How Do Mathematicians Learn Mathematics?

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In this paper, we present preliminary results of a study investigating how experts make sense of unfamiliar *mathematical phenomena. Specifically, 10 mathematics graduate students and professors thought aloud and responded to questions as they read and tried to understand an unfamiliar, but accessible, mathematical proof in the domain of geometric topology. Interview data was coded for experts' descriptions of their own understanding of the mathematical phenomena presented, and for instances in which experts systematically linked different types of knowledge to make sense of new information. While experts' pre-existing and post-interview understandings of the topics presented varied, general strategies for connecting specific types of information to make sense of the proof were identified across participants.*

Introduction

Typically, studies of expert mathematical activity are based on experts' descriptions of their own mathematical practices, experiences and discoveries, or observations of highly performing advanced students and experts as they work with mathematics that they have already considered in depth (Tall, 1991; Vinner, 1991; Dubinsky, 1992; Sfard, 1992; Schoenfeld, 1985; Wilensky, 1991; Lakoff & Nunez, 2000). Little work has been done, however, on understanding how experts think about and make sense of an *unfamiliar mathematical idea as it is first introduced*.

Just as it cannot be taken for granted that mathematics as a field of study is reflective of mathematics as a cognitive activity (Papert, 1980; Tall, 1991; Lakoff & Nunez, 2000); it cannot be taken for granted that the ways in which one describes, understands, or even misunderstands an idea with which they have considerable formal experience is indicative of the processes by which that idea was first acquired. If we are to describe expert *acquisition* and *building* of ideas, we must take all aspects of knowledge into account: not only the structure, description and use of knowledge that experts already possess, but also how experts acquire and build new mathematical knowledge. With this study, we attempt to investigate how this process of knowledge acquisition and construction occurs as an expert engages with a common tool of his discipline – a mathematical proof.

Theoretical Framework

This study aims to describe the nature of mathematical knowledge – both its *structure*, and how that knowledge is built and enacted in authentic expert *practice*. Therefore, this section will concentrate on major theories of expert knowledge, expert practice, and how these theories inform the design and analysis of the study.

The Structure of Expert Mathematical Knowledge

It is well established that the knowledge possessed by novice and expert mathematicians In novices and students, this collection of experiences, images, features and examples is described as a *concept image* (Vinner, 1991) or "informal knowledge" (Schoenfeld, 1985). As these individuals move to expertise, however, these collections of experiences are described as *encapsulated* or *reified* into a mathematical object with associated processes (Tall, 1991; Sfard, 1992). This distinction – between possessing mathematical knowledge that is better described as encapsulated and object-like entities versus a collective store of experiences and informal understandings – is cited as a hallmark of expertise and advanced mathematical thinking. We cannot assume, however, that every new mathematical idea that an expert encounters is automatically understood as such a formal abstraction. While the encapsulated view of mathematical expertise is particularly well-suited for analysing expert practice as it related to well-known mathematical ideas with which one has had considerable experience, it does not necessarily illuminate the process by which such encapsulation occurs, or how new information about an idea is incorporated into an existing organization of knowledge.

Instead, we use the notion of understanding as *connection* (Skemp, 1976; Papert, 1993) and expert knowledge as *dense connection* between mathematical rules, examples, images, everyday experience, and other resources an expert encounters during study. These rules, examples, and so forth can vary from person to person as a result of one's background knowledge, experiences, their own interpretation of the mathematics as presented in disciplinary materials (Wilensky, 1991). This is not to discount the usefulness of the encapsulated view of expert knowledge – indeed, , but rather to illuminate the *mechanism* by which that encapsulation occurs – in other words, to find out how mathematical ideas unknown to an expert can eventually come to be understood by them in a formal way.

