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ORIGINAL ARTICLE

Coordinating visualizations of polysemous action: values added for grounding proportion

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Abstract We contribute to research on visualization as an epistemic learning tool by inquiring into the didactical potential of having students visualize one phenomenon in accord with two different partial meanings of the same concept. 22 Grade 4–6 students participated in a design study that investigated the emergence of proportional-equivalence notions from mediated perceptuomotor schemas. Working as individuals or pairs in tutorial clinical interviews, students solved non-symbolic interaction problems that utilized remote-sensing technology. Next, they used symbolic artifacts interpolated into the problem space as semiotic means to objectify in mathematical register a variety of both additive and multiplicative solution strategies. Finally, they reflected on tensions between these competing visualizations of the space. Micro-ethnographic analyses of episodes from three paradigmatic case studies suggest that students reconciled semiotic conflicts by generating heuristic logico-mathematical inferences that integrated competing meanings into cohesive conceptual networks. These inferences hinged on revisualizing additive elements multiplicatively. Implications are drawn for rethinking didactical design for proportions.

This article builds on a paper presented to the Special Interest Group on Research in Mathematics Education (SIG RME) at the 2012 annual meeting of the American Educational Research Association (AERA).

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> I didn't pay enough attention to change of perspective. …. The subject deserves a more systematic treatment, which I do not dare undertake. …. Learning processes are marked by a succession of changes in perspective which should be provoked and reinforced by those who are expected to guide them. (Freudenthal, 1991, as cited in Streefland, 1993, pp. 132–133)

1 Introduction

Hold your hands level in front of you. Now, raise them slowly in parallel, with the right hand moving faster than the left hand. You are performing a bimanual action that enacts a particular operatory schema. Let us try another activity. Hold your hands level in front of you. Raise them slowly in parallel, with the right hand slightly higher than the left hand, and steadily increase this vertical interval. You are performing the same bimanual action as before, only that it now enacts a different operatory schema. Try explaining how a single bimanual action enacts two different schemas, and note how this modest exercise—comparing two visualizations of a single action—rapidly develops into an exploration of core content.

This article aims to contribute to research on visualization as an epistemic learning tool by inquiring into the didactical potential of having students visualize one phenomenon in two different ways such that each evokes a partial meaning of the same mathematical concept (see also Godino et al. 2011). We propose that when learners reconcile this ambiguity, they do so by generating heuristical logico-mathematical inferences that integrate competing meanings into cohesive conceptual networks. The paper considers the theoretical plausibility of this proposal in light of learning sciences literature and evaluates its empirical validity by interpreting data from a pilot implementation of an exploratory design for proportion.

We selected the mathematical content of proportion for this study due to debates over its instruction. In particular, this content is suited for investigating the coordination of visualizations due to the enduring pedagogical question of how students might ground multiplicative conceptualizations of ratio in additive conceptualizations of proportional progression. We hope to demonstrate that a visualization approach to design, instruction, and analysis can contribute to this important debate.

1.1 Visualization and learning from a sociocultural, embodied-cognition approach

Arcavi (2003) offered an expansive definition of visualization:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (Arcavi, 2003, p. 217)

We are inspired by this work and seek to promote its presence in scholarly dialogs on mathematical cognition and instruction. At the same time, informed by the sociocultural and embodiment turns in mathematics-education research, and working with emerging technologies that challenge our tacit assumptions about pedagogical activities, we wonder whether Arcavi's definition might now be pruned. We thus offer several emphases and differentiations, as follows.

1. *Modality*. Whereas the term "visualization" might suggest a theoretical focus on the visual modality alone, we conceptualize it as multimodal (Arzarello and Robutti 2010).
2. *Activity*. Whereas "visualization" might suggest a form of passive perception, we conceptualize it as active perceptuomotor coupling with the affordances of the environment (Roth 2010). This view resonates with genetic epistemology and in particular the construct of a schema that emerged from micro-ethnographic studies of cognitive development (Piaget 1968).
3. *Phenomenology*. Whereas "visualization" might suggest a psychological construction of immediately available proximal stimuli, we conceptualize it as treating either actual or mentally simulated images (Goldstone and Barsalou 1998; Hommel et al. 2001; Kirsh 2010), which may be deployed in a variety of media and semiotic systems.
4. *Ontology*. Whereas "visualization" might suggest an object—a thing in the world—we conceptualize it as an orientation—an implicit, malleable perceptuomotor grip on the world (Dreyfus 2002). When we orient our attention toward perceptual information with the goal of drawing inferences, visualizing is the natural epistemic orientation (Bakker and Derry 2011). Linguistically speaking, visualization is more verb than noun.
5. *Epistemology*. Whereas "visualization" might suggest an articulated judgment or inference, we conceptualize it as pre-articulated mental content available to reflection and elaboration, such as through objectification (Radford 2003).

Thus, visualization is a naturalistic epistemic mode invoked in attending to and drawing inferences from perceptual displays, including resources offered to students as bearing didactical potential for content learning. Visualization in the disciplines is mediated to novices socioculturally through vicarious or direct participation in the enactment of historically evolved, goal-oriented collaborative practices for managing resources in shared perceptual fields (Alač and Hutchins 2004; Goodwin 1994; Jay 1988; Rogoff 1990; Stevens and Hall 1998).

1.2 Action as an ambiguous object

Arguably, the mind evolved to make sense of moving objects at least as much as static displays. Making sense of other agents' actions, in particular, relies on tacitly simulating their enactment (Gallese et al. 1996). To do so, the perceiver draws on available schemas in accord with the actor's assumed goal (Csibra and Gergely 2011). As such, generating meanings for perceived action is both enabled and delimited by a person's repertory of schemas. When more than a single schema is available and suitable, nuanced shifts in attention may evoke different meanings for an action. As I watch you raise your hands, I cannot know for certain what you mean by it and what schema you are enacting. The action is ambiguous. I may even watch my own hands rising and, as I am doing so, toggle between alternative schemas.

