role in students' bridging between S.A.M.P.L.E.R. and the other pillars of ProbLab. The 9-block features in S.A.M.P.L.E.R. as samples of size 3 by 3. Students taking 3-by-3 samples from the S.A.M.P.L.E.R. population may construe the greenness of the population in terms of 9-blocks, and this interpretation may help students bridge from statistics to both theoretical and empirical probability.

Students may bridge between statistics and *theoretical*-probability by comparing between the S.A.M.P.L.E.R. population and the combinations tower. Specifically, students may construe a S.A.M.P.L.E.R. population as a collage from "the right side" (more green than blue) or "the left side" (more blue than green) of the combinations tower.

Students may bridge between statistics and *empirical*-probability using 9-block distributions: the act of sampling a 9-block from the S.A.M.P.L.E.R. population is meaningfully related to generating a random 9-block, e.g. in the 9-Blocks interactive model. Both in S.A.M.P.L.E.R. and in the 9-Block model, the user expects to receive a 9-block but does not know which 9-block will appear on the interface. So students may think of the S.A.M.P.L.E.R. population as a collection of many 9-blocks. This may support students in developing and using sophisticated techniques for evaluating the greenness of the S.A.M.P.L.E.R. population. Specifically, students may attend to each 9-block sample individually, and learn to use histograms so as to record sample values as distributions. Otherwise, students often count up all the green little squares they have exposed and then divide this total by the total number of exposed squares, in order to determine the greenness of the population. Such a strategy, albeit effective, misses out on a learning opportunity, because it does not mathematize the variety of samples as a distribution—it "collapses" the variation.

2.4.2 Learning with S.A.M.P.L.E.R.

Whereas participatory simulation activities (PSA, Wilensky & Stroup, 1999)) may take many trajectories, depending on facilitators' goals and learners' age and interest, we have found it useful to describe "typical" implementations of our PSAs when we have a clear idea both of the age group and the broader curricular context (see HubNet participatory-simulation guides at http://ccl.northwestern.edu/netlogo/hubnet.html). Such descriptions have helped teachers, and in particular teachers who are new to networked classrooms, prepare for facilitating the PSA in their own classrooms. The following description is based on focus-groups- and pilot-classroom studies of S.A.M.P.L.E.R. (Abrahamson & Wilensky, 2004b) and a ProbLab implementation that included S.A.M.P.L.E.R. (Abrahamson & Wilensky, 2005a-e).

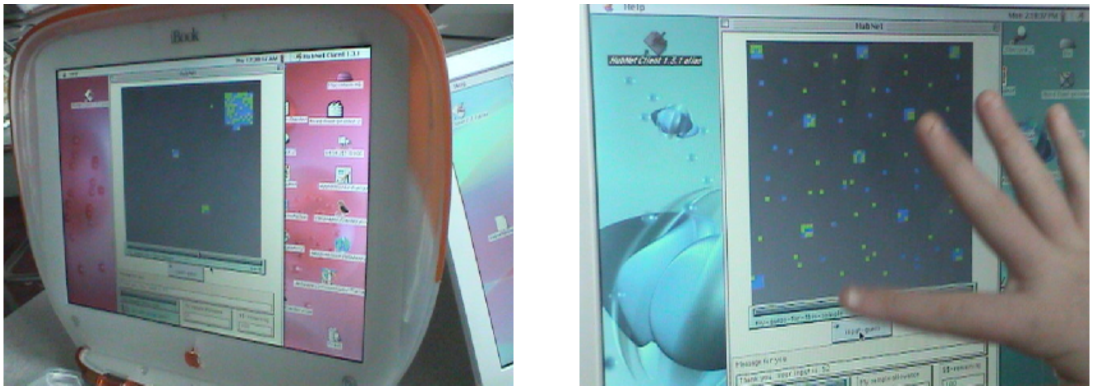
The implementation of S.A.M.P.L.E.R. follows three stages: introduction (server only), student-led sampling and analysis (server only); and collaborative simulation (clients and server). Typically, the first two stages take between half an hour and an hour, depending on student age group. The third stage may take between one and three periods, depending on student engagement and the teacher's flexibility in "weaving into" the PSA other ProbLab activities, such as NetLogo models, that may challenge students to reason carefully and thus deepen and enrich the discussion.

Introduction. The activity begins with the facilitator showing students a population of green and blue squares (the population is entirely exposed). Even before the nature of the activity has been introduced or relevant vocabulary has been explained, the facilitator asks students to describe what they see. We find it useful to elicit students' ideas, whether fanciful or mathematical, even before we explain the activities. First, collecting student ideas allows us furnish future facilitators with potential points-of-departure for classroom discussion—these teachers may wish to consider student ideas prior to the lessons so as to anticipate the mathematical content in these ideas and thus be prepared to facilitate classroom discussions. Second, student spontaneous ideas help us improve the learning environment by providing students with tools for pursuing their ideas. Third, student ideas may point to necessary changes in the design's activity plan so as to enable students to pursue learning paths that depart from their intuitive understandings. Fourth, allowing students to express ideas that, *traditionally,* are not considered mathematical, enables greater inclusion in this mathematics lesson, because students who do not consider themselves mathematically able may feel more comfortable when there is does not appear to be a "correct answer" and ideas need not be couched in terms of numbers. Fifth, by crediting students for expressing what they see, we hope to instill in students a greater trust of their perception judgment, because such judgment is an important personal resource in these activities that include visualization displays of proportional relations between colored areas.

Students offer their interpretations of what they are seeing. The teacher then asks students how green the population is, and students discuss the meaning of the question, offer intuitive responses, reflect on the diversity of responses in their classroom, articulate personal strategies, and develop more rigorous strategies and suggest how they could be implemented in the computer environment. The teacher facilitates the discussion by reminding students of mathematical content they had studied in the past that appears relevant to students' intuitive strategies. In doing so, the teacher introduces mathematical vocabulary that will help students communicate during the activity. For instance, a student might say, "It's too much to count all of the little squares—if only we could just look at one little place and decide with that," the teacher may respond, "So you want to focus on just a *sample* of this entire *population* of squares—how should we decide what a good sample is that will allow us to make a *calculated guess* or *predict* the greenness in the entire population?"

Student-led sampling and analysis. The teacher creates a new population that is not exposed. A student uses the teacher's computer, which is functioning as the "server" of the activity, to take a single sample from the population. To determine the size of this sample and its location in the population grid, the student–leader takes suggestions from classmates, asking individuals to warrant their suggestions. Once the sample is taken, by clicking with the mouse on a selected point in the population, students discuss the meaning of this sample in terms of the goal of determining the population's greenness. For example, if a 5-by-5 sample has 4 green squares and 21 blue squares, students may want first to describe it mathematically, e.g., "The ratio is 4 to 25" (correct), and then draw conclusions from this sample, e.g., "There are 16 green squares on the whole screen, because 4/25 is like 16/100" (partially correct). Students then debate over the location and size of another sample, further discussion ensues based on this new sample, and then yet more samples are taken. The teacher encourages students to keep a record of the *data* and to draw conclusions from the accumulated data. For instance, let us assume that students have taken ten samples each of 25 squares and have received the following data, couched in terms of the number of green squares in each sample: 8, 4, 4, 9, 21, 6, 4, 8, 9, 7. What are we to do with these data? Sum them all up? Decide that the answer is "4," because "4" occurred more than any other number? Ignore the "21," because it does not fit with the others? Calculate the average—8—and state that 8% of the population is green? Perhaps we should conclude that, seeing as the samples are inconsistent, these data are useless? The teacher guides students towards effective procedures by recording all the ideas and then exposing the population and discussing with students which procedure yields the best results over repeated trials.

Collaborative simulation. The teacher creates a new unexposed population and, through the server's interface, enables students' sampling functionalities. Students each take samples. The total number of little squares students may expose is limited by a "sampling allowance," for instance a total of 125 squares, that is set by the facilitator, from the server. This allowance is "replenished" between rounds. To optimize the gain from their limited sampling allowance, students each strategize the size and number of their individual samples as well as the location of these samples on the population grid. Figure 7 (see below) illustrates two different strategies students often use. One student (see Figure 7a) worked in the "few–big" strategy, spending the allowance mostly on a single location where the student took an 11-by-11 sample (a 121-block). Students who operate thus often say they are trying to create a reduced picture of the entire population. Some of these students choose to take the large sample from the center of the population (and not from a corner as in Figure 7a) and say that the center is the most representative location for the whole population. They also suggest that they can more readily calculate the proportion of green in their samples if they take just one sample and not many. Another student (see Figure 7b) worked in the "many–small" strategy, spending the sampling allowance by scattering samples of size 3-by-3 (9-blocks) and 1-by-1 (1-blocks) in a more-or-less uniform pattern across the population. Students operating thus often say they are trying to cover as much ground as possible, in case there is variance in the population that could not be found through a single large sample. Also, the "many–small" students are more likely than the "few–big" students to use averaging methods in analyzing their sampling data. Classroom discussions address individual techniques for maximizing the utility of the limited sampling resources and for making sense of the data.



a. *b.*

Figure 7. Examples of student sampling strategies: “few–big” and “many-small.”

At the end of each round, students use a slider to indicate their guess for the population’s greenness, e.g., 83%, and press a button to input this guess to the server. A histogram that shows all students’ guesses is thus projected on the overhead screen. Often, this histogram approximates a bell shape. The teacher exposes the population and then “organizes” it so that the population’s true value of greenness is evident. Whereas individual students may be up to 20 or more percentile points off mark of the true value, the *mean* of the histogram—the “class guess”—is often less than 5% away. Moreover, often no student has input the value of the classroom mean guess—it is indeed only the guess of the classroom as a whole.

