Appropriate Tools:
On Grounding Mathematical Procedures in Perceptual Intuitions

Dor Abrahamson, University of California, Berkeley

Abstract: I report on a design-based research case study in the area of middle-school probability that served as a context for investigating whether students can build meaning for the disciplinary tools they are taught to use, and if so, what personal, technological, and interpersonal resources may support this process. The topic of binomial distribution was selected due to robust literature documenting students' apparent 'misconceptions' of expected likelihoods. Li successfully built upon his event-based intuitive sense of likelihood in developing the outcome-based notion of sample space. Utilizing cognitive-science, sociocultural, and cultural–semiotic theoretical models of mathematical learning, the construct 'semiotic leap' is developed herein to explain Li's insight as appropriating an available artifact as a means of warranting his intuitive inference.

Objectives

Can students construct meaning for the mathematical procedures they learn to use in school? What cognitive, technological, and social mechanisms might facilitate this learning process? In particular, what roles may semiotic tools—material objects, diagrams, symbols, speech, gesture, etc.—play in a guided learning process in which students come to see mathematical content as enhancing their perceptual intuitions? And to the extent that students can use perceptual judgments to perform intuitive inferences that are aligned with normative mathematical knowledge, what would the implications of an intuition-based learning process be for the adequacy of theories of learning that are based exclusively either on the cognitive sciences or on sociocultural theory?

This proposed paper communicates the case study of Li, a 6th-grader, who was guided to build on his intuitions of likelihood as he took first steps in learning fundamental notions of probability within the context of a single interview-based tutorial interaction. I will claim that despite initially experiencing stark discrepancy between his intuitive notions and the procedures that he performed, Li eventually succeeded in coordinating these resources, albeit this coordination was abductive, embryonic, and still unstable.

Theoretical Background

This study is aligned with the constructivist perspective whereby: (a) mathematical understanding is viewed as developing from naïve intuitions that are persistent yet become qualified, calibrated, refined, and reorganized, on the basis of feedback in diverse contexts, into the complex structures that are manifest as mastery in a domain; and therefore (b) educators should identify, embrace, and work with students' intuitions (e.g., diSessa, 2008). Accordingly, the learning materials, activity sequence, and facilitation of this study, as well as the data analysis, are all oriented toward treating mathematical learning as an individual's negotiated reconciliation of personal schemata for mathematical situations with mediated engagement of cognitive artifacts (Greeno, 1998; Saxe & Esmonde, 2005; Sfard, 2002; Stetsenko, 2002; Stevens & Hall, 1998).

Due to its canonical perception as counterintuitive, probability appeared to present an ideal disciplinary topic for examining relations between intuition and learning. In particular, people tend to think that a coin flipped four times will more likely land on "HHHT" than on "HHHH," whereas according to probability theory these outcome sequences are equiprobable (adapted from Tversky & Kahneman, 1974). I propose to interpret students' "P(HHHT) > P(HHHH)" responses as resulting from a "legitimate reconstruction of the problem" (Borovcnik & Bentz, 1991; see also Chernoff, 2007). Specifically, I submit that intuitive judgment of outcome frequency privileges the *number* of occurrences of the binomial values 'H' and 'T' (or the ratio of H's to T's) at the expense of attention to their order, thus encoding 'HHHT' as "3H, 1T." Moreover, this intuitive judgment would underlie a correct answer to a question that ignores the outcome order, because "3H, 1T" is indeed more likely that "4H"—it is four times as likely (compare the four possible outcome sequences with exactly three heads—HHHT, HHTH, HTHH, THHH—with the single possible outcome sequence that has four heads, HHHH) (related work: Amit & Jan, 2007; Drier, 2000; Iversen & Nilsson, 2007; Jones, Langrall, & Mooney, 2007; Kazak & Confrey, 2007; Pratt, 2000; Wilensky, 1995, 1997).