Research on the perception and interpretation of ambiguous visual information has by and large focused on static figures external to the perceiver (Tsal and Kolbert 1985). Instead, the ambiguous phenomena discussed in this article are goal-oriented spatial–dynamical physical actions performed by the perceivers themselves or their collaborator in their attempts to solve a manipulation problem. Such physical actions, for example manually grouping objects into a set, can be designed to bear the pedagogical potential of grounding socially valued cultural notions, such as the arithmetic operation of adding. One may thus learn mathematical content via developing visualizations of actions. Moreover, one may learn by reflecting on different visualizations of an ambiguous action.[1]

1.3 Ambiguity as a condition and catalyst of learning

Researchers of didactical interactions have noted the favorable contribution—even discursive necessity—of ambiguity, looseness, or vagueness in interlocutors' indexical reference (Foster 2011; Mamolo 2010; Newman et al. 1989; Rowland 1999; Sfard 2002). In some sense, instructional communication is the discursive activity of generating, determining, and removing ambiguity (Isaacs and Clark 1987). Applying theory to practice, Abrahamson and Wilensky (2007) engineered ambiguous artifacts whose subtly alternating visualizations engender for the student productive semiotic conflict between conceptually complementary mathematical notions. Unlike Jastrow's classical duck-*or*-rabbit figure, these ambiguous objects were thus polysemous.

1.4 Polysemy

Polysemy is the quality of a sign having multiple meanings. Fillmore and Atkins (2000) develop this construct within the discipline of cognitive linguistics as it pertains to the phenomenon of contextual variation in word sense. Unlike ambiguity per se, polysemy requires that *the various senses of a sign share a common core and that linking these senses contributes to forming a network of meanings.* We are interested in exploring the conjecture that, under appropriate facilitation, *latent polysemy of appropriately designed perceptual stimuli may contribute to forming a network of mathematically coherent meanings*. If it turns out that indeed students are able to coordinate different senses of a polysemous object and, in so doing, generate new mathematical meaning, how might we model this cognitive feat theoretically?

2 Methods

This study analyzes selections from empirical data collected in the context of a design-based research (DBR) project investigating the emergence of mathematical understanding from guided perceptuomotor interaction. DBR is an investigative approach into cognition and instruction wherein learning theory and instructional materials are co-developed simultaneously, reciprocally, and iteratively, all driven by conjectures pertaining to some latent didactical potentiality (Confrey 2005). Reporting on findings from the project's first study, this article presents *generative case studies*, small-scale explorations suitable for pioneering the complexity of new pedagogical terrain, such as with evolving technological media (Clement 2000). Whereas findings from these studies are delimited in their generalizability, they can potentially inform future experimental designs that attempt to replicate, control for, and hone observed behaviors. As such, generative case studies may induce scholars at least to consider the plausibility of new ways of conceptualizing design, teaching, and/or learning. This section furnishes as much details as relevant to contextualizing the circumstances of the case studies reported in Sect. 3.

2.1 Participants

Participants were 22 volunteering 9- to 11-years-old (Grades 4–6) students from a private K-8 suburban US school (33 % on financial aid; 10 % minority students). The school did not have an advanced mathematics curriculum, so that the students' formal exposure to mathematical content was on par with public schools: Students in Grade 4 and 5 had not studied ratio and proportion at all, whereas students in Grade 6 had had minimal exposure to the contents. Yet whereas the set of individual interviews spanned the entire school year, we did not witness across the students any progress in their proportional reasoning or use of vocabulary. In fact, never did a single student mention spontaneously the words "ratio" or "proportion," and not once did they perform arithmetic procedures that might have suggested an application of that content. As generic as the design was, they were not able to use any proportionality content to model the situation. For the purposes of this study, we therefore did not disqualify any students from our pool of volunteer participants and considered all participants to be novices to the deep meanings of the target content (see also results from pre-tests, below).

2.2 Materials: rationale and build of the mathematical imagery trainer for proportionality

Our design problem was students' poor understanding of rational numbers (Lamon 2007). A chronic symptom of this problem is naïve application of additive forms of reasoning to what in effect are multiplicative systems. For example, students looking at the sequence "1/2, 2/4, 3/6" find it difficult to accept that these three sign-compounds represent equivalent quantities, because they attend to the

[1] Gestures, too, are dynamical manual actions perceived in the visual modality and bearing semiotic content, yet we will be focusing on pragmatic manual actions operating on objects in the environment and affecting their properties.

different internal differences between the numerator and denominator—1, 2, and 3, respectively—yet cannot integrate these into a visualization of equivalence.

Building on the embodied/enactive approach to mathematical cognition and instruction (Lakoff and Núñez 2000; Pirie and Kieren 1994), we assume that some mathematical concepts are difficult to learn because everyday experiences do not occasion opportunities to embody and rehearse the body-based dynamical schemes underlying those specific concepts. Our response is to augment student experience via engaging them in activities with carefully designed objects, a pedagogical technique harking back to the work of education luminaries such as Rousseau, Froebel, Montessori, Diénès, and Gattegno. Specifically, we conjectured that students' naïve application of additive reasoning implicates absence of multimodal dynamical images as personal meanings for proportion-related signs. In line with Radford (2003), we further assumed that learners objectify presymbolic meanings via appropriating available semiotic–cultural forms as ad hoc means of enactment, discourse, and reasoning. We were thus charged with designing both an experience that evokes conceptually relevant meanings and a method of introducing relevant semiotic artifacts for objectifying these presymbolic meanings (Abrahamson 2012).

We decided to design an experience in which a changing physical interval, analogous to a numerical difference, is associated with an unchanging sensory stimulus, analogous to equivalent quotients (as in 1/2, 2/4, 3/6, etc.). Students would have to discover this association by controlling the size of an interval between two locations in space in an attempt to preserve some constant visual feedback. We further wanted students to draw on their multiplicative fluency in articulating this association in proto-proportional forms.

Our design solution is the Mathematical Imagery Trainer for Proportion (MIT-P; see Fig. 1). MITs are interactive technological devices supporting the discovery, enactment, and rehearsal of perceptuomotor operatory schemas that the designer identified as bearing semiotic potential for the visualization of targeted mathematical notions (Abrahamson et al. 2011). The MIT-P was engineered to foster presymbolic dynamical operatory strategies pertaining specifically to the mathematics of proportional equivalence.

Using remote-sensing technology, we engineered an interactive system that converts physical hand position (in real space) into virtual object position (on a computer screen). As the user moves the hands up and down in physical space, a virtual object "mirrors" each hand's position on a large screen directly opposite the hand. Similar to standard commercial games, the expectation is that new players quickly perceive the virtual objects as ready-to-hand functional extensions of their body—the hand-to-object coupling should recede from consciousness, so that the player may focus on the problem-solving activity (Clinton 2006).

The device registers the precise height of each hand above the desk and, employing a continuous function, computes the quotient of these heights. When these heights (e.g., 10 and 20 cm) match a ratio set on the interviewer's console (e.g., 1:2), the screen is green. If the user then raises her hands in front of the display at appropriate rates, and within an adjustable "tolerance zone" that accommodates for users' fine-motor skills, the screen will remain green; otherwise, such as if she maintains a fixed distance between her hands while raising them, the screen will gradually turn yellow and then red. Participants were tasked first to make the screen green and, once they had done so, to keep the screen green while moving their hands up and down.