An optional feature of S.A.M.P.L.E.R. is that students begin each round with 100 personal “points.” When students input their guess, they also commit either to their personal guess or to the classroom mean guess. Once students have input their guesses, each student has some points deducted according to the error of the guess they had committed to. For instance, based on her samples, Maggie input “70%” and committed to her personal guess. Assuming the true value of greenness turns out to be 50%, Maggie will lose 20 points. But, assuming that the class’s mean guess is 55%, had Maggie committed to the class guess, she’d have only had 5 points deducted. The juxtaposition of personal and pooled accuracy often engenders a pivotal moment in the activity: as individuals, students each can view themselves as a single data point on the histogram, but as an aggregate, the classroom embodies a distribution. It could be that this identity tug-of-war, “me vs. classroom,” that is stoked by personal stakes in the guessing game and by social dynamics around this game, provides opportunities for students to ground the ideas of distribution and mean.

Once the classroom guesses have been plotted as a histogram and the true value of greenness has been exposed, volunteer students go up to the front of the classroom, explain the histogram, analyze the accuracy of the classroom guess, and respond to their classmates’ questions. In particular, students share their personal sampling- and data-analysis strategies in a collaborative attempt to improve on the accuracy of the classroom mean guess on a subsequent round.

Following several practice rounds, the facilitator may challenge students by decreasing the sampling allowance so that students each have limited *personal* information about the population. Some local as well as classroom-level spontaneous conversation may emerge, through which students coordinate their sampling so as to maximize the total exposed area in the population (because it would be redundant to take multiple samples from the same location). If students conclude that it is better, individually, to "go with the group guess," can the group somehow collaborate to ensure higher accuracy?

Some students believe that, once a new population is created and students have taken samples, it is better first to discuss their estimations and then input guesses rather than first to input their guesses and then discuss the distribution. These students argue that by first discussing, the group can decide on a single guess, thus minimizing the range and variance of the distribution and thus, *ostensibly*, achieving higher accuracy.

With the description of the S.A.M.P.L.E.R. participatory simulation activity—in theory and in practice—we have completed the Design section. The next section, Theory, will refer back to the Design section as context for elaborating on our learning-axes and bridging-tools theory.

3 Theory

In this section we further explain our constructs 'learning axes' and 'bridging tools,' using ProbLab's activities as context. This elaboration will address some of the intellectual roots of these constructs. Also, we will discuss, from the perspective of these constructs, dimensions of student resources that factor in student construction of understanding, e.g., perception and classroom dynamics and discourse culture.

3.1 Learning Axes

The National Council of Teachers of Mathematics (2004) standards state that:

> Instructional programs from prekindergarten through grade 12 should enable all students to
>
> - recognize and use connections among mathematical ideas;
> - understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
> - recognize and apply mathematics in contexts outside of mathematics.

This focus on building concepts by making connections is one of five NCTM meta-content foci, along with problem solving, reasoning and proof, communication, and representation (NCTM, 2004). Building connections as a way of learning was the focus of Wilensky (1991, 1993, 1997a), who critiqued prevalent mathematics-education pedagogy for not enabling students to connect to new mathematical knowledge using their previous mathematical knowledge or "extra-school" experiences. This thesis resonates with the constructivist perspective on learning (Piaget, e.g., 1952) and it's implications to mathematics education (e.g., Papert, 1980, 1991; von Glasersfeld, 1987). Essentially, each student has to construct individually—*using previous knowledge*—an understanding of mathematical concepts; designers and teachers can help by providing learning environments that foster student mathematical reasoning and by supporting students' navigation in these environments and group discussions towards a broad set of learning goals.

The learning-axes design approach to students' mathematical learning is a one possible design operationalization for Wilensky's (1993) *connected mathematics* project, and specifically, the *connected probability* project. The learning-axes approach is an attempt to strike a balance between, on the one hand, under-specifying target content of learning environments to the extent that students may not avail of useful learning paths inherent in these environments, and, on the other hand, over-specifying content to the extent that students do not have opportunities to achieve deep understanding through "re-inventing" the mathematical constructs. Given supportive learning environments, young learners are inclined to reinvent core aspects of commonly-used mathematical, computational, and, in general, quantitative–symbolical artifacts and procedures (Papert, 1980; diSessa, Hammer, Sherin, & Kolpakowski, 1991; Bamberger, 1991; Abrahamson, Berland, Shapiro, Unterman, & Wilensky, 2004; Abrahamson & Wilensky, 2005c). The learning-axes approach attempts to leverage students' capacity to reinvent mathematics by articulating domain-analysis principles for locating the conceptual ingredients that may foster student reinventing of target concepts, as we now explain.

A plausible route towards designing learning environments that strike a balance between under- and over-specifying mathematical content begins with conducting a domain analysis, in which the researcher deconstructs a mathematical concept into its *ingredients*. By "ingredients" we do not mean graphic constituents of the concept's symbolical notation, as in, "A fraction has a numerator that is written on top of a line and a denominator that is written below the line," nor do we mean an operational/functional definition, as in, "A fraction, marked *a/b*, is a quotient expressed in terms of the dividend, *a*, and the divisor, *b*." Rather, by "ingredients" we are referring to the cognitive constructs—schemas, implicit cognitive skills, or ways of seeing and operating on the world—that would need to be adjusted so as to construct the novel concept. For instance, to understand the meaning of a fraction, a student might need to successfully align her counting skills and measuring skills in some context requiring accuracy—she would need to account for the partial units of measure that hinder a straightforward quantification of some extensive substance in terms of a standard unit of measure. 'Counting' and 'measuring' serve, in terms of our approach *and in terms of this specific learning situation*, as two edges of a learning axis—these edges define a space of potential learning in which a student may reconcile these locally-competing schemas so as to construct the concept of fractions, given suitable context.

In some respects, by searching for psychological ingredients of mathematical concepts, we are looking for the historical antecedents, dialectics, and contexts that engendered the invention of new mathematical knowledge that became our cultural heritage. Our hindsight enables us to choose for our classrooms the ingredients and activity contexts that may be most conducive to students' replication of aspects of this historical process. Specifically, we attempt to create for students a problem space for which two different notions each appears useful, but it is not initially clear how these notions may be combined to solve the problem. For instance, in the context of an engineering-and-construction activity, a student may want to use a 1 centimeter-long stick to measure the exact length of a longer stick of a length that is greater than 3 centimeters but less than 4. To measure the length of the longer stick, the student might count how many times the shorter stick fits into it. A problem may rise when the student cannot name the remainder beyond 3 centimeters in terms of the centimeter-unit stick. The student might describe this remainder as a partial unit, e.g., "a bit," and might mark the shorter stick to indicate the length of this partial unit. Conceivably, such a scenario may contribute towards this student being predisposed to appropriate millimeter units as a useful enhancement of her invention. So problem spaces that give rise to learning axes include objects that both contextualize the problem and stimulate the complementary notions that will contribute to the solution of the problem. Therefore, learning axes are anchored within concise and shared classroom objects of manipulation and discourse. When a single object evokes two or more conceptual ingredients—when it stimulates student reasoning along a learning axis, we call such an object a *bridging tool*.

We now further explain how we use the learning-axes approach to inform the design of bridging tools, and then we broaden the discussion by examining perceptual and social dimensions that contribute to student learning probability and statistics w/ bridging tools.

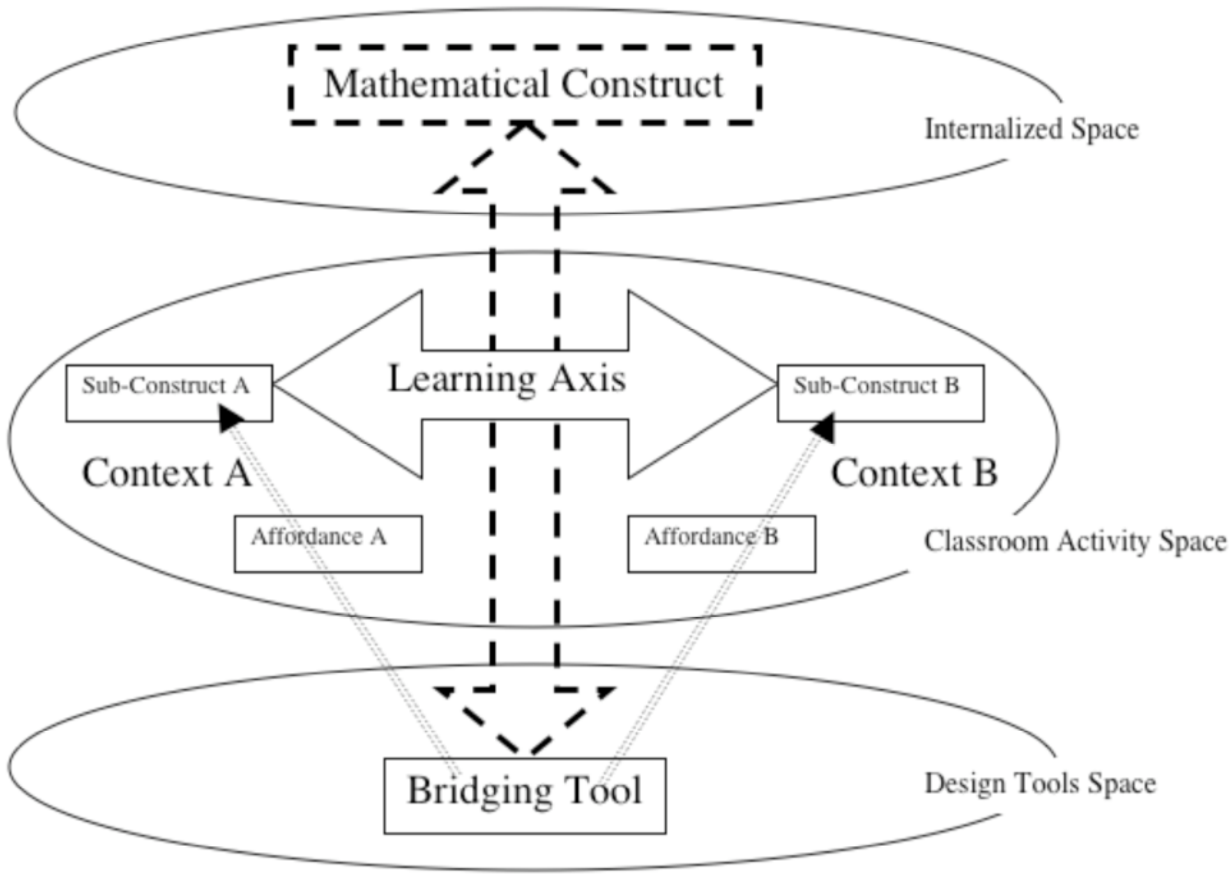


Figure 8: Learning axes and bridging tools: Student construction of mathematical concepts is viewed as a problem-driven reconciliation of competing interpretations afforded by a single bridging tool. Bridging tools are designed through domain analysis of mathematical concepts.