Methods: An Intuition, a Procedure, and an Activity to Potentially Connect Them

Learning Tools. The marbles box (see Figure 1) contains a mixture of hundreds of marbles of two colors, with equal numbers of each color (green and blue), and the *marbles scooper* is a utensil for drawing out of this box samples of exactly four marbles. These devices thus constitute a random generator of type "urn," a mathematical artifact which is widely used and/or referred to in probability literature, only that: (a) the marbles box is an open urn, so that the color ratios are exposed for perceptual inspection; and (b) the scooper's structural properties impose constraints on the possible spatial configuration of the independent outcomes within the compound-outcome sample. I assumed that affixing the locations of the independent outcomes might create opportunities for a student and facilitator to co-attend to the phenomenal property of order as a potentially meaningful parameter, i.e., as bearing on the question of anticipated outcome distribution. Moreover, the built-in 2-by-2 configuration could serve as a ready template for determining the set of "what we could get," i.e., for building the sample space. Students were further provided with an ample stock of cards, each bearing an empty 2-by-2 matrix (see Figure 2), as well as two crayons (green and blue). Furthermore, I marked one edge of the matrix with a thicker line, as a means of cueing the idea that a rotation permutation might be interpreted as a discernable outcome.

Figure 1. The marbles-scooper random generator (set at $n = 4$, $p = .5$)

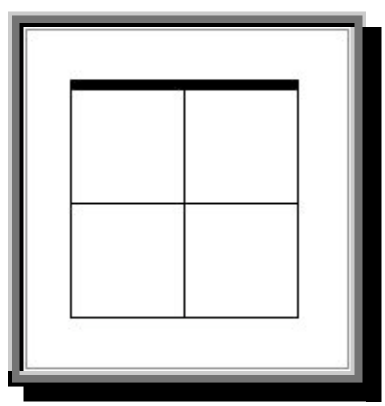


Figure 2. A card for constructing the sample space

Participant and Procedure. Li, a 6th-grade student, was one of 28 Grade 4 – 6 participants in a study conducted at a suburban school (Abrahamson & Cendak, 2006). Li, who was rated by his mathematics teachers as a 'high achiever,' was verbose, articulate, and argumentative. His generally engaged disposition throughout the interview as well as the contents of his observations were not atypical with respect to the majority of participants in this study, yet Li's comfort and confidence in expressing his beliefs, even as these were shifting, were especially illuminating of his reasoning process. The study consisted of conducting a one-to-one semi-structured ~30-minute clinical interview, which implemented a prepared protocol (diSessa, 2007; Ginsburg, 1997): First, we present the marbles-scooping equipment, demonstrate its mechanism by letting the student scoop several times. We then ask the participant, "What do you think will happen when I scoop?"; Next, we present the cards and crayons and guide the participant to color in "all the different scoops we could get" and assemble this sample space into a *combinations tower*, a histogram-shaped structure (see Figure 3).

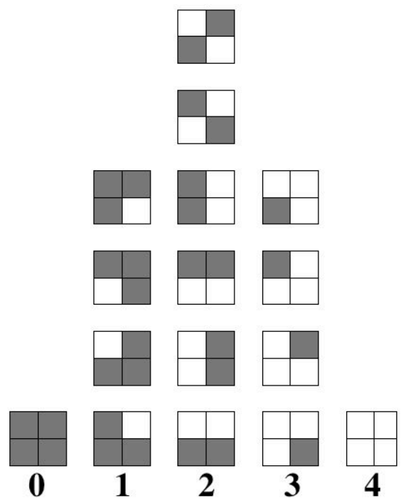


Figure. 3. The combinations tower

Data Collection and Analysis. All interviews were audio/video-taped for subsequent analysis. We worked in the traditions of collaborative microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1991; Siegler & Crowley, 1991) and grounded theory (Glaser & Strauss, 1967).

Results and Discussion: Analysis of a Case Study

To demonstrate the nature of the analytic work reported in the full paper, I shall briefly focus here only on a single 3-minutes episode, wherein, I claim, Li reinvented classicist probability. Prior to this episode, Li—similarly to the other 27 participants—had guessed, on the basis of visually scrutinizing the marbles-scooping random generator, that the empirical distribution in hypothetical experiments would bear a plurality of outcomes of type '2 green, 2 blue' (hence, 2g2b - see Abrahamson, in press). However, in explaining his reasoning, Li switched to claiming that 2g2b has a one-in-five chance (which I interpret as a case of 'ontological imperialism,' Bamberger & diSessa, 2003, i.e., a dire consequence of prematurely imposing upon learners semiotic tools that they cannot as yet use as means of expressing their preformulated notions). Accordingly, in building the sample space, Li created only five cards (the bottom row of Figure 3, above). However, through the following exchange, once he has created the combinations tower, Li—similarly to all-but one of the other 27 participants, will come to assert that the *entire* collection of cards should be considered in determining the expected outcome distribution (Abrahamson, in press).