2.3 Procedure

2.3.1 Pre-intervention assessment

Two pre-tests measured students' fluency with proportions. The test materials consisted of a set of cards with either a pair of numerals (in Test 1, see Fig. 2) or images of hot-air

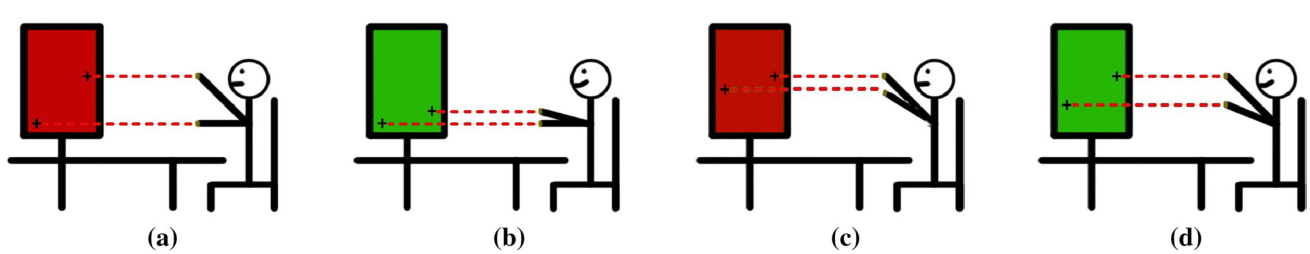


Fig. 1 The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 ratio, so that the favorable sensory stimulus (a *green background*) is activated only when the right hand is twice as high along the monitor as the *left hand*. This figure encapsulates the study participants' paradigmatic interaction sequence toward discovering the proportional operatory scheme: **a** while exploring, the student positions the hands incorrectly (*red* feedback); **b** stumbles on a correct position (*green*); **c** raises hands maintaining constant distance between them (*red*); and **d** corrects position (*green*). Compare **b** and **d** to note the different vertical intervals between the virtual objects (color figure online)

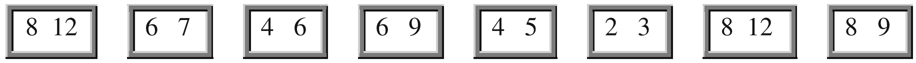


Fig. 2 Some of the cards used for the pre-intervention "numbers" task

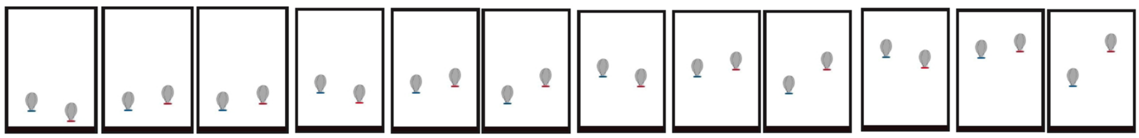


Fig. 3 Some of the cards used for the pre-intervention "balloons" task

balloons (in Test 2, see Fig. 3). Prior to placing the cards on the desk, the interviewer shuffled the cards and stated that some of the cards in the set could be selected and arranged so as to form a sequence of four cards that "makes sense" or "tells a story". The number pairs were such that both same-difference sequences (e.g., [2, 3] [4, 5] [6, 7] [8, 9]) and proportional sequences (e.g., [2, 3] [4, 6] [6, 9] [8, 12]) could be formed, and the hot-air balloon cards were analogously either of same-distance or proportional sequences. After students created a set of cards, we encouraged them to consider another set.

All students used the cards to create either same-difference/distance sets or some idiosyncratic set. Four students created also a proportional set; however, they explained their set as two parallel "count by" sequences (e.g., [2, 3] [4, 6] [6, 9] was "2, 4, 6" alongside "3, 6, 9").[2] Moreover, not once throughout all our interactions with the students both in the pre-test and throughout the intervention did any of them initiate the use of vocabulary related to advanced multiplicative concepts, such as the words "ratio," "rate," or "proportion." Thus even if these students had ever studied proportionality, for example outside of the school curriculum, this knowledge was inert—it was not in the form of actionable schemes for making sense of new phenomena (see Bereiter and Scardamalia 1985).

2.3.2 *Intervention*

Immediately following the pre-test, students participated in an explorative, task-based, semi-structured tutorial clinical interview (Goldin 2000). This form of interview was initially developed by Piaget. Working from a prepared protocol (see Appendix A online at http://tinyurl.com/zdm-viz), the interviewer asks the children to solve problems involving generic objects. The children are asked to manipulate these objects in an attempt to effect some goal state that the interviewer describes, such as an analysis, synthesis, or copy of these objects, or to predict the effect of changing one property of a situation on another property, such as to predict whether changing the spatial configuration of a set of elements changes the set's cardinality. Interviewers will often ask children to explain their reasoning, either as they perform these actions or immediately after. Interviewers might also ask a few follow-up questions, both from the protocol and extemporized questions, to ensure the validity of the findings as well as probe for new directions of research. The rationale for this methodology is that the researchers conceptualize participants' behaviors and utterances as making manifest their conceptual knowledge, including their implicit notions and beliefs. Through repeated observations with multiple students as well as careful post-session analyses of videography collected during these interviews, the researchers gradually begin to discern and articulate dimensions of variation and gradations of competence among the participants vis-à-vis the task specifications. These observations, in turn, lead researchers to hypothesize models of cognition that would account for apparent patterns in cross-sectional developmental trajectories.

Our interviews are longer than classical Piagetian interviews, because we wish to evaluate pedagogical activities that might become experimental classroom units, which are more extended and elaborate than the mini-tasks that cognitive-developmental psychologists have classically used in their laboratory studies. Our interviews could thus be conceptualized as a sequence of short interviews, with the child's new competence developed through each section then becoming their entering knowledge or skill for the next section. Analyses therefore take into account that knowledge may emerge through the actions of measuring it.

Thus, given our interest in education and not just learning, we conceptualize the role of interviewers as not only parts of the experimental design—a would-be disinterested instrument that only measures the phenomenon—but inherently as agents who change the phenomenon even as they are observing it. This conceptualization of the interviewer as both tutor and researcher aligns with the sociocultural perspective on the irreducibility of educational events to the behaviors of the student alone: learning

[2] These particular students are not featured in the three case studies.