3.2 Bridging Tools

'Bridging tools' (Abrahamson, 2004a, 2004b; Fuson & Abrahamson, 2005) are classroom pedagogical artifacts and activities that tap and stimulate students' previous mathematical knowledge, situational understandings, and kinesthetic schemas and link these reciprocally to formal mathematical representations. Figure 8 (above) uses the *apprehending-zone* model, (Abrahamson, 2004a; Fuson & Abrahamson, 2005), a mathematics-education model of design, teaching, and learning, as a theoretical framework for illustrating bridging tools. A bridging tool

is created in the Design Tools Space (see Figure 8, lower tier). Through participating in the Classroom Activity Space (see Figure 8, middle tier), students construct meanings for the bridging tool and link these meanings. A unique attribute of bridging tools is that each bridging tool is designed to evoke at least two meanings that are complementary in understanding the target concepts. Each of these meanings is an affordance of the tool within some activity context, and each affordance supports a subconstruct of the target domain (see Figure 8, the dashed arrows rising from the bridging tool). Students negotiate and reconcile these complementary meanings to construct a new mathematical concept in their Internalized Space (see Figure 8, top tier; note vertical axis). For example, students interpreted the multiplication table either as columns composed through repeated adding of fixed values (e.g., +3, +3, +3, etc.) or, traditionally, as a tool for locating products through cross-product referencing (factor x factor = product). These alternative interpretations were conducive to students' grounding in the multiplication table an *additive–multiplicative* understanding of multiplication and, through that, an additive–multiplicative understanding of ratio and proportion (Abrahamson, 2004; Fuson & Abrahamson, 2005; the term 'additive–multiplicative' was first introduced in Fuson, Kalchman, Abrahamson, & Izsák, 2002).

Our design of bridging tools is informed by cognitive, pedagogical, and socio-constructivist assumptions and motivations that have led us to regard learning tools as more than 'computation devices' for carrying out solution procedures or 'scaffolds' towards some alleged 'abstract' understandings. We assume that mathematical instruments can play pivotal roles in mediating to students a cognition of the domains in which the instruments are applied. Specifically, bridging tools can potentially embody and convey dilemmas and solutions inherent in a mathematical domain. Using bridging tools, students may potentially emulate thought processes that the designer sensed are conducive to the construction of central ideas for the target domain.

By focusing on bridging *tools* as organizing mathematics-education learning environments rather than on the mathematical *concepts*, we wish to foreground a design principle that learning environments should create opportunities for students to construct new ideas, and that presenting students with completely "baked" ideas may defeat the objective that students themselves construct the concepts (see, e.g., von Glasersfeld, 1990). The classroom activities and classroom episodes that we present in this paper attempt to convey the plausibility of designing for learning opportunities rather than designing directly for concepts. Bridging tools play a pivotal role in suspending a concept-driven pedagogy of definitions, formulas, and word problems. Using bridging tools, students are to experience the challenges inherent in understanding mathematical concepts and initiate discussion of these challenges. So bridging tools are 'precocious,' in the etymological sense of the adjective—they are 'under cooked' or 'half baked' and require learners' active participation to become "well done" as new mathematical constructs—personal understandings that are sufficiently shared in the classroom (see also Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997, on the *emergent perspective on learning*).

One assumption of our design framework, coming from Abrahamson (2004) is that students can construct mathematical concepts as reconciliations of the dual interpretations inherent in a bridging tool. The idea of learning as reconciliation is not new. In 1837, William Whewell wrote the following words about students' intuitive understanding of the fundamental axioms of geometry: "The student's clear apprehension of the truth of these is a condition of the possibility of his pursuing the reasoning on which he is invited to enter" (Whewell, 1837/1989, p. 40). Learning, according to Whewell, is the process of individual students grounding formalisms in their intuition, and this learning process is fostered through discourse. Following Whewell, we attempt to help students ground formal ideas in their perception and implicit understandings (Wilensky, 1993; Abrahamson, 2004). The idea of understanding-in-action that is a hallmark of constructivist pedagogy (e.g., von Glasersfeld, 1987) can be seen as rooted in phenomenology (e.g., Heidegger, 1962) and in Gibson's (1977) construct 'affordance' that is widely used in the learning-sciences literature. We wish to extend the idea of learning as reconciliation by submitting that reconciliation can transpire not only between intuition and formalism but also between two intuitions grounded in one and the same object—the learning occurs as constructions when learners attempt to reconcile two competing interpretations of a phenomenon in the context of some designed activity (see Poincaré, 1903/2003, Polanyi, 1967, and Steiner, 2001, on the "mental combinatorics" of mathematics creativity; see also Piaget, 1952, on how the idea of volume arises in conservation tasks; see Minsky, 1985, on hierarchies in mental structures; see Case & Okamoto, 1996, on central conceptual structures). The object, and, later, an internalized image of this object, is an "external representation" of a target concept in that the object carries the reconciled coordination between the rival intuitions (see diSessa & Sherin, 1998, on coordination classes).

Following, we demonstrate students' two catalysts for potential learning that is enabled by the bridging tools and activity contexts we designed for ProbLab: perceptual schemas and cultural practices.

3.3 Perceptual Schemas and Cultural Practices That Catalyze Bridging

We have discussed the domain of probability and statistics in terms of the inherent dualities of the domain that may account for its challenges. Also, we have explained 'learning axes' and 'bridging tools,' the analytic–pragmatic constructs that inform our design of mathematics learning environments. These learning environments, and specifically, student interaction with bridging tools within a designed activity context, are to foster tension between competing action-based interpretations of such tools, so that student learning may play out as reconciliation between these competing interpretations. This section brings together and elaborates on our theory and design: We demonstrate how cognitive tension can be fostered either by stimulating competing perceptual schemas afforded by bridging tools or by stimulating competing personal and interpersonal motivations stemming from the classroom activities and interacting with tacit classroom microcultural practices. A later section will present data of student behavior that can be interpreted through these theoretical lenses.

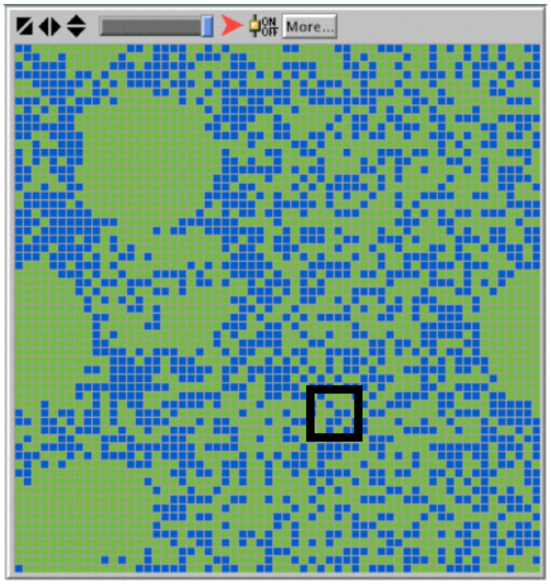


Figure 9. The S.A.M.P.L.E.R. population stimulates both global and local judgments.

An example of perceptual schemas stimulated by the design of S.A.M.P.L.E.R. is student responses to the S.A.M.P.L.E.R. population (see Figure 9, above). The S.A.M.P.L.E.R. population is designed so as to afford an immediate global gauging of its 'greenness' value just by virtue of gazing at the population and mapping the greenness onto a 0-through-100 scale. For instance, the population in Figure 8 "looks" about 60% green. Yet one could attend closely to the population and count its squares one at a time. That is a different type of affordance of one and the same visual display. In particular, the grid that is superimposed on the population renders the otherwise amorphous blobs enumerable. So the global and local types of attending are each stimulated by the design. These affordances compete for the user's attention. Both are useful for the task at hand—determining the population's greenness—but it is not initially clear how these interpretations may coexist. In and of themselves, there may be little learning gain for students who engage either in eyeballing the gestalt or in counting up little squares. But these two competing interpretations of the perceptual data define a learning axis—in between them lies a potentiality of learning. Thus, the design creates a tension and a possible negotiation between the bottom-up measuring (enumeration) of elements in the population and the top-down sense of the population's central tendencies. The construct that bridges the global and local interpretations of the population is the 'sample' (see dark frame in Figure 8). In S.A.M.P.L.E.R., the 'sample' feature anchors the mathematical construct that emerges from the negotiation of the dual affordances of the population. The 'sample' both focuses/hones the tension between these complementary schemas and paves the way to reconciling these tensions: a sample can "feel" mostly green, but it can also be counted up so that this feeling (the proportional judgment) can be corroborated using basic counting skills.