Expert Mathematical Practice

If expertise is characterized by encapsulated or densely connected knowledge that can be deconstructed and reconstructed in a number of ways (Tall 2001), then it is not only the structure of knowledge, but also the act of identifying, manipulating, and coordinating that knowledge that is an important component of expertise. For example, Schoenfeld (1985) showed that experts are more likely to monitor their own progress when solving problems, and that experts often employ general-purpose problem solving heuristics unknown to novices. Sierpinska (1994) notes that a distinction should be made between one's *resources for understanding* and *acts of understanding*, in which such resources are put to use in order to solve a problem or make sense of some mathematical idea. Duffin and Simpson (2000) refer to one's ability to not only *have*, but to *build* and *enact* knowledge.

In the context of describing knowledge and understanding as connections between existing and acquired knowledge resources, it makes sense to also investigate the *strategies* by which such connections occur. How do experts select and coordinate existing and new pieces of a given mathematical phenomenon in order to make sense of, or create, new mathematics? Are the combinations of elements used to build, as opposed to enacting knowledge, different?

Research Questions

1) What is the nature of knowledge acquired by experts as they encounter and make sense of a new or unfamiliar mathematical idea?
2) How is this newly acquired knowledge employed by experts to build an adequate understanding of some new or unfamiliar mathematical idea?

METHODS

Participants

10 participants, including 8 professors (assistant, associate, and full) and 2 advanced graduate students from a variety of 4-year universities in the Midwest participated. Preliminary analysis of 3 interviews is included in this paper. Participants were identified primarily through university directory listings, and contacted via email to see if they would agree to be interviewed. In the email, participants were told that we were interested in how experts reason about mathematics, and that they would be provided an unfamiliar proof and asked to discuss the ideas presented within.

Protocol

Students and professors who wished to participate were given semi-structured clinical interviews using a think-aloud protocol (Ericsson & Simon 1993; Chi, 1997; Clement, 2000). Each was provided with the same mathematics research paper (Stanford, 1998) – not directly related to any of the interviewees' specific fields of research – selected for its relative accessibility in terms of complexity and vocabulary. They were asked to read the paper and try to understand it such that they would be able to teach it to a colleague. They were also asked to describe what they understood of the mathematical ideas presented as they read, if this did not come up naturally in the course of the interview. Interview data was videotaped, transcribed, and coded using the TAMSAnalyzer software. The coding system is discussed below.

Proof

The research paper provided to participants (Stanford, 1998) concerns *links,* which can be thought of informally as arrangements of circles of rope that are entwined with one another, and the conditions under which those circles can be pulled apart. If a link has the property that when any single circle is removed from the arrangement, the rest can be pulled apart, that link is said to be *Brunnian*. If, as a result of the entwining of circles, one circle passes over (or under) a different circle, this is called a *crossing*. If part of one circle passes over another circle and is rearranged so that it then passes under the other circle, this is called *changing crossings*. Finally, if all circles in a link are arranged such that there are *n* distinct collections of crossings that, when changed, make the loops fall apart, the link is said to be *n-trivial.* A *trivial* link is one for which all circles can be pulled arbitrarily far away from one another (or, can be reduced to a point without touching one another). The proof establishes a systematic relationship between the properties that make a link *Brunnian* and *n-trivial*, such that any Brunnian link can be described as *(n-1)-trivial*.

DATA

Two sets of codes were used. First, any descriptions of mathematical knowledge were classified as one of six distinct categories: a *parent*, *fragment*, *example*, *construction*, *prototype*, or *definition*. Although these categories were derived for this study specifically and thus may be an artifact of the structure and content of the task provided to participants, we believe that it is applicable to additional domains of mathematics. To illustrate this, each category description includes a real example obtained from interview data, and a hypothetical example to illustrate how each category would apply to possible descriptions of even numbers.

Parent. Aspects of some mathematical idea that are inherited from more familiar experiences or understandings related to the idea under consideration.