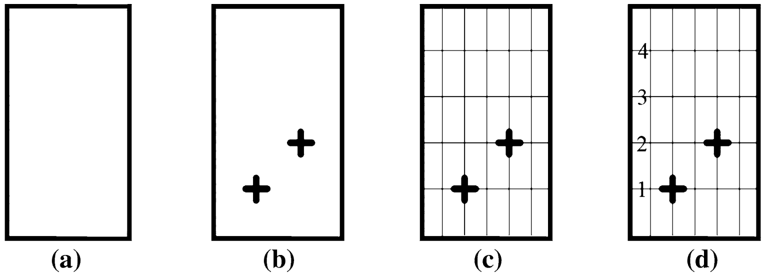


Fig. 4 MIT-P display configuration schematics, beginning with **a** a blank screen, and then featuring a set of symbolic objects incrementally overlain by the facilitator onto the display: **b** cursors; **c** a grid; and **d** numerals along the *y*-axis of the grid. These schematics are not drawn to scale, and the actual device enables flexible calibrations of the grid, numerals, and "correct" ratio

is *per force* a teacher-and-student co-enactment of a cultural practice: any elicitation of understanding necessarily affects the understanding, and so any observation on learning is also an observation on teaching (Newman, Griffin, and Cole 1989). This theoretical position on the nature of pedagogy demands a hermeneutic treatment of the empirical data gathered during the sessions: we make sense of participants' phenomenology and reasoning in light of their presumed interpretation of the discursive content "modulo" the tutors' implicit pragmatic and socio-epistemic framing of the task-based interaction (Barwell, 2009; Guba and Lincoln 1982). At the same time, we are also very much interested in understanding the tutors' professional practice, and in particular how they foster mathematical discovery (see Abrahamson et al. 2012).

Researchers enter interviews with prepared protocols, and yet their in situ responsive follow-up probes may be critically informative of student reasoning (Ginsburg 1997). One such new probe that emerged in the course of conducting our set of interviews proved so informative that we supplemented it into the protocol itself as the last item and implemented it with the remaining 18 participants. This probe prompted the participants to compare among alternative visualizations of the polysemous action. For this study, we focused on data gathered during those latter interviews. The interview lasted approximately 70 min, and students worked either as individuals or in collaborating pairs.

We used the following activity sequence. At first, a 1:2 ratio was set as the condition for green, and no feedback other than the background color was given (see Fig. 4a). Soon after, cursors were introduced on the screen that "mirrored" the participants' hands (see Fig. 4b). Next, a grid was overlaid on the display monitor as a frame of reference for the participants to quantify aspects of their effective interaction strategies (see Fig. 4c). In time, numerals were overlaid along the grid's vertical axis to highlight latent quantitative relations (see Fig. 4d). In addition to the 1:2 ratio, we worked with students on 1:3 and 2:3 ratios. This article treats the 1:2 data only, where students made their initial discoveries and were not constrained by their limited fluency with rational-number arithmetic, such as modeling non-integer multiplicative relations.[3]

In general, participating students were seated in front of the technology between a lead researcher and an apprentice researcher. All interviews were videotaped, and additional data included the researchers' field notes.[4] For most of the interview, students manipulated both cursors, but occasionally one of the researchers took control of one or two of the devices so as to guide the investigation. Regarding the specific question of how students came to coordinate multiple visualizations, we detail below results of our *post facto* analysis of tutor prompts. The following breakdown goes beyond the interview protocol per se, in that it draws from actual events as they enfolded in semi-spontaneous conversation between the tutor and participant during the empirical phase of the study that enacted the protocol.

For the most, the tutor reminded the participants what they themselves had said and done earlier in the interview, at times highlighting the fact that the participants had used more than a single strategy. More specifically, the tutor did one of the following:

- Recounted two strategies that the participant had previously articulated (3 cases)
- Presented two strategies as potentially different/similar/related (7 cases)
- Suggested to explore whether one strategy could be used to describe another (1 case)

[3] Another resource was an interactive ratio table. When it is layered onto the screen, students effect green by typing numerals into the tables' cells. This resource is not relevant to the article and so will not be treated further.

[4] In Abrahamson et al. (2012), we present an elaborate table of the interviewers' tutorial tactics that we determined *post facto* from analyzing the videography. A list of tutorial tactics is different from an interview protocol, because it describes an instructor's domain-general dialogical mechanics in service of implementing educational interaction.

- Presented one strategy and then asked whether there was another (1 case)
- Spoke about an earlier strategy in a way that highlighted a new feature (1 case)
- Asked a general question that did not allude directly to any strategy (2 cases)
- Did not refer at all to the participants' strategies (3 cases)

2.4 Data analysis

Our chief methodological orientation toward empirical data is collaborative, intensive micro-ethnographic analyses of multimodal behaviors observed in videographed interactions among students and instructors (Nemirovsky 2011; Schoenfeld et al. 1991). Using grounded-theory techniques (Strauss and Corbin 1990), these iterative, systematic analyses of the entire data corpus gradually give rise to the development of new constructs germane to the research questions. Appendix B (see online at http://tinyurl.com/zdm-viz) provides the coding scheme we developed and used for identifying students' interaction strategies. Typically, our investigative and expository efforts converge on brief excerpts from this videography that we agree upon as paradigmatically illuminating of the constructs.

This study focused on interactions that occurred when students had already enacted both "additive" and "multiplicative" solutions (see next section) and were then asked to reflect on relations *between* these solutions. Yet because these different solutions were enacted in essentially the same dynamical hand gestures, we were in effect asking the students to reconcile competing visualizations of a polysemous bimanual action. How might students accomplish this reconciliation? How might this activity contribute to learning mathematical content?

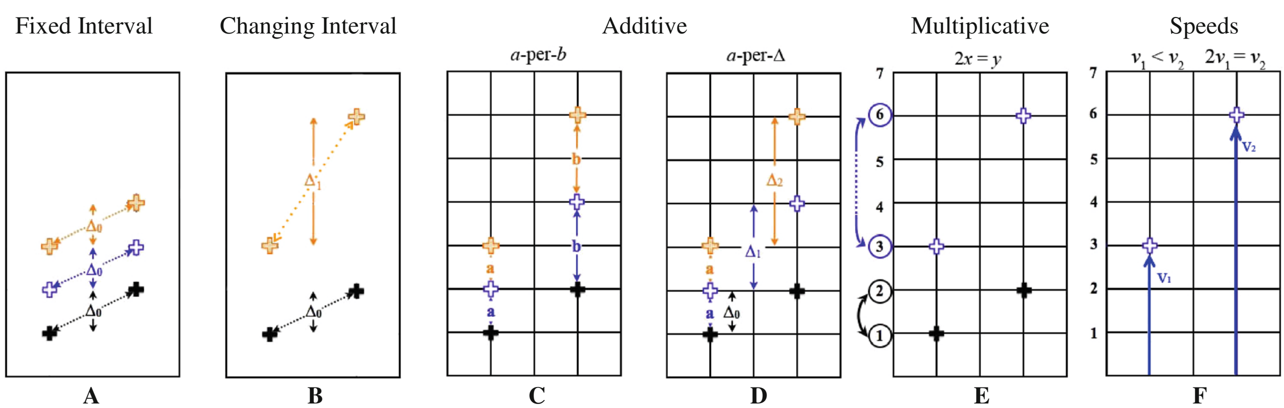
3 Findings: students coordinate additive and multiplicative visualizations of a polysemous action