It is not imperative for the bridging-tools design framework that students' mental action of bridging be instantiated in a new element that is introduced into the bridging tool, e.g., a 'sample.' That is, the learning activity need not be 'reified' (see also Confrey & Costa, 1996). Rather, bridging can result in new ways of *attending* to a bridging tool. By foregrounding aspects of the tools and coordinating these, bridging is a new way of *seeing* mathematical tools.

An example of personal and interpersonal interdependencies that play out in S.A.M.P.L.E.R. is students' concern for their own stakes in the S.A.M.P.L.E.R. game: how their data contribution compares to those of their classmates and how these comparisons reflect on their social status.

Students in a classroom do not operate in a social void. Learning is ultimately an individual experience, yet it is set within and stimulated by the interpersonal space (what Brousseau, 1986, calls the *milieu*) wherein taken-as-shared ideas are constructed (Cobb & Bauersfled, 1995). However, this web of meaning making that is largely self-organizing may be porous—some students may be left behind. Students know about the importance of teamwork. They play competitive group games, and the importance of collaboration in non-sporty spaces is often preached in the form of slogans. Yet how often does a classroom have impact on straggling individuals? If they do not participate, one cannot diagnose their difficulty let alone engage them in conversation. One way of inviting struggling students into the classroom activity is to use tools that equalize all students' voices and demand all students' attention and contribution in a single classroom venture. The HubNet technology attempts to do just that (see Abrahamson & Wilensky, 2005c).

Participatory simulations are specifically designed to leverage the classroom social dynamics in joint inquiry into mathematical or scientific phenomena. In particular, HubNet participatory simulations use technology to process individual students' actions onto a central processor and to project these onto the classroom screen in the form of mathematical representations. Thus, the social and conceptual spaces are superimposed.

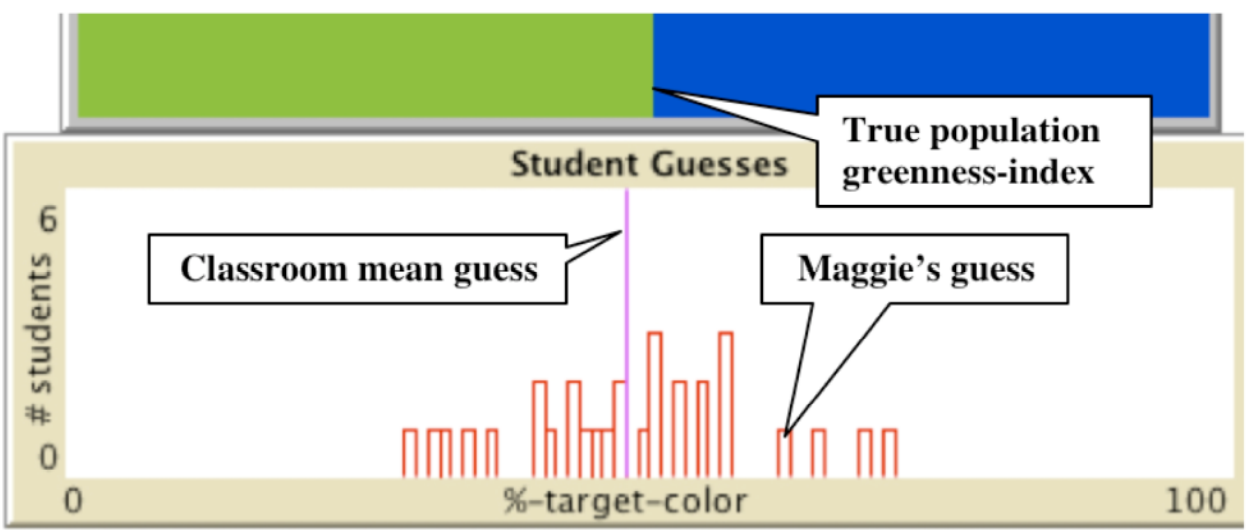


Figure 10. Fragment from the S.A.M.P.L.E.R. interface showing the histogram of student guesses.

The S.A.M.P.L.E.R. histogram (Figure 10, above) collects students' individual guesses. Each student samples from different locations in the population, and so each student is likely to have a different estimate for the population's greenness value. Unless a student has erred in a calculation, any guess is a valid reflection of that student's data. So variability in the S.A.M.P.L.E.R. population is mirrored in the student distribution—students are plotted onto the histogram, so to speak, and each data point contributes to the overall accuracy of the classroom guess. The "classroom = distribution" analogy is an authentic reflection of collaborative statistical practice that is simulated in S.A.M.P.L.E.R.

The histogram portrays the overall profile of classroom work (the distribution and its mean). At the same time, it is also a trace of each student's sampling and compiling of data (e.g., Maggie's guess, see Figure 9, above). Individual student actions that differ from the emergent "normal" behavior—for instance a guess of 81% that is several standard deviations away from the classroom mean—stand out as "deviant" mathematical actions and invite interpretation and evaluation. Such student commentary is articulated in terms of working in S.A.M.P.L.E.R.: Is it possible that this student in fact worked well?; Did this student hinder or help the overall accuracy of the class?; Can a student both be "way off" and at the same time have contributed to the accuracy? Thus, the S.A.M.P.L.E.R. the classroom interpersonal dynamics stimulate attention shifts: the histogram is viewed one moment as a collection of individual guesses and then as an integral entity. The histogram operates as a bridging tool on the axis between individual and collaborative action to support student construction of distribution, range, and mean.

This section concluded the expanded introduction of our design and theory. Following, we delineate the methodology of our study and then present results from an implementation of ProbLab in a middle-school classroom. The discussion will present data episodes from S.A.M.P.L.E.R. in terms of the learning-axes and bridging-tools perspective.

4 Methodology

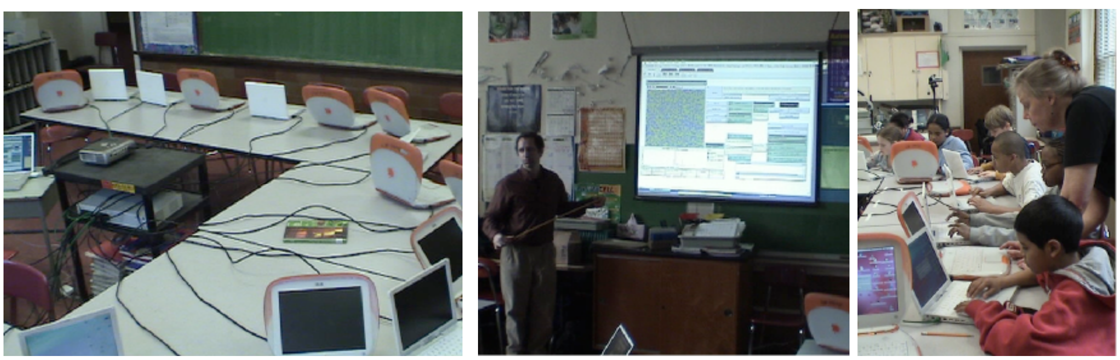
The implementations were opportunities to study students learning probability and statistics. At the same time, these were opportunities for us to test our design. Specifically, analysis of students' spontaneous actions and discussion informed iterative modification of the design (see Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, on the rationale of design-based experiments).

4.1 Participants

S.A.M.P.L.E.R. was enacted in two 6^{th}-grade classrooms (n = 20; n = 18) in a Grades 6 – 8 middle school in a very heterogeneous urban/suburban district (school demographics: 47.6% White; 35.1% African–American; 15.2% Hispanic; 2% Asian; .2% Native American; 29.3% free/reduced lunch; 4.7 ESL). At our request, the teacher informed us of the students' mathematical achievement as based on schools tests, grouping students into "high-," "middle-," and "low-" achievement groups. Although students' work in ProbLab was not expected necessarily to reflect their work in traditional curricula, these groupings pointed out to us some of the students we would want to pay special attention to. Also, these groupings allowed us to evaluate whether our design enables students who had been struggling in traditional curricula to participate with understanding in our activities (see Abrahamson & Wilensky, 2005b, 2005c). The S.A.M.P.L.E.R. lessons took place during the second of two weeks of implementing ProbLab in each of these classrooms (first week: 2 * 80 min. periods work on the combinations tower interspersed with work on NetLogo models; second week: 3 * 80 min. periods work on S.A.M.P.L.E.R., for a total of five double-period lessons per classroom). In these implementations, NetLogo models were operated by the teacher and discussed by the students.[5] The teacher was a Caucasian female teacher in her third year as a teacher, with a background in a health-related field. The research team on site included four graduate students completing their doctoral studies in the Learning Sciences. The first author who was also the lead designer of the ProbLab activities, took an active role in co-facilitating the lessons with the teacher. The other team members' roles included collecting video data and field notes, eliciting student ideas through on-the-fly interviews during classroom activities, and addressing software and hardware issues that often occur in technology-based pilot studies.

4.2 Procedure

The first author ran the activities, and was helped by the teacher in facilitating discussion and by two design-research colleagues in technical support and data collecting. Other than introductions and summaries that were facilitated by the leader, lesson time was dominated by individual and group work, with occasional classroom discussions, some of which were spontaneous and others initiated by the facilitators. Students were encouraged to lead discussion from the front of the classroom, using a pointer to explain the computer interfaces projected on the overhead screen and calling on other participating students.



a *b* *c*

Figure 11. Optional classroom layout in S.A.M.P.L.E.R.