> Ted: Some guy I knew in grad school did some sort of knot theory things, and had I don't know, lots of little lines that were supposed to represent little loops and he'd move them around and see if he could make them look more complicated or less complicated. And so I'm thinking it's somehow related to that, but I don't have a good sense.
>
> Even numbers as a type of integer: “Well, I know that even numbers are a kind of number, so I can perform operations like adding and subtracting with them”.

Fragment. Components, pieces, or relations that comprise the building blocks of a mathematical idea or object; or ways to divide the idea into smaller, easier to manage pieces.

> Ana: “the rest is trivial whatever that means, and I'm assuming that means the disjointed circles.”
>
> Groups of two as fragments of an even number: “They are made up of groups of 2”

Example. Specific instantiations of the idea being considered that are immediately available to an individual, either via recall or because it is provided.

> Mike: "Yea, the Borromean rings should be... I know enough... they have that property… that when you unlink a component, you get a trivial knot."
>
> The number "10" as an instance of an even number: "I know that 10 is one"

Prototype. Special instantiations of the idea being considered that are assumed to be representative of a more than one single example or instance of the idea.

> Joe: so, so my definition was sort of, construct a canonical example and say this is, any Brunnian link is isotopic to this brunnian link, so.. it's like a representative of equivalence classes of brunnian links, so…
>
> The last digit of a number as an indicator of evenness: "every number that ends with 0"

Construction. Ad-hoc example, usually developed by combining fragments, examples, and/or prototypes in some way.

> Mark: Okay. And so if I got something like that [forms circle with one finger] and [interlocks with other finger] something interchanging here, if I remove one of the links the other two come apart, then that's what they're talking about .
>
> An even number as constructed from fragments: "They are made of groups of 2, and 6 is three groups of 2, so 6 is even".

Definition. Complete descriptions of the behavior, structure, or properties of the focal mathematical idea, which accounts for all instances of the idea.

> Joe: …he's saying if we have n-components of brunnian… whenever [turning page] you look at… whenever you throw away one of the components you have something trivial.
>
> A formal definition of an even number: "Any integer multiplied by 2."

Second, experts' responses to interviewer questions and their think-aloud statements while reading the proof were coded as *questions*, *solutions*, or *explanations*. For example, if an expert simply states that she does not understand some aspect of the proof, that statement would be coded as a *question.* On the other hand, if the expert is not immediately familiar with some aspect of the proof, but is able to use definitions, examples, or other features described within the proof to arrive at an explanation of that aspect, that statement would be coded as a *solution*. Finally, if an expert asserts that she was already familiar with phenomena described in the proof and simply describes that existing understanding, that statement would be coded as an *explanation*. While there are some differences, these codes map particularly well to previous theories of mathematical understanding – notably, Duffin and Simpson's (2000) classification of understanding as *building*, *enacting*, and *having* understanding. Each question, solution, and explanation could contain any number of parents, fragments, examples, and so forth.

Question. Participant does not understand some aspect of the proof.

> Joe: Okay. [takes pencil] Okay, so they're saying something about… n-triviality, I've never heard of that…
>
> Interviewer: Do you have any idea of what that might mean?
>
> Joe: Not a clue.

Solution. Participant is unfamiliar with some aspect of the proof, but is able to use other components of the proof such as definitions, examples, and so forth to arrive at an explanation.

> "So he's saying here are these crossings, these are in one set and these two are in another set. But then what does it mean to change them? [pause] So suppose I picked um one corresponding to A, what am I supposed to do what does that mean to change them? I wonder if it means to go from an up crossing to a down crossing, so let's try. […] oh yeah, see I do think I'm right, because that circle is disengaged by changing these two crossings okay. So changing crossings means going from up crossing to down crossing. Until two pages later when we'll realize that that's wrong." (Ana)

Explanation. Participant is familiar with some aspect of the proof, and readily describes their understanding of that aspect.