In this section, we present and analyze vignettes from a set of videographed tutorial interviews with the objective of elaborating and evaluating the following thesis: *A polysemous physical solution is a potentially useful resource for mathematical didactics, because reconciling its disparate visualizations requires integrating conceptually complementary meanings.*

First, we describe students' solution strategies for the make-the-screen-green problem. These strategies varied in their visualizations of the bimanual solution action. Next, we contextualize and elaborate on these strategies by presenting vignettes from three paradigmatic cases of students who reasoned about relations among various strategies and, in so doing, generated new mathematical meanings.

3.1 Students' solution strategies

Figure 5 offers schematic representations of the main solution strategies we observed for the MIT-P problem across all the participants. For illustrative clarity, all the diagrams depict solutions for the same ratio, 1:2, that is, the right cursor (RC) needs to be twice as high along the monitor as compared to the left cursor (LC). Whereas these



Note on abbreviations: LC = left-hand cursor; RC = right-hand cursor; Δ = magnitude of interval between cursors (vertical and diagonal variants); v = velocity

Fig. 5 Student-generated solution strategies for the make-the-screen-green problem (the case of a 1:2 ratio): **A** fixed interval—maintaining Δ constant regardless of RC-and-LC elevation (incorrect solution); **B** changing interval—modifying Δ correlative to RC-and-LC elevation; additive, either **C** co-iterated composite units—both LC and RC either ascend or descend at respective constant values *a* and *b* (*a*-per-*b*), or **D** LC rises by *a* (usually 1), RC by 1 box more than the previous Δ; **E** multiplicative—relocating to a next "green" position as a function of the height of only one of the cursors (given LC at *x* and RC at *y*, $2x = y; x = ½ y$), e.g., determining LC *y*-axis value, then doubling to find RC, or determining RC value, then halving for LC; and **F** speeds—LC and RC ascend/descend at different constant velocities ($v_1 < v_2$) or RC velocity is double LC velocity ($2v_1 = v_2;\ v_1 = ½\ v_2$)

Tab. 1 Frequency of solution strategy according to grade level

Grade	Fixed interval	Changing interval	*a*-per-*b*	*a*-per-Δ	Multiplicative	Speeds
4	1	1	2	–	3	2
5	2	10	6	6	9	6
6	2	3	5	1	5	5
Totals	5	14	13	7	17	13

strategies are phenomenologically distinct, they are functionally equivalent and mathematically commensurate. That is, the subjective experience of performing varies across the strategies, and yet they bear similar effects and could be derived from each other through a chain of inferences.

Initially, the participants attempted to effect a green screen by moving their arms about, with some students waving one arm at a time, some students waving both arms uniformly up and down, some cautiously exploring only at the bottom of the screen, and others exploring the entire vertical extent. Eventually, all students chanced upon a "green" position of the hands. Prompted, they next sought and found another such position. Concurrent with these first successes, the students realized they should pay attention not to each hand's individual location along the screen, but to the relation *between* the hands.

All participants articulated a strategy relating the cursors' position (how high the pair is elevated) and interval (the distance between them, hence "Δ"). Initially hypothesizing that Δ should remain constant ("Fixed Interval"; see Fig. 5a), ultimately they inferred that Δ should vary with elevation ("Changing Interval," see Fig. 5b). This inference is of central interest to us, because it could potentially ground the meaning of proportional equivalence (e.g., 1:2 = 2:4), in which within-ratio differences vary across the equation (c.f. 1:2 = 2:3). Table 1 provides a breakdown of the strategy occurrence frequencies according to grade level. We will now overview each of the strategies themselves.

3.1.1 Fixed interval

Initially, students articulated a strategy relating the hands' elevation above the baseline and Δ. In particular, once students had found an initial "green spot" and were prompted to find another, they all believed that Δ should remain constant as they move. A typical statement a student made was, "I think it's just, like, you stay the same distance apart"—a fixed-interval strategy. However, students eventually inferred from the feedback that this strategy does not constitute a valid solution.

3.1.2 Changing interval

Students inferred that Δ should vary correlative with the pair's height above the baseline. They stated, for example, "The higher you go, the bigger the distance".

Overlaying a grid onto the screen caused students to revisualize the cursors' positional properties. Incorporating this new frame of reference into their operatory schemas, they shifted their attention from Δ per se to construing two topical cursor locations. These revisualizations were concomitant with determining a strategy for recursively relocating the hands/crosshairs, as follows.

3.1.3 a-per-b

Building on Cobb and Steffe (1998), we name this proto-ratio strategy "*Co*-iterated Composite Units". This strategy involves moving the hands/cursors by their respective constant rates. Students state, for example, "For every one I go up on the left, I go up two on the right". This manipulation may be enacted sequentially so as to facilitate accurate execution.

3.1.4 a-per-Δ

Another recursive strategy we observed is, more accurately, "*a*-per-(Δ ± 1)". Here, too, the left hand paces iteratively by a constant (composite) unit, yet the right hand moves in relation to the previous *interval*. For example, students stated that "the distance grows by one every time". By necessity, this strategy requires sequential rather than simultaneous hand motions, because the new left-hand location must be established prior to determining the new right-hand location relative to it.

Overlaying numerals onto the grid caused students both to construe cursor locations as heights above the baseline and to recruit their arithmetical knowledge. As a result, they yet again changed their strategy, as follows.

3.1.5 Multiplicative (2x = y, x = ½ y)

This strategy is non-recursive, that is, one need not attend to previous pair locations so as to determine new locations: Given one cursor's numerical position index (e.g., LC at "2"), the other cursor's "green" position is determined computationally (e.g., by doubling to "4"). Using multiplication, one student stated, "[you] double the number that the left one is on, and you put the right one on that number"); using self-adding, another student explained, "One plus one is two, two plus two is four".