4.3 Technical Issues

The classroom was arranged for this implementation in a horseshoe shape, with the opening towards the front of the classroom (see Figure 11a, above, for a view of the classroom from the back-left corner). Twenty students seated in a horseshoe shape at individual laptop computers were connected through a switch to the facilitator's computer. In the center of the horseshoe we positioned the facilitator's laptop, angled so as to allow eye contact with most of the students (see Figure 11a, on left). The facilitator's laptop computer was wired via a 24-port switch to all students' laptop computers, Macintoshes running OS9 operating system. The switch was in a shelf of a wagon (see Figure 10a, left of center), and on top of the wagon was the overhead projector, also connected to the facilitator's computer. We ran S.A.M.P.L.E.R. in NetLogo 1.3.1.[6] The facilitator, whether teacher, researcher, or student, often stood at the opening of the horseshoe to present the overhead projection (Figure 11b). During the lesson, the facilitator had

[5] In later implementations of ProbLab, students operated the NetLogo models individually.
[6] The current NetLogo environment that is in its 2.x generation does not support Macintosh computers that predate OSX.

access to all students, and they could consult each other (Figure 11c). So all students and the facilitators could see each other and the overhead projection, and only a single electrical cable, running from the horseshoe center to a socket on the wall, was within the area students moved in. This cable was secured to the floor with duct tape.

4.4 Data Collected

Two video cameras filmed all lessons: one was carried by a researcher to capture student work and discussion and the other one was typically positioned on a tripod at the back of the classroom as a backup but occasionally carried by another researcher. During individual work, the researchers interacted with students, asking them to explain their strategies, actions, and thoughts. Every day, the design team wrote extensive field notes during and immediately after the lessons and during a first run through the video data (as the tapes were being digitized). Verbal and electronic communications within the design team and with the teacher were recorded to track the rationale of day-to-day modifications of the design. A posttest was administered as another measure of students' understanding and so as to elicit students' feedback on the experimental unit.

4.5 Data analysis

Four individual researchers examined the videotaped classroom data and the daily field notes. Each researcher marked in the tapes episodes they sensed could be of significance for understanding and improving the learning potential of students participating in the classroom activities. In research meetings, we discussed our selected episodes, many of which were chosen by more than a single researcher. These episodes tended to portray students who either had difficulty understanding the activities or made insightful comments about the meaning of mathematical representations. Our discussions resulted in a delineation of facilitation emphases for clarifying and enhancing the activities (see Abrahamson & Wilensky, 2005e for a subsequent study that applied these conclusions). Also, we came to articulate student insight as resulting from a negotiation between competing meanings either of a single design element or coming from two different elements. Building on the bridging-tools approach and on our emergent domain analysis, we further developed the learning-axes approach reported in this paper.

5 Results

In this section, we first discuss student entrance understanding as reflected in their initial sampling strategies. Next, following a brief report of posttest results, we show that some students' limited fluency in multiplicative concepts and procedures constrained their learning in S.A.M.P.L.E.R.

5.1 Students' Initial Understandings and Sampling Strategies

The S.A.M.P.L.E.R. activities began after students had participated in the combinations-tower activity, in which students created and assembled 9-blocks, and after students had observed the NetLogo simulations that included randomly generated 9-blocks. So the 9-block had become contextualized by different activities in different media, and participating students often referred to 9-blocks in classroom discussions. (See in section 2.4, explanations of the S.A.M.P.L.E.R. design and a typical lesson scenario.)

In the introduction of the S.A.M.P.L.E.R. activity, the teacher showed a revealed population and asked students for their interpretations. Students' interpretations ranged from fanciful to "mathematical." Fanciful interpretations included "Superman," "a mushroom house," "a man with a shield," "a messed-up face," a "teddy bear with eyes," "a mountain," and "an elephant with a nose." Mathematical interpretations used the 9-block as a reference point. For instance, students said that the population could be thought of as a collection of many 9-blocks. Students wondered how many 9-blocks might fit into the population. This stimulated a discussion of methods for determining the number of squares in the population. Thinking in a different direction, one student suggested that we look at the entire population as a single "1000-block," and that what we were looking at was just one of many different combinations of this 1000-block. This interpretation appears to apply ideas that arose from the combinatorial-analysis and empirical-probability activities to the entire population of squares..

Next, to introduce the sampling activity feature, the facilitator used a population that was entirely revealed but for which the greenness value was not disclosed. The facilitator asked the students how one could determine the greenness of the population. Students said that, in principle, one could count up all the green squares in the population and divide this number by the total number of squares in the population. However, students said, there are too many little squares to count, making this strategy unfeasible. Other students suggested that it might be useful to focus on a single area of the population and count up the green squares in it. The facilitator reiterated this idea, calling that area a "sample." The question on the table then became, "If we could only take a single sample, where should we take it from?" The following transcription illustrates classroom discussion about sampling.

> **Student 1**: It would be better if there were a way to get a random spot. [for the sample]
> **Researcher**: A random spot?
> **St. 1**: Yeah, because if you *chose* somewhere, you might think, "Mmm, this one has a lot of green, let's do it there."
> **Res**: But what if *randomly* the computer gives me a place with a lot of green or a lot of blue?
> **St. 1**: Well, then that's what you've got to guess on.
> **St. 2**: [You should put the sample] in the middle, a little higher… it seems a little sort of balanced.
> **St. 1**: But that's just what I'm saying. If you try to find something balanced, it's going to be around 50% no matter what.

These students' exchange reflects a pivotal quandary of statistics—is the sample sufficiently representative of the population, and what measures can we take to ensure that it is? A feature of the design that supported this conversation was that the facilitator could toggle between a view of the whole population and a view of different samples. Thus, students could gauge whether various suggested samples were sufficiently representative of the population. Most students did not use proportion-based mathematical vocabulary, possibly because they were not fluent in its application to novel situations (see section 5.3, below). Yet, the visualization features of the learning environment enabled these students to communicate about proportionality qualitatively.

a. *b.*

Figure 12. During student individual work, the teacher speaks with each student.

The lesson continued with students working on their individual computers. Students took samples from the population, inputted their guesses to the server, and examined results once the population and its true greenness value was revealed. The teacher worked with individual students as they participated in these activities. In Figure 12a, above, the teacher is working with one of the students she had listed as high achieving in mathematics. They are interpreting that student's guess for the population's greenness as compared to the true value. In particular, the student is showing the teacher that she had guessed correctly—the green–blue partition in the population is precisely where the student had predicted it would be. The student explains to the teacher her sampling strategy. In Figure 12b, above, the teacher is working with one of the students she had listed as low achieving in mathematics. The student had taken samples from the population and had input a guess that did not seem to reflect all the samples he had taken. The teacher is discussing with the whether it would help for him to consider all samples in determining the greenness of the population. These classroom data demonstrate both that the S.A.M.P.L.E.R. activity enables immediate feedback to the teacher and helps the teacher elicit specific student difficulty. Also, these data demonstrate one way that PSA integrate group- and individual work: the framework of the activity is collaborative, but to participate successfully in this collaboration, students must each achieve an understanding of the activity.

5.2 Post-Tests

Students' responses on the post-intervention questionnaire revealed a wide range in classroom experiences in the unit. Responding to an item requesting their favorite sampling strategy, many students said they enjoyed spreading their samples all over the screen and then counting up the total number of green squares, dividing this number by the total number of exposed squares, and calculating this quotient as a percentage. Of these students, some thought that it is better to take single-square samples so as to maximize the spread of squares. Other students said that distributing their samples "randomly" was a better strategy as compared to distributing them systematically. Many students thought it is highly efficient to divide the sampling task between many students by allocating specific sampling areas to specific students—this strategy, they wrote, maximizes the total exposed squares.

Another item asked students whether it is better to commit to one's own guess or to commit to the group guess. That is, which of these two strategies ensures better long-term results? Students' answers varied, and they depended on the students' mathematical ability. High-achieving students preferred going alone, unless they were very unsure of themselves, whereas lower-achieving students preferred to trust the group guess. So the lower-achieving students were those who believed that the compiled guess is a more accurate measure of the statistical data as compared to an individual guess. This finding is somewhat counter-intuitive. One might expect that the higher-achieving students and not the lower-achieving students would be those who gain this mathematical insight. Possibly, the higher-achieving students are those who more often suffered from their classmates' "wayward guesses," i.e. off-mark input that resulted from incorrect analysis. So the accuracy of students' individual guesses resulted both from a *random factor*—the specific samples each student exposed—and from a *skill factor*, students' individual mathematical competency reflected in their ability to calculate a percentage. In section 5.3 we elaborate on this point.

In their written responses, all students referred in one way or another to the distribution and range of the guesses, couching these in terms of 'left,' 'right,' average, and balancing ("it evens out"). We interpret this finding as indicating that the S.A.M.P.L.E.R. activities created a shared classroom artifact that carried shared meanings, experiences, and vocabulary. Such shared mathematical images could serve as helpful anchors in future classroom discussions.

Yet another item asked students whether one should first input a guess and only then discuss the input or first discuss and then guess. Many students thought that discussing first might either confuse you or bias the group guess—that a wider distribution guaranteed more accuracy of the classroom group guess. We interpret this finding as indicating that students experienced how an aggregation of random outcomes can nevertheless effect higher accuracy than would a "centralized command" (see also Wilensky, 1997a, 2001; Surowiecki, 2004).

Finally, students varied in what they considered to be a "good guess." Some students were happy to be several percentage points off the true value, whereas other students were more critical of their guesses (for a more detailed report on students' spontaneous sampling strategies, see Abrahamson & Wilensky, 2004a).[7]

[7] For a report on student responses to the probability data, see Abrahamson and Wilensky (2005a).