Res: You see this [gestures toward the entire sample space], and you say the chance of getting, uhhh, two… the chance of getting, uhhh, a… scooping something with two-green is one-out-of-five.
Li: /5 sec/ Well, actually… /3 sec/ yeah [one-out-of-five]!
Res: Ok.
Li: /2 sec/ Actually, /7 sec/ it kinda seems like it would be six-out-of-sixteen.
Res: Huh! Ok, so what… so… 'One-out-of-five' now went to 'six-out-of-sixteen.' What….how…
Li: Well, it's like…
Dor: That's quite a difference!
Li: Yeah. It… /10 sec/ Well, there are sixteen… /4 sec/ Well, actually… /10 sec/ No, it's still—I think it still would be one-out-of-five.
Res: Mm'hmm. So if I scoop, about a fifth of the time I'll get a… something with two.
Li: Two of each.

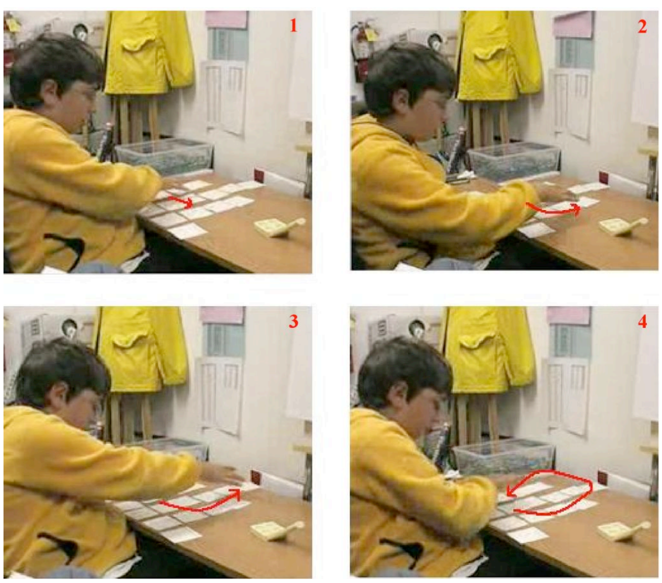


Figure 4a. Li sees the permutations as irrelevant to the task

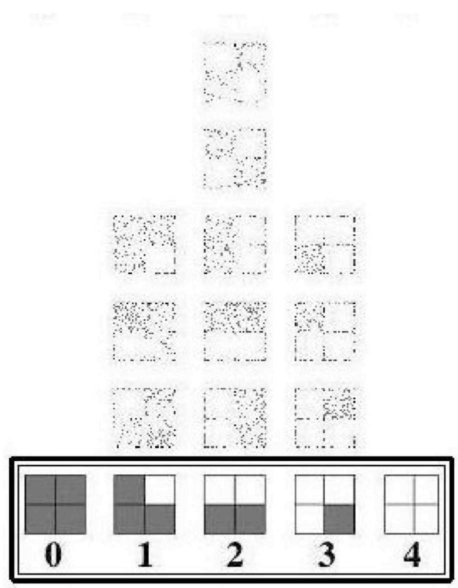


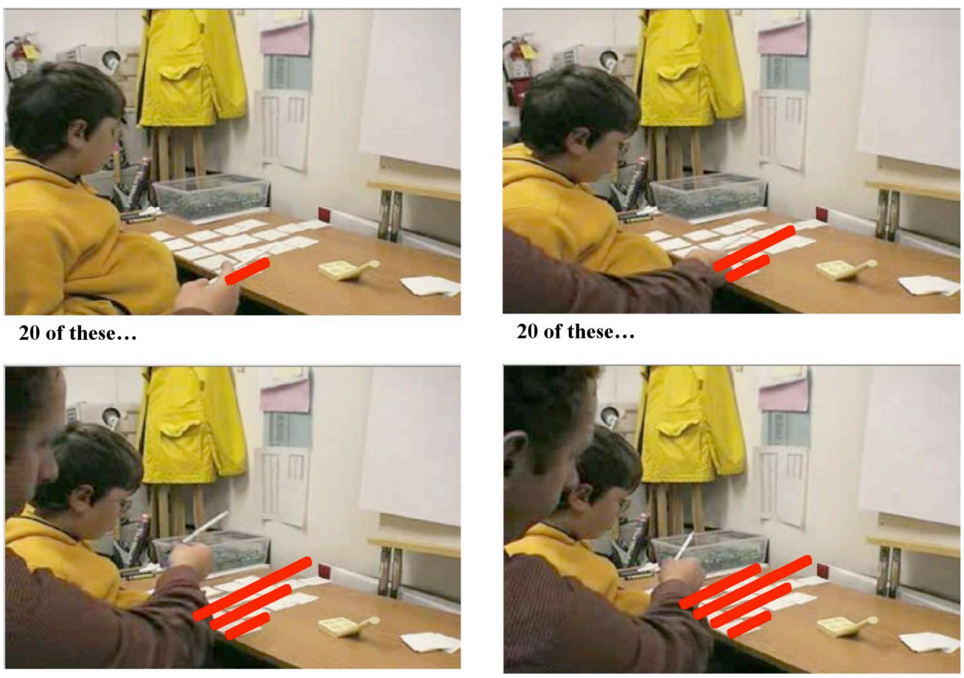
Figure 4b. A schematic representation of Li's perceptual construction of the sample space

Res: Ok. So…
Li: 'Cause like, these [indicates all 11 cards above the bottom row of five cards] don't really matter. [see Figures 4a & 4b]
Res: In what sense?

Li: Well, if you're looking to… /4 sec/ Well, if the placement mattered [gestures back and forth between the scooper and the 11 cards], these *would* matter, but these [eleven cards] are all the same thing. These [within the 1g3b column, indicates the three cards above the bottom card] are the same thing as this [points to the bottom card] except for the placement [repeats gesture pattern for the 2g2b column and the 3g1b column]. So it's these same original five [indicates bottom row] or, like, any one of these [indicates that any of the cards above the bottom row could replace its respective bottom card]… /3 sec/ that matters.