> [reads] Note that n-trivial implies n-1 trivial for n > 0. [done reading] Which of course, if you remove one link and it's trivial, and then you remove another link, well it's already trivial. You're expanding on your triviality, you're feeling… really trivial. (Greg)

Results

First, experts varied dramatically in the amount of background knowledge they possessed and employed when solving problems, and this affected the resources that they had available to make sense of the proof. However, the *types* of knowledge employed for different expert statements were relatively consistent: for example, experts are much more likely to refer to examples, constructions, or prototypes when working on a solution than when asking a question or explaining an already understood component of the proof. Similarly, experts are more likely to refer to multiple definitions within explanations rather than within questions or solutions – a sign, perhaps, of mature understanding in which an expert is able to, as expected "… link together large portions of knowledge into sequences of deductive argument" (Tall, 1991, p. 4).

Statement Types

While all participants exhibited a relatively even distribution of statement types – that is, they all explained, asked questions, and found solutions during the interviews

– the number of elements of the proof that they brought together within each statement type varied. Notably, for all three participants that have been analysed, more elements of the proof were brought together in solutions than in any other type of statement, and less elements were brought together for explanations. This may indicate that experts build the most knowledge – that is, they make the most connections between knowledge elements – when they do not fully understand all of the components needed for a given mathematical idea. Similarly, the small number of elements involved in a given explanation may reflect the object-like, unitary nature of well-understood (or, depending on perspective, well-connected) knowledge commonly described in studies of expert performance with ideas with which they have had a great deal of experience.

	Ana	Mark	Joe	All
Explanations	42%	35%	43%	41%
Questions	32%	29%	28%	30%
Solutions	26%	35%	29%	41%
Elmts. per Q	2.31	2.00	2.41	2.24
Elmts. per S	2.60	3.16	3.20	2.99
Elmts. per E	1.33	1.60	2.21	1.75

Table 1. Statement types and elements per statement type by participant

Patterns Within Statement Types

In addition to patterns in the number of elements mentioned in a given type of expert statement, there were also patterns in the *type* of elements mentioned in various types of statements. Below, we discuss two of these patterns: the high frequency of direct connections between fragments (F) and definitions (D) or between more than one definition (DD) within explanations (DF~E; DD~E), and the high number of embodiments (E; examples, constructions, or prototypes) that accompany connections made between fragments and definitions within solutions (FED), and the low frequency of questions in these categories asked by Mark – a function, perhaps, of his increased use of parent references to make sense of unknown elements within the proof.

Patterns	Statements	Ana	Mark	Joe	All
DD~E	Explanations	50%	0%	18%	12%
	Questions	8%	0%	29%	19%
	Solutions	0%	17%	27%	19%
FD~E	Explanations	50%	60%	27%	46%
	Questions	25%	0%	43%	23%
	Solutions	10%	17%	33%	20%

FED	Explanations	6%	0%	9%	5%
	Questions	0%	0%	0%	0%
	Solutions	20%	33%	13%	22%

Table 2. Frequency of multiple definitions (DD~E), definitions and fragments (FD~E), and definitions and fragments with an embodiment (FED) within statements

Embodiments (E) – items coded as examples, constructions, or prototypes – were much more likely to occur within *solutions*: statements in which experts did not readily understand an idea presented in the proof, but were able to build an understanding using other pieces of the idea. When embodiments did occur, they did so within the context of a solution. It might be that, when participants did not readily understand how two definitions, or a definition and a fragment were related, they used embodiments to help form those connections, which later manifested as well-understood explanations without those scaffolding embodiments (DD~E and FD~E).

Conclusion

In this paper, we presented a framework, coding system and data illustrating how experts acquire and synthesize knowledge to make sense of new and unfamiliar mathematical concepts. Preliminary results suggest that though expert knowledge is often described as encapsulated or object-like, it may be experts' familiarity and systemic interaction with a given idea – rather than an experts' status as an expert – that results in this organized structure. Instead, expertise may lie in the ways that experts combine and scaffold their knowledge in order to identify, acquire, and build dense connections between components of a mathematical idea.

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