5.3 Mathematical Fluency as a Bottleneck in Students' Mathematizing Their Intuitions

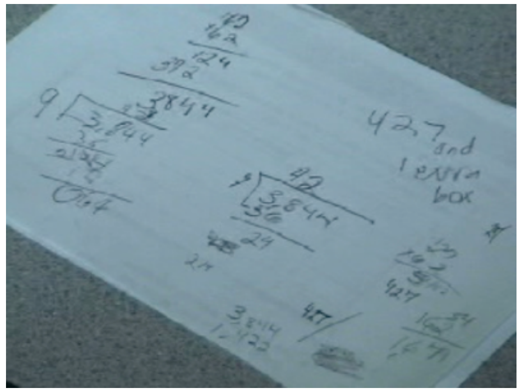


Figure 13. Students use mathematical calculation to determine the greenness of the population based on the samples.

Many students' initial methods for using samples to compute the population's greenness were naïve, and students' calculation procedures were cumbersome. For example, one student counted the number of squares in the top row to be 62. She then counted and found the number of green squares in that row to be 27. She stated that the top row appeared representative of other rows, and so she would base her prediction for the population's greenness on that row. Finally, rather than stating the population greenness directly on the basis of this 27:62 ratio, she performed a calculation that appeared to confuse the procedures for adding fractions and for multiplying fractions: first multiplied the 27 and the 62 each by 62, got the equivalent ratio 1674:3844, and then restated this ratio as a percentage, to get 43.5% (see Figure 13, above).

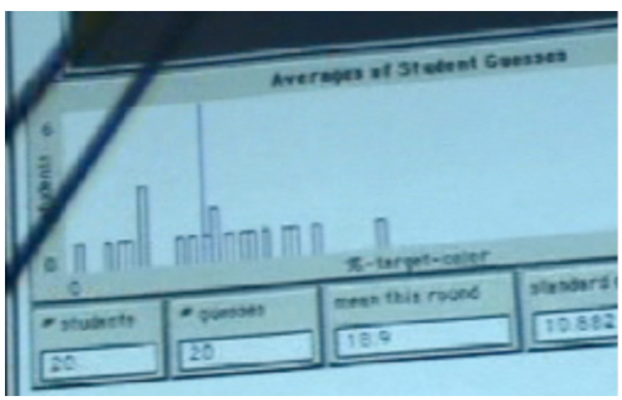


Figure 14. Histogram of student guesses supporting discussion of central-tendency indices.

Many of the other students in this 6th-grade classroom initially had difficulty understanding the difference between the absolute and relative (proportional) number of green squares in their samples. For instance, a student who guessed 11% as the population greenness had found 11 green squares out of a total of 200 squares he had exposed. These students also had difficulty in calculating equivalent fractions. So S.A.M.P.L.E.R. activities constitute opportunities both to ground proportional reasoning and to practice calculating proportional equivalencies. Similarly, students for the most part did not understand the concept of 'mean' beyond knowing to calculate it. Specifically, they did not know whether, given a set of values and their mean, there should necessarily be an equal number of values lesser than and greater that the mean. The classroom was discussing a histogram (see Figure 14, above) that plotted all students' guesses as well as the mean of those collective guesses (the tall thin line). The facilitator asked students whether there should necessarily be the same number of students who guessed above (to the right of the mean) and below it (to the left of the mean). We expected students to be able to differentiate between the constructs 'median' and 'mean,' but students could not do so in the context of a histogram. S.A.M.P.L.E.R. activities are opportunities to contextualize and discuss differences between central-tendency indices. In Figure 14 (above) there are 9 guesses lesser than the mean and 11 greater than it. The long thick diagonal line is a pointer that a presenting student is using to explain this display to the classroom. As it turned out, the mode (the tallest column) was very close to the correct answer—closer than the mean or median were.

The variety of tasks performed by a group of students who produce individually and then share statistical data affords opportunities for examining the unique function of these indices. For example, in Abrahamson and Wilensky (2004b), students elected to calculate the *mean* of eighteen samples taken from the same population, but when the necessary calculations were performed by ten students and four different means were reported, students elected to trust the *mode* of these means...

This remainder of this paper focuses on five learning axes in the S.A.M.P.L.E.R. design, the bridging tools designed to stimulate dilemmas along these axes, and the statistics ideas students construct in reconciling these dilemmas: (a) local-vs.-global interpretation of the S.A.M.P.L.E.R. population stimulates the construction of 'sample'; (b) theoretical-probability- vs. statistical interpretation of a collection of 9-blocks stimulates the construction of 'sample distribution in the population'; (c) theoretical- vs. empirical-probability interpretation of the combination tower stimulates the construction of 'sample space'; (d) range-vs.-cluster interpretation of a histogram stimulates the construction of 'variance' and 'balance'; and (e) individual-vs.-social interpretation of a histogram stimulates the construction of 'sample mean' and 'distribution.' For each learning axis, we demonstrate through classroom data students' negotiation between the poles of the axis. The paper ends with a summary of our approach, a brief discussion of tradeoffs inherent in the learning-axes approach to design, and a "recipe" for designing effective bridging tools.

6 Discussion: Analysis of Student Learning Through the Theoretical Lenses of Learning Axes & Bridging Tools

In earlier sections, we have demonstrated how we use the learning-axes theoretical framework in our analysis of mathematical domains and how this analysis, in turn, informs our design of bridging-tools for classroom activities. In this section we demonstrate how the learning-axes- and bridging-tools perspective also can be used as lenses for analyzing classroom data. Using the same lenses both towards classroom implementations and following these implementations helps us evaluate the efficacy of our activity design. In particular, we select episodes in the data that, to our judgment, demonstrate student insight and inventiveness, and we work to articulate this insight in terms of the bridging tools the students were working with, the underlying learning axis stimulated by this bridging tool, the statistical construct that the axis potentially coheres as, and the activity that contextualized the student's work. Following, we will examine five brief classroom episodes so as to explain five of the learning axes, bridging tools, and statistical constructs that are enfolded in the design and are enabled through student participation in the classroom activities. The structural elements of these episodes will be summarized in a table in section 6.6.

6.1 Episode One: The Local–Global Learning Axis Coheres as a Sample

Within the design sequence, an introductory activity engages students in determining the greenness of a population that is completely revealed (students see all of the tiny squares in the green–blue mosaic). This activity occurs at a point before students have discussed sampling and before the facilitator has enabled the model's sampling functionality. Nevertheless, students performed quasi-sampling actions. Students: (a) attended to selected spatial locations in the S.A.M.P.L.E.R. population; (b) used counting actions to inform their sense of the greenness within these selected locations; and (c) coordinated information from these samples so as to determine the population's global value of greenness.

Upon close attending to students' verbal descriptions and gestures, it appears that their guesses were informed both by counting tiny squares ("local" actions) and by eyeballing the *entire* population and assigning to it a greenness value ("global" actions). So students were using two different methods: local enumeration actions and a global perceptual judgment. Importantly, students did not appear, initially, to be aware that they were using two different methods, nor did they appear to coordinate these methods as complementary. Yet, through discussion with their peers and the facilitator, students have opportunities to connect between these personal resources, as the following transcription demonstrates.

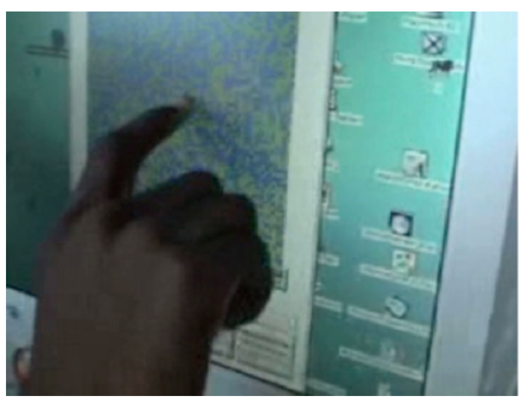


Figure 15. Devvy explaining how he determined a value for the population greenness.

Researcher: [standing by the student, Devvy, who is working on his individual laptop computer] **What are you doing here?**
Devvy: [gazing at the S.A.M.P.L.E.R. population, index finger hopping rapidly along adjacent locations in the population; see Figure 15, above] **Counting the squares.**
Res: **What did you come up with?**
Dev: [hands off screen, gazing at screen; mumbles, hesitates] **Around 60 or 59 percent.**
Res: **Sorry… so, show me *exactly* what you're counting here.**

Dev: **Green squares,** [right index on screen, swirls in one location, then hops to another location, unfurling fingers] **'cause it says, "Find the percentage of the green squares."**

Res: **Uhm'hmm**

Dev: **So if you were to look at it** [left hand, fingers splayed, brushes down the whole population and off the screen] **and sort of average it out,** [touches the 'input' button] **it'd probably equ...** [index on population, rubbing rapidly up and down at center, using little motions and wandering off to the left and then down] **it'd probably go to 59 or 60.**

Res: **And how did you get that number?**

Dev: [index strokes population along diagonal back and forth] **Because it's almost even, but I think there's a little bit more green than blue.**

Devvy's actions are not statistically rigorous—he is not taking equally sized samples, nor is he systematically counting the number of green squares in each sample or methodically averaging values from these counts. But his actions are *proto*-statistical (see L. B. Resnick, 1992)—without any formal background in statistical analysis, Devvy is going through the motions of statistical analysis, if qualitatively: skimming the population, attending to selected locations, comparing impressions from these locations, and determining a global value. Albeit, Devvy appears to acknowledge the tenuousness of his methods in qualifying his suggested strategy as "*sort of* average it out."