Res: /5 sec/ So, I mean, this issue of 'placement,' that seems to be what the… It's not just, like, you and me deciding, "Let's use placement" or "Let's not use placement." It's, like, How does that relate to the situation in the world, like, the scooping?—Should we care about placement or not? And it seems like you're saying… "not."

Li: Uhhm, yeah.



20 of these… **20 of these…**

20 of these… **And 20 of these.**

Figure 5. "20 of these, 20 of these…" (overlays indicate highlighted columns)

Res: Ok. So your prediction is that if we scooped, say… I donno, 100 times, we'll get about 20 of these, 20 of these, 20 of these, 20 of these, 20 of these? [each of the "20 of these" utterances is accompanied by a gesture, pen in hand, toward a column in the combinations tower, beginning with the right-most, 4g column, and moving to the left (see Figure 5)].

Li: /5 sec/ Actually, no. I would… I'm going back to… there's, out of all the possibilities you could get, six-out-of-sixteen are two-and-two, and these [indicates the 0g and the 4g cards] are only one-out-of-sixteen, so… Like, what I was saying—"one-out-of-five

> chance"—that would mean… /6 sec/ 'Cause, [vehemently] *you'll get* these [hand sweeps up and down the 2g column] more than these [holds up the single 4g card], 'cause there's six of these and there's only one of these.

In the full paper (Abrahamson, in press), I bring to bear empirical findings and constructs from the cognitive sciences, socio–cultural theory, cultural semiotics, and pragmatics—which, I propose, should be viewed as complementary—so as to explicate the epistemological nature and underlying mechanisms of Li's fragile insight (Fauconnier & Turner, 2002; Gelman & Williams, 1998; Grice, 1989; Hutchins, 2005; Radford, 2003; Stavy & Tirosh, 1996; Tsal & Kolbet, 1985; Xu & Vashti, 2008).