3.1.6 Speeds

Finally, some students described a strategy whereby the left and right hands move simultaneously, up or down, each with a different respective constant velocity. Some analogized their hands to moving vehicles, whereby each hand moves at a constant speed, v_1 and v_2, respectively, stating that the "right hand moves faster than the left," or that "Like this [LC] one's going 20, and this one's [RC] going 50. And they have to keep on going…" ($v_1 < v_2$). Other students noted that RC moves at double the speed of LC ($2v_1 = v_2$) or vice versa ($v_1 = ½\ v_2$)

Note that when students shifted from one effective strategy to another, there was no indication that they reasoned about these strategies as similar, neither via logico-mathematical commensurability (i.e., that they are inter-derivative) nor transitive equivalence (via a common effect). At least, they never articulated any chain of inferences from one strategy to the next. Granted, when the symbolic artifact (the grid and then the numerals) was introduced onto the workspace, students identified in it utilities for enacting, explaining, or evaluating their existing strategy; yet in the course of instrumentalizing these new affordances, the students created a new strategy that appeared to preserve little of the previous strategy (e.g., compare changing interval and *a*-per-*b*). The students did not look back to contemplate this emergent transition—they ceased to employ the old strategy, because they judged the new strategy as more efficacious (Abrahamson et al. 2011). This behavior should be expected, because the researcher did not pose for them this reflective task. Only later in the interview did we ask the students explicitly to reflect on relations among their strategies.

All in all, 8 out of the total 18 participants engaged in coordinating conceptually complementary visualizations either spontaneously or in response to the interviewers' explicit request, 7 were either focused on a single strategy or did not articulate a coherent coordination, and 3 were not asked to coordinate due to logistical challenges, such as time constraints or technical difficulties with the experimental technology.

When students coordinated among their different interaction strategies, they did so for the most in relation to the 1:2 ratio. In addition to that, we witnessed 12 occasions of students coordinating strategies for other ratios. We coded seven of these as unsupported by the interviewer and five as supported. Given that not all interviews arrived at these ratios, we view these results as promising.

We now turn to present vignettes from three cases of students responding to this protocol item.

3.2 Coordinating visualizations generates new meaning: the roles of polysemous association and heuristic inference

The following three sections are presented as portraying paradigmatic cases of students coordinating conceptually complementary visualizations of perceptual elements in a problem space. These coordinations, our analyses will suggest, sprouted as heuristic causal inferences reconciling semiotic conflict and, in so doing, generated new meanings relevant to the conceptual domain. In each of the following case studies, we offer a synopsis of events leading up to the focal episode. The episodes themselves are transcribed, with relevant gestures annotated to clarify utterances.[5]

3.2.1 Visualizing a pair of iterated composite units multiplicatively

We begin with the case of a student who coordinated between the *a*-per-*b* (1 per 2) and $y = 2x$ strategies by realizing that the within-pair multiplicative relations in *a*-and-*b* and *x*-and-*y* were identical, that is, $b/a = y/x$. She first noticed the constant multiplicative relation (double) across all [x y] "green" pairs she had found. She then visualized the pair of composite units a and b itself as bearing the same relation (double).

Shani is a female Grade 5 student identified by her teachers as low achieving. During the pre-grid interview phase, Shani attended to the diagonal, not vertical, interval between the cursors. Perhaps for this reason, she determined the Changing Interval strategy later than other students, only once both the grid and numerals had been overlaid and she had recalibrated to the vertical interval. Shani then said, "They're getting farther apart as it goes up".

Dor, the interviewer, took control of LC. He raised it one unit at a time, with Shani controlling RC in an attempt to make the screen green. Shani's attention vacillated between RC and Δ. She discovered both *a*-per-*b* and *a*-per-Δ and appeared to waver between their corresponding visualizations.

Dor handed LC back to Shani. The transcription below begins at the point when Dor asked Shani to summarize the sequence of "green" value pairs. Dor implicitly suggested a particular syntactical form and vocal cadence for this summary ("1 and 2, …"). In response, Shani physically reenacted the sequence on the screen, using *a*-per-*b*. While doing so, she read off the numerical values of each successive "green" pair. As we will see in the transcription below, Shani suddenly became aware of the constant within-pair multiplicative relation for making green.

[5] See http://tinyurl.com/zdm-viz for video clips from the vignettes.

Consequently, she shifted from the additive strategies a-per-b and a-per-Δ to a multiplicative strategy $y = 2x$. She explained this new strategy in terms of the multiplicative relation between a and b.

Dor: So, what else can you say about those numbers? 1 and 2,...
Shani: 1 and 2,… then 2 and 4,… 3 and 6. Hey wait. Um, oh, it's… [fidgets body, becomes animated] It's all doubles. The bottom number, like time… times two is the top number [she points at the monitor.] We had, like, 1 and 2, then 3 and 6, then, um, then 4 and 8, then 5 and 10.
Dor: Interesting. Huh. So you've said...
Shani: Because they're each going…'cause this [RC]… this one's always going up by two, and this [LC] one's going up by one, which would mean that.
Dor: Which would mean that *what*?
Shani: That, uhm, this one [RC] is always double this [LC].

Thus, Shani coordinated additive and multiplicative visualizations of the rising-hands polysemous action by inventing a multiplicative visualization of its additive elements. This coordination was mediated by the emerging semiotic and inferential function of the numerical values she uttered (e.g., "1 and 2"). At first, the numerical values were mere appellations ("read outs") indexing the successive paired locations on the grid she had found to satisfy the task objective (green screen). Yet then ("Hey, wait"), applying arithmetic fluency and inductive reasoning, each and all of the paired values suddenly became associated via an invariant internal relation apparently indicative of a general pattern (i.e., "It's all doubles"). In particular, Shani revisualized the iterated composite units a and b of the a-per-b strategy as related to each other multiplicatively. Shani thus inferred causal relations between additive and multiplicative solution strategies by alternating between two visualizations of paired elements in the perceptual field: as iterated cumulative motions along parallel linear axes or as running totals of these added values.

Soon after, the grid and numerals were removed, and Shani was challenged to enact green again. She found this difficult. As we see below, Dor then physically guided Shani's hands in performing a continuous and dynamical enactment of green. While her hands ascended, Shani noticed that RC was moving faster than LC.

Dor: I want to see if we can—if I can track your hands all the way up in green. I'm going to find a green place… [Dor holds Shani's wrists and raises them].
Shani: So this one should be… So my right should be moving faster
Dor: Oh, I see
Shani: So that it can make… be going up two spaces on the grid,…. while the other one is only going up one
Dor: Oh, I see

Thus, Shani linked the $2v_1 = v_2$ and a-per-b visualizations, implicitly interpreting speed as rate.