Whereas Devvy's spontaneous local and global methods are as yet disconnected, both methods are grounded in the same object, the S.A.M.P.L.E.R. population. This 'common grounds' constitutes the platform or arena upon which Devvy may negotiate the competing mental resources. Devvy may have already begun building a micro-to-macro continuum by attending to mid-level clusters of tiny squares, i.e. "samples" (see Levy & Wilensky, 2004, on the role of the mid-level constructions in student reasoning about multi-agent phenomena). Through participating in the S.A.M.P.L.E.R. activities, Devvy's proportional judgments could possibly be connected to his acts of counting. Yet, at this point in the classroom activities, this student's limited fluency in applying proportional constructs does not enable him to quantify his proportional judgment in terms of the local data. Therefore, he begins with a local narrative but, when pressed for an exact answer, he switches to a global approximation.

In summary of this episode, 6th-grade students have personal resources that are relevant to statistical reasoning. The S.A.M.P.L.E.R. PSA stimulates these resources and supports student coordination between these resources. Specifically, in the context of determining the greenness of the S.A.M.P.L.E.R. population—the bridging tool in this episode—students invent sampling as an action that reconciles enumeration and perceptual judgment.

6.2 Episode Two: The Theoretical–Statistical Learning Axis Coheres as a Distribution

a. *b.*

Figure 16. Students can learn about distributions by coordinating between a "population" of green-or-blue squares (on left) and a related combinatorial sample space (on right).

Luke is seated to Devvy's right.[8] He, too, is looking at the S.A.M.P.L.E.R. population (see Figure 16a, above). Luke responds to Devvy's guess of 59%. In his observation, Luke refers to the combinations tower that students had built during the previous week and is now attached to the wall near him (see Figure 16b, above):

> **Luke**: Ok, the reason I think '50 percent' is 'cause when you make that tower [turns in his seat to face the combinations tower], it's gonna be equal for green and blue [equal total numbers of green and blue squares]. So if this [the S.A.M.P.L.E.R. population] were to be all of the combinations, it'll be equal green and blue—50%. [the correct answer was indeed 50%]

The S.A.M.P.L.E.R. population and the combinations tower are physically distinct objects in the classroom. Luke's insight is that we can couch the S.A.M.P.L.E.R. population of thousands of squares in terms of discrete 9-blocks. The connection that Luke builds between these objects is not associated with any particular *new* object. It is grounded in and facilitated by the bridging tool "9-block," yet it is essentially not about 9-blocks per se but about the *distribution* of 9-blocks in the population. This, at this point in the unit where 'distribution' had not been named or otherwise symbolized. So combinatorics-based construal of a population may provide basic

[8] We chose to discuss two episodes that are consecutive in our video data so as to demonstrate variability in student mathematical fluency coming in to the design. Also, the episode shows the flexibility of the design in engaging and stimulating understanding at different levels.

tools for statistical analysis.[9] If Luke had not participated in constructing the combinations tower, he may not have been able to use it as a resource for his insight (see *project-before-problem*, Papert, 1996).

6.3 Episode Three: The Theoretical- vs. Empirical-Probability Axis Coheres as a Sample Space

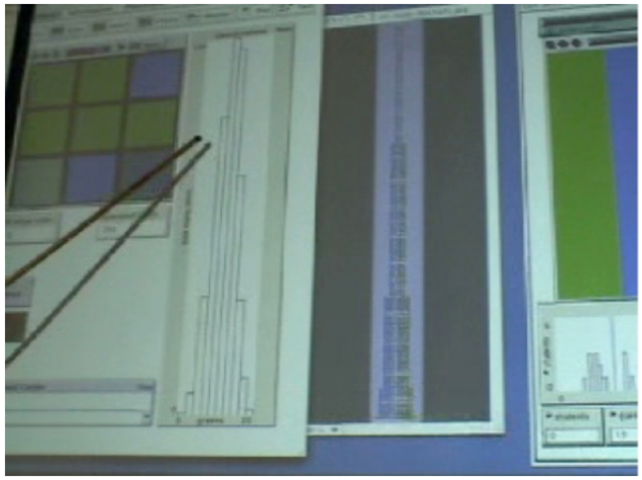


Figure 17. A student explaining why a probability experiment (on the left) produces a histogram that resembles the representation produced through combinatorial analysis (in center).

On the last day of our intervention, we asked students to address what is perhaps the most powerful idea of the domain of probability, the idea of prediction: distributions of *randomly* generated outcomes gradually come to resemble the *anticipated* distributions produced through theoretical analysis. Figure 17 (above) features the 9-Block model on the left and a compute-generated picture of the combinations tower (center). As the 9-Blocks experiment ran, Emma volunteered to explain why one of the central columns in its outcome distribution was growing taller than most other columns.[10]

> **Emma**: "Maybe because there's more of that kind of combination. Just basically, because if there's 512 different combinations, and we know that there's more [possible combinations] in the middle columns, [then] even though there're duplicates, there's still going to be more combinations in the middle columns. [The student is now using a pointer to explain what the class is watching on the screen] Even though these patterns [in the empirical live run, on left] may have duplicates in this [as compared to the combinations tower, center], it's still counting all the patterns, so it's going to have the same shape…. It's going to be the same shape, because it's basically the

[9] Luke's combinatorial–statistical link also fits populations that deviate from 50% green: one can speak of a 73%-green population as being "from the green side" of the combinations tower.
[10] This specific episode from the implementation of ProbLab does not directly describe a S.A.M.P.L.E.R. activity but is relevant to the discussion of learning axes and bridging tools.

same thing. Because in the world there are more patterns of these than there are of the other ones."

Emma's insight is that the tower and the distribution are alike in shape. Moreover, she explains why these two representations *should* be alike in shape. This, despite the possibly confusing fact that the empirically generated representation records many more samples than the 512 9-blocks in the static combinations tower. Emma's assertion that "in the world" there are *relatively* more of one type of pattern as compared to the other suggests that she is attending to the proportions between counts and not to the absolute difference between them. That is, in comparing between the representations, Emma first came to think of the combinations tower as a theoretical-probability tool. The combinations tower is not just the collection of all combinations. Rather, it represents propensity—it is a template for gauging relative frequencies through multiplicative comparison. It is thus a bridging tool for grounding the idea of a sample space.

6.4 Episode Four: The Range-vs.-Cluster Learning Axis Coheres as Variance

For the last day of enacting S.A.M.P.L.E.R., we designed a competitive game between the two study classes. A monitor on the screen tracked the students' average score from round to round (how many "point" the classroom had on average), and the class that had the highest score at the end of five rounds was to be the winner. Also, we disabled students' choice between committing to their own guess or the group guess—all students had to go with the group guess. These combined circumstances engendered a higher-than-usual collaboration in the classroom. The following conversation occurred during a third round in one of the classrooms.

All students had taken their full quota of samples and were discussing how to process the collective classroom data. Becky has been walking around the classroom, observing her classmates' screens and adding up the total green squares on all of these screens. Now, she has just rushed to the teacher, with the following idea: (1) students should each call out their personal guess for the population greenness, but they should not input that guess; (2) someone should calculate the average of these guesses; and (3) all students should input this average. This way, Becky contends, the class as a whole would minimize the error, which she describes as the collective distances from the true value in the population, and would thus minimize the loss of points. Jerry replies that this strategy is redundant and error prone—that all students should just input their own guess and let the computer calculate the mean automatically. [Technically speaking, Jerry is correct—that is precisely what the procedure does.] Becky disagrees. She is vehement, pensive. The question on the table is whether the error of the mean (Becky) is the same as the mean of all errors (Jerry). In particular, Becky warrants her claim with her main concern that outlying guesses would be detrimental to the mean of errors, whereas Jerry thinks that the two calculations are commensurate. It is as though Becky is worried about the variance of the distribution, whereas Jerry reckons that the variance is irrelevant for the task at hand.

Becky: If we're closer to the average, won't the average be closer and we'll [lose less points?]

Jerry: It's the same thing, because this is just like adding up the whole classroom—we're adding it up on the computer.

Becky: If we get this [= if we first calculate the class average independently of the computer] and then people change their guess to be closer to the average…

Jer: [Change] our guess?

Becky: Yeah.
Jer: No, [points to screen] [inaudible]
Becky: I'm adding up how many [blue] they have [and then subtracting to determine how many green they have]
Jer: I know, but [the computer is finding the average] so it will be the same answer. It will be more precise, though.
Becky: I'm adding up how many blues.
Jer: I know, but still, then we reverse it. [because the proportion of green and blue complements 100%]
Becky: And then if they... and people can change their answer closer [to the average]
Jer: But then, well, if we all just put it in [= input our guesses], then since we're all going with the group guess, it'll all... we'll all go with the average, so it'll be the same thing. It'll be a precise average, down to the decimal.
Becky: Yes, but *couldn't we get less points taken off if people changed their guesses so it's closer to the average, since the average will be more precise?*
Jer: It won't matter, 'cause we're going with the group guess, so they'll automatically guess the average. We all are guessing the average, no matter what. We have no choice—we're guessing the average, since we're going with the group guess, and the group guess is the average. We're all guessing the average. We're all guessing exactly the average, down to the millionth.
Becky: Ok, Jose got only 4 blue. His average will be really high up, won't that change the average?
Jer: Yeah, but still, it still takes the average. [Becky rushes back to her seat]

What's in a mean? Should it reflect the *range* of the sampling distribution? If both the mean and the range are important in some activity, perhaps some new mathematical construct is needed that captures both ideas? The context of the guessing game and in particular the high stakes invested in the classroom mean created an opportunity to ground in the histogram, which served as a bridging tool, the idea of variance, an index of the sampling distribution that had not been discussed in the classroom forum. Students naively assume that the tighter the cluster of a set of guesses, the higher its accuracy. Whereas this intuition is sensible and is reflected in statistical measures of confidence, a dense and a sparse cluster of guesses may be as accurate *as a whole*, and in fact a sparse set of guesses may be more accurate than a dense one. Even if these issues are not resolved immediately, the issues are raised through cogent argumentation grounded in personally meaningful mathematical reasoning—the conventional tools are problematized and the domain is complexified.