Conclusion, Implications, and Significance

Students can construct personal meaning for the mathematical procedures they learn to use as problem-solving tools, even when they initially do not understand the rationale of these cultural artifacts. For researchers, a full understanding of such learning-through-using processes requires an integrated theoretical perspective encompassing both the cognitive sciences and sociocultural theory. That is, analyses of artifact-mediated learning as participation in apprenticeship (Lave & Wenger, 1991), discursive activities (Cobb & Bauersfeld, 1995; Sfard & McClain, 2002), or acculturation into reflective praxis (Radford, 2006), should be complemented with attention to individuals' struggle to align intuitive and cultural resources. Individual learning transpires at the nexus of complex bottom-up and top-down dialectical processes (Clancey, in press; diSessa, 1993).

An epistemological commitment of the proposed integrated theoretical perspective, as well as a concomitant heuristic design framework, is that some mathematical concepts can be learned through a process of initially "meaningless" tool use that is nevertheless crafted so as to lead up to students' abduction of analytic procedures. One effective strategy for facilitating students' grounded appropriation of cultural tools is to design mathematical situations that tap students' intuitive schemes and then facilitate activities through which students take *semiotic leaps*, i.e., come to see the tools as semiotic means for warranting their preformulated intuitions, even before these processes are inscribed symbolically. Such guided learning can ultimately be as meaningful for students as other types of discovery-based processes practiced in constructivist curricula, because the students experience personal invention of the procedure-*as-instrument* even in the midst of learning to perform this procedure with ready-made tools (see Vérillon & Rabardel, 1995, on 'instrumental genesis').

Specifically for the topic of basic probability, this study constitutes empirical support for the conjecture that students' event-based intuitive expectation of likelihoods in experiments with random generators is qualitatively in accordance with mathematics and, moreover, that this intuition can ground the mathematical procedure of outcome-based combinatorial analysis. Thus, the intuitive sense of likelihood, which Tversky and Kahneman (1974) regard as a bias-prone heuristic, is in fact a useful cognitive resource that can and should be embraced in the teaching and learning of the binomial. This, counter to decades of curriculum research that has been struggling to eradicate learners' naïve intuition of likelihood.

References

Abrahamson, D. (in press). Orchestrating semiotic leaps from tacit to cultural quantitative reasoning—the case of anticipating experimental outcomes of a quasi-binomial random generator. *Cognition and Instruction*.

Abrahamson, D., & Cendak, R. M. (2006). The odds of understanding the Law of Large Numbers: A design for grounding intuitive probability in combinatorial analysis. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the Thirtieth Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 1-8). Charles University, Prague: PME.

Amit, M., & Jan, I. (2007). *A learning environment that promotes concept construction: The case of probability*. Paper presented at the annual conference of the American Educational Research Association, Chicago, IL.

Bamberger, J., & diSessa, A. A. (2003). Music as embodied mathematics: A study of a mutually informing affinity. *International Journal of Computers for Mathematical Learning, 8*(2), 123-160.

Borovcnik, M., & Bentz, H.-J. (1991). Empirical research in understanding probability. In R. Kapadia & M. Borovcnik (Eds.), *Chance encounters: Probability in education* (pp. 73-105). Dordrecht, Holland: Kluwer.

Chernoff, E. J. (2007). Sample Space Rearrangement (SSR): The example of switches and runs. In T. Lamberg & L. Wiest (Eds.), *Proceedings of the Twenty Ninth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 433-435). Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Clancey, W. J. (in press). Scientific antecedents of situated cognition. In P. Robbins & M. Aydede (Eds.), *Cambridge Handbook of Situated Cognition*. New York: Cambridge University Press.

Cobb, P., & Bauersfeld, H. (Eds.). (1995). *The emergence of mathematical meaning: Interaction in classroom cultures*. Hillsdale, NJ: Lawrence Erlbaum.

diSessa, A. A. (1993). Towards an epistemology of physics. *Cognition and Instruction, 10*(2-3), 105–225.

diSessa, A. A. (2007). An interactional analysis of clinical interviewing. *Cognition and Instruction, 25*(4), 523-565.

diSessa, A. A. (2008). A bird's-eye view of the "pieces" vs. "coherence" controversy. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 35-60). New York: Taylor & Francis - Routledge.