3.2.2 Visualizing a changing interval multiplicatively

Next, we present the case of a student who coordinated two effective strategies—Changing Interval and Multiplicative—by inventing a multiplicative visualization of the changing interval (Δ).

Liat is a Grade 6 female participant identified by her teacher as middle achieving. With the cursors visible on the screen, Liat initially discovered that to make green she must position RC higher up along the screen as compared to LC. Only once the grid was introduced did Liat articulate a Changing Interval strategy: "If it's farther up, then they have to be more apart". Liat then manipulated the cursors up and down the screen accordingly, creating a continuous green feedback. After introducing the numerals onto the screen, a researcher probed Liat's reasoning by holding RC level at some value and asking Liat to predict the corresponding "green" location of LC. Initially, Liat's predictions were approximate. For instance, when RC was held at 10, Liat predicted that LC should be "a little bit higher than 6". After some further structured exploration, Liat initiated the multiplicative strategy "half of the number".

Next, Liat operated both cursors. Below is a transcription of the episode in which she discovered a logical relation between Changing Interval and Multiplicative. As we shall see, this coordination hinged on visualizing Δ relative to RC as "half the number".

Dor: So when this one [LC] is at "1", where will that one [RC] be?
Liat: This [LC] is at "1" [raises LC to the horizontal Gridline 1], and this [RC] would be at "2", I think. [raises RC to horizontal Gridline 2; screen turns green]
Dor: Ok. Now, if you wanted to go up—
Liat: Oh, then it would… and then it gets… then it gets farther apart, because the number gets bigger.
Dor: Which number gets bigger?
Liat: The number—well. Hmmm. 'Cause it's farther apart, every time it goes—the… the… the right nu… the number that the right is on—it gets higher, so they have to be farther apart, because half of it is bigger than the number before it.

Liat's inferential reasoning process appears to be as following. Liat: (a) adopted the interviewer's inquiry method of raising a cursor one unit at a time and then finding the corresponding "green" location of the other cursor; (b) knew that the LC should be half as high as RC; and (c) inferred that therefore Δ—which at 1:2 is equivalent to the height of LC—should change correlative to the height of RC. That is, Δ = ½ RC, so Δ grows with RC.

The interviewer, for whom this inferential reasoning was new, still required clarification.

Dor: Half of it is bigger than the number before it? …
Liat: Because if this one [RC] is at 4, then this one [LC] has to be at 2, because it's half, and then if I were to put the right one on 5, then…and I would have to…I'd have to put it [LC] on 3-and-a-half… or 2-and-a-half, and then… So it gets farther apart, because there is a bigger difference between the numbers.
Dor: Ah, so the one on the right went up, like, went up one?
Liat: Yeah, so there is a bigger space between the left and the right.
Dor: Huh, is that something that keeps on happening? Or not? Oh. Uh huh...
Liat: So I go to 6 [RC], and then to 3 [LC]. So there is a bigger space between the right and the left one.

Similar to Shani, Liat coordinated additive and multiplicative visualizations of the rising-hands polysemous action by inventing a multiplicative visualization of its additive elements. Namely, she revisualized the spatial interval between the hands (Δ) as half the total height of the right hand ($x = ½ y$) and perceived this changing difference as an *a/b* fraction of the increasing total. Also similar to Shani, Liat built her reasoning as a causal argument, explaining the non-multiplicative as resulting from the multiplicative.

3.2.3 *Integrating complementary meanings of a polysemous object*

Finally, we discuss an excerpt from a paired-student interview, in which a "high achiever" 6th-grade dyad coordinated the ½-per-1, $x = ½ y$, and $2v_1 = v_2$ strategies via agile visualization shifts among different partial meanings of the mathematical object "half". "Half" is a polysemous sign whose meanings include measure (e.g., half a unit) and fraction-as-operator (e.g., half as much of/as something). The dyad's achievement, below, is in implementing this polysemy as alternative visualizations of the *a*-per-*b* action, and in so doing they demonstrated flexible conceptualization of the Whole.

Working with the grid under the 1:2 ratio setting, the students measured Δ at various "green" locations, with Eden operating LC and Uri operating RC. They soon agreed that "the higher you go, the more boxes it is apart," the Changing Interval strategy. They next shifted to the *a*-per-*b* strategy. In particular, they co-discovered that to maintain green, they should progress hand-per-hand at vertical intervals of ½ (Eden) and 1 (Uri), both either ascending or descending the screen. (Note that ½-per-1 is proportionate to 1-per-2.) As Eden concluded, "For every box he [Uri, RC] goes up—you [Eden, LC] have to go up half". Similarly, once the *y*-axis numerals were overlaid on the display, the students shifted to the Multiplicative strategy "halving", that is, LC should be placed by the numerical value that is half as much as the RC value.

The transcription that follows documents the dialog immediately after Dor had asked Eden and Uri whether their ½-per-1 and $x = ½ y$ strategies are related.

Eden: Uh, well, I think they basically mean the same thing, because if I go up one half [LC] and he goes up one [RC], it's the same thing as he being up twice as much as me.
Uri: Yeah
Eden: So if he's up at 10 and I'm up at 5, I still move up half as much as him

Eden thus switches adroitly between half-as-measure, where the Whole is a grid unit of 1, and half-as-operator, where the Whole is an ad hoc unit of 10. The aligned image structure of "half" across these visualizations appears to serve Eden as an inferential hinge for evaluating these proportional relationships as similar. As in the cases of Shani and Liat, the dyad integrated polysemous visualizations of the solution strategy by revisualizing additive elements multiplicatively. The exchange then ensued, with "half" and its reciprocal "double" hinging further.

Dor: Ah, I see. So there're two ways of speaking about this idea of "half"
Eden: So you can say I move up this much or he moves up that much. It's like I move up half or he moves up twice as much as me. Maybe he moves up two times faster.

Thus, Eden generated and integrated yet more meanings for the polysemous bimanual action—the notion of two velocities related multiplicatively.

4 Conclusion

Paraphrasing an old adage, we submit that coordinated meanings are greater than the sum of their parts. In particular, coordinating competing visualizations of polysemous action bears greater semiotic potential than each visualization per se. We have demonstrated that linking up conceptually complementary visualizations can create opportunities to build more robust and integrated understandings of content, such as grounding multiplicative conceptualizations of proportional equivalence in iterated additive actions.