6.5 Episode Five: The Personal–Social Learning Axis Coheres Around the Histogram Mean

During the second day of implementing S.A.M.P.L.E.R., a unique moment of potential learning occurred at a point where students had all input their guesses for the population's greenness (see Figure 18, below). Whereas most of the students' guesses clustered around the classroom mean of 47.2% (the tall thin line), there was a lone guess of 81% far off to the right. The true value of the population was 50% green, as indicated by the contour between the colored areas immediately above and to the right of the mean. So the outlying guess was instrumental in "pulling" the classroom mean up, to the benefit of the many students who had committed to the group guess with the understanding that this mean is generally more accurate than their personal guess. As it turned out, students were initially under a knee-jerk impression that the outlying guess was detrimental to the precision of the group guess ("How can it be good if it's way off?"). This moment of potential learning is delicate—the facilitator must assess whether exposing the outlying student may be conducive to classroom learning and in improving that student's peer esteem as a mathematician.

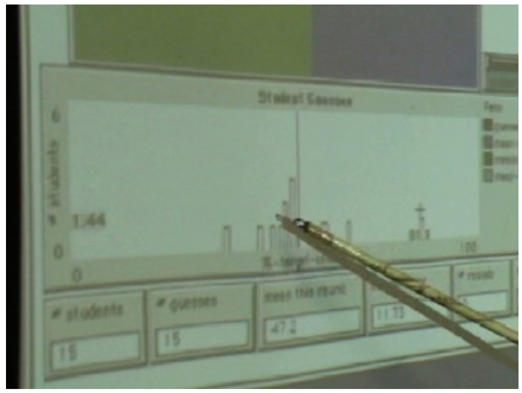


Figure 18. Histogram of students' guesses for the greenness of a population.

Teacher: Do we know whoever that is for sure? [who guessed 81%]
Researcher: We can figure out. [walks over to facilitator's computer to determine the value of that guess—it turns out to be 81%, and Jade identifies herself as the guesser]
Teacher: Oh, it was Jade. Ok. [Jade had been identified by the teacher as low achieving in mathematics]
Res: [to Jade] Ok, you put in 81. Now, this is something very interesting. This is really really interesting. Now, on the one hand... so... ok, so... [addresses class] What do you think of Jade's guess? .... More hands -- what do you think of that guess?
Jade: Terrible. [laughing, somewhat uncomfortably, preempting anticipated ridicule]
Riv: She probably just uncovered a lot of… more green than blue when she was clicking.
Res: Ok, I'll... let's get some more input... Jonathan?
Jonathan: I think that it's a good thing that she guessed so high, because otherwise the average would have been lower.
Res: Could you come up and explain that? [Jonathan walks over to the screen, uses a pointer, see Figure 18, above]
Jon: Uhhm... because the average includes everyone's guess, so that, say she guessed, like, down here [on the far left side of the distribution] like in the 40's or the 30's, well then the average would have been lower, and the average would have been farther away from the actual thing. So like...'cause... if she moved it [her guess] like down here [to the 40's], the average would have been lower, because the total before you divide would have been lower. So, the lower the total before you divide, the lower the number would be. The average would be, like, more down here -- it would be farther away from the actual... from the actual guess... from the actual answer.
Res: So, so Jonathan, people who went with the group guess, what should they think about Jade's guess?
Jon: They should, like, thank her for guessing so high, 'cause that's what got them—that's what got them close enough to the actual answer.

Students: Thank you Jade, thank you Jade.

Jade's episode is an example of how a facilitator working in a networked classroom can tap the classroom's social dynamics to ground mathematical understanding in an authentic interaction. The histogram serves as a bridging tool between Jade's individual guess, which she constructed within her computer environment, and the classroom distribution and mean. If it were not for the entire distribution of guesses being available as a shared display—if, for example, we had been working only with monitors showing various central-tendency outputs—this moment could not have been spun through social tension into learning and new esteem for a student who, apparently, struggles in mathematics. Also, based on students' evidently limited understanding of 'mean,' coming into this design, Jade's episode may have afforded the classroom an opportunity to construe the mean in a new way that was more meaningful than the algorithm for computing it.

6.6 Summary

Table 1.
Learning Axes, Bridging Tools, and Statistical Constructs in S.A.M.P.L.E.R.

Episode	Learning Axis	Bridging Tool	Context	Statistics Construct
1	Local vs. global	Population	Determine the greenness	Sample
2	Theoretical probability vs. statistics	9-Blocks	Determine the greenness	Sample distribution in the population
3	Theoretical- vs. empirical probability	Combinations tower	Explain similarity	Sample space
4	Range vs. cluster	Histogram	Engineer group guess	Variance, balance
5	Individual vs. classroom	Histogram	Judge outlying guess	Sample-mean distribution

We have discussed five classroom episodes that we have interpreted as cases of student and/or classroom negotiation between some pair of affordances of designed objects within some activity context (see Table 1, above). In each case, a different aspect of the design constituted a bridging tool between these affordance antipodes. Student interaction with this bridging tool within the classroom forum supported coordination between schemes, was instrumental in achieving the design-facilitated classroom tasks, and reflects common domain-specific practices. Students' mental constructions around the design-embedded bridging tools is the core objective of our learning environments: these technology-facilitated environments tap, shape, and coordinate students' implicit skills by providing objects (artifacts) around which conceptual structures—concepts-in-action—cohere as useful bits of knowledge.

7 Conclusion

We conclude this paper with a discussion of tradeoffs of the learning-axes approach to design. Following, we propose implications of the work for the design of learning environments. We end by pointing to several future directions that we expect this work will take.

7.1 Tradeoffs of the Design

Designs that foster student learning as a negotiation between polar embedded affordances run the risk of "congestion," i.e. too many new tools and associated terminology that are liable to yield confusion rather than learning. The structure of the ProbLab experimental unit is indeed a complex design that involves concrete and virtual media, personal and collaborative problem solving, art, artisanship, and arithmetical calculation. The designed bridging tools are given to multiple interpretation, so many viewpoints co-exist in the classroom space that is bustling and boisterous. The "free-range students" exchange fragments of information and within their clusters invent fanciful topological and procedural constructs. Yet it is in this very richness—in this complexifying of the classroom—that lies the greatest potentiality of collaborative learning, and networking the classroom is one way of harvesting this richness, as we have attempted to demonstrate. To harvest this richness, facilitators need to be comfortable in leading exploratory discussion that uses the multi-media resources. Leading such exploration calls for a deep understanding of the mathematical content and of possible learning trajectories enabled by the learning environment.

7.2 Implications for Design in Mathematics Education

Learning mathematics is a process of coordinating mental action models into new schemas. On these new schemas ride mathematical terminology, symbolical notation, and solution procedures. The action models do not become coordinated haphazardly. Rather, the coordinating is grounded in objects in the learning environment and stimulated by some task that problematizes the object. A set of heuristics follow from the work reported herein for designers of mathematics learning environments: (a) analyze the target mathematical domain to identify its key concepts; (b) determine the action models and situational contexts inherent in making sense of these key concepts; (c) formulate hypotheses as to the learning challenges inherent in reconciling different domain-specific action models; (d) identify or create "ambiguous" (hybrid) objects affording these competing action models; (e) embed these objects in activities that bring out the ambiguity; and (f) design a learning environment that stimulates individual students to struggle with the ambiguity and facilitates student argumentation.

a. *b.*

Figure 19. Not every ambiguous figure is a bridging tool.

Note that not every ambiguous figure is a bridging tool in the sense that we have been using it to discuss design for mathematics learning environments (see Figure 19, above). If I say 'duck' and you say 'rabbit' (Figure 19a, above), we might learn something about visual perception. But if I say "2 rows of 3 X's" and you say "3 columns of 2 X's" let's not call the whole thing off—rather, we might learn something about mathematics.

7.3 Future work

Future work on ProbLab, and specifically on S.A.M.P.L.E.R., is expected to introduce new challenges as we work with schools to introduce our pedagogical perspectives: (a) *Scheduling*: A design problem is to foster and support student engagement and re-inventing of central ideas for our target domain of probability and statistics within a reasonable time frame; (b) *Responsiveness, generativity*: A technological and research problem is to create a design that anticipates as many student ideas as possible, so that responses to these ideas are embedded in aspects of the design; and (c) *Training, dissemination:* A professional-development problem is to provide teachers with an accumulated repertory of students' ideas and valued suggestions for nurturing these ideas as well as with opportunities to experience these learning environments through participation in workshops.

Future theoretical development of the learning-axes and bridging-tools perspective will involve further design work. Thinking of other mathematical domains in terms of pillars and learning axes may suggest a need for new activities and new bridging tools. Also, we will attempt to improve and elaborate on the current definitions of the construct. One direction will be in understanding how different learning axes interact when they are grounded in the same bridging tools. Learning axes may be orthogonal, but they may interact in complicated ways. For instance, it may be useful to engage the individual-vs.-social learning axis in supporting students work along the local-vs.-global axis. Finally, we have limited the present discussion of learning axes to the domain of mathematics, but the learning-axes approach may be useful as meta-design principles for other domains.

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