Drier, H. (2000). Children's meaning-making activity with dynamic multiple representations in a probability microworld. In M. L. Fernandez (Ed.), *Proceedings of the 22nd annual meeting of the North American Chapter of the International Group for Psychology of Mathematics Education* (Vol. 2, pp. 691-696). Columbus, OH: Eric Clearing House.

Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. New York: Basic Books.

Gelman, R., & Williams, E. (1998). Enabling constraints for cognitive development and learning: Domain specificity and epigenesis. In D. Kuhn & R. Siegler (Eds.), *Cognition, perception and language* (5th ed., Vol. 2, pp. 575-630). New York: John Wiley and Sons.

Ginsburg, H. P. (1997). *Entering the child's mind*. New York: Cambridge University Press.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine Publishing Company.

Greeno, J. G. (1998). The situativity of knowing, learning, and research. *American Psychologist, 53*(1), 5-26.

Grice, P. (1989). *Studies in the way of words*. Cambridge, MA: Harvard University Press.

Hutchins, E. (2005). Material anchors for conceptual blends. *Journal of Pragmatics, 37*(10), 1555-1577.

Iversen, K., & Nilsson, P. (2007). Students' reasoning about one-object stochastic phenomena in an ICT-environment. *International Journal of Computers for Mathematical Learning, 12*(2), 113-133.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realities. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909-955). Charlotte, NC: Information Age Publishing.

Kazak, S., & Confrey, J. (2007). Elementary school students' intuitive conceptions of random distribution [Electronic Version]. *International Electronic Journal of Mathematics Education*, *2*. Retrieved July 23, 2008 from www.iejme.com.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.

Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education, 31*(5), 602–625.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37-70.

Radford, L. (2006). Elements of a cultural theory of objectification. *Revista Latinoamerecana de investigacion en Matematica Eduactiva, Special Issue on Semiotics, Culture and Mathematical Thinking*, 103-129 (Trans. H. Empey).

Saxe, G. B., & Esmonde, I. (2005). Studying cognition in flux: A historical treatment of "fu" in the shifting structure of Oksapmin mathematics. *Mind, Culture, & Activity, 12*(3&4), 171–225.

Schoenfeld, A. H., Smith, J. P., & Arcavi, A. (1991). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 55-175). Hillsdale, NJ: Erlbaum.

Sfard, A. (2002). The interplay of intimations and implementations: Generating new discourse with new symbolic tools. *The Journal of Learning Sciences, 11*(2&3), 319-358.

Sfard, A., & McClain, K. (2002). Analyzing tools: Perspectives on the role of designed artifacts in mathematics learning [Special Issue]. *Journal of the Learning Sciences, 11*(2&3).

Siegler, R. S., & Crowley, K. (1991). The microgenetic method: A direct means for studying cognitive development. *American Psychologist, 46*(6), 606-620.

Stavy, R., & Tirosh, D. (1996). Intuitive rules in science and mathematics: The case of "more of A - more of B". *International Journal of Science Education, 18*(6), 653-668.
Stetsenko, A. (2002). Commentary: Sociocultural activity as a unit of analysis: How Vygotsky and Piaget converge in empirical research on collaborative cognition. In D. J. Bearison & B. Dorval (Eds.), *Collaborative cognition: Children negotiating ways of knowing* (pp. 123-135). Westport, CN: Ablex Publishing.
Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 107-149). New York: Cambridge University Press.
Tsal, Y., & Kolbet, L. (1985). Disambiguating ambiguous figures by selective attention. *Quarterly Journal of Experimental Psychology, 37*(1-A), 25-37.
Tversky, A., & Kahneman, D. (1974). Judgment under uncertainty: Heuristics and biases. *Science, 185*(4157), 1124-1131.
Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77-101.
Wilensky, U. (1995). Paradox, programming and learning probability. *Journal of Mathematical Behavior, 14*(2), 231-280.
Wilensky, U. (1997). What is normal anyway?: Therapy for epistemological anxiety. *Educational Studies in Mathematics, 33*(2), 171-202.
Xu, F., & Vashti, G. (2008). Intuitive statistics by 8-month-old infants. *Proceedings of the National Academy of Sciences, 105*(13), 5012-5015.