To coordinate competing visualizations of a perceptual field, a learner draws concurrently both on "bottom up" and "top down" resources. The learner: (a) aligns the visualizations structurally (cf. Markman and Gentner 1993) by selecting a single case in point and applying both visualizations to this case; yet in so doing (b) imports relevant domain knowledge to constrain the selection of candidate cases. The learner then reconciles the competing meanings via heuristic logico-mathematical inferential reasoning (see also Fauconnier and Turner 2002, on cognitive mechanisms supporting conceptual blending).

The validity of these conclusions as well as the following implications is limited by the explorative nature of our experimental design. However, we hope to have suggested useful directions for future research on visualization in mathematical learning.

4.1 Implications for theory of learning: conceptual roots in informal grounding

The students' coordinations were onset by powerful yet loose inferential reasoning. Whereas these insights appear to be useful for content learning, important conceptual work still lies ahead to formalize their argumentation.

Analyzing Shani's insight, for example, one is liable to gloss over the mathematical tenuousness of her inference. Why, precisely, do successive *a*-per-*b* actions "mean" that all numerical values relate by a constant *b/a* relation? Just because the co-iterated *a* and *b* composite units *themselves* relate by *b/a*, does that imply that their respective running totals relate likewise? The reader is invited to sketch a convincing argument from Shani's premise to her inference. Can one do so without resorting to the distributive property of multiplication over addition? Is not that property a critical inferential link that is absent in Shani's reasoning or, at least, in her verbalization? What are the implications of this tenuous inference for instruction?

Granted, as constructivist designers we are the first to laud these students' achievement. At the same time, as educational researchers we are intrigued by the nature of their inferential reasoning. Whereas we do not have evidence to support the claim that this reasoning endured or transferred beyond the interview session, we believe that these efforts reflect precisely the type of quantitative reasoning called for by progressive mathematics-education researchers (e.g., Arcavi 1994; Thompson 1993). For example, Shani's sense that particular iterated-adding actions implicate particular multiplicative relations could serve as a naïve yet productive epistemic substrate grounding the formal proof she is yet to learn.

4.2 Implications for theory of multiplicative reasoning: not so spindly

Mathematics-education researchers have long been debating optimal cognitive groundings for multiplicative concepts. Whereas mathematicians state axiomatically that 3 x 4 is simply $4 + 4 + 4$, and whereas some mathematics educators stipulate that instruction of multiplicative concepts should accordingly build on this iterated composite-unit model (Cobb and Steffe 1998), other scholars in the field seek alternative, non-additive entries into multiplicative concepts, such as splitting and folding. In particular, Jere Confrey—a champion of non-additive foundations to multiplicative concepts—has critiqued multiplicative models grounded in additive conceptualizations as creating "spindly networks of mathematical reasoning on ratio, rate, and later functions" (Confrey 1998, p. 40).

Based on our empirical findings in this study, we concur with Confrey that the additive-to-multiplicative coordination involves rickety heuristic leaps. At the same time, these leaps appear to offer children useful cognitive foundations for the content (Resnick 1992). Given appropriate didactical design, students can informally coordinate conceptually complementary visualizations of situated elements, which in turn enables them to ground multiplicative conceptualizations of proportionality in the more familiar additive forms. Our interaction design explicitly targeted students' naïve tendency to visualize number pairs additively, which has been the bane of rational-number instruction, by creating physical embodiments for these numbers and positioning the spatial interval between them as the critical object of manipulation. By later interpolating symbolic artifacts into the problem space, we enabled students to signify these bimanual actions multiplicatively. Then by yet later asking students to compare these multiplicative strategies to their initial additive strategies, we enabled the students to generate idiosyncratic conceptual bridges between additive and multiplicative foundations of proportion.

Across all our data, when students achieved the coordination of additive and multiplicative forms of reasoning about proportional progression, they did so by alternating between two visualizations of elements in the perceptuomotor field. Per Vergnaud (1983), these visualizations highlighted either the "scalar" axis, that is, the between-ratio dimension connecting *a:b* to their respective running totals *x:y,* or the "functional" axis, that is, the within-ratio dimension between *a* and *b*, as between *x* and *y*. The scalar visualization was additive, in the sense that each element *a* and *b* was iterated independently as measured against the grid lines, whereas the functional visualization was multiplicative, in the sense that the relation between *a* and *b* was expressed as a factor or quotient. The students in our study linked these visualizations heuristically.

We are fully aware that heuristics will not suffice for a formal knowledge of multiplicative concepts, and we have yet to furnish more evidence for the efficacy of this activity. Notwithstanding, we submit that spontaneous synoptic coordinations lend students a sense of grounding, which we view as an important didactical objective. We are thus heartened both by the students' reasoning and by this design that appears to create opportunities for students to ground multiplicative conceptualizations of proportionality in embodied additive strategies. In later years, students who participated in designs such as these should revisit this relation using the distributive property of multiplication over addition as expressed in algebraic forms.

4.3 Implications for designs for proportion

This study, which utilized a non-routine technology-based design for proportion, paves avenues for questioning previous pedagogical approaches to this content regardless of the media available to the classroom. When teachers wish for students to connect additive and multiplicative interpretations of proportionality, they usually begin by generating running totals via iterated addition, and then they help the students see how these tabulated running totals can be re-construed multiplicatively (Kaput and West 1994). The findings of this study suggest an alternative or complementary route. Students may be able to discern the multiplicative relations inherent to a proportional system even prior to tabulating running totals, by focusing on the composite units *themselves*, re-seeing their relation multiplicatively, and realizing inductively that this relation stretches across the emerging equivalence class of number pairs.

A proposal to accentuate *within*-ratio relations in the instruction of proportion is well aligned with phenomenological approaches to the didactics of mathematics education (e.g., Freudenthal 1983), because the essential meaning of a proportion is arguably in its functional, not scalar, dimension. For example, what is experienced as "the same" across a set of proportionally related quantity pairs is not so much the dynamical action of iterating the *a* and *b* composite units *from* one pair *to* the next as much as the ratio *between* the running totals *x* and *y*. These psychophysical experiences of constancy over changing quantity—a particular shade of green across volumes of blue and yellow paint, a particular flavor of a cocktail mix across glass sizes, a particular balance of orchestral sound across amplitudes of strings and woodwinds—enable learners to co-opt a sensation of identity as the epistemic grounds of proportion.

In our study, the Mathematical Imagery Trainer provided a technologically contrived proxy of this psychophysical experience of identity across change by creating a set of otherwise nondescript bimanual locations on a screen, all associated with the same, otherwise arbitrary sensory information, the color green. By signifying mathematically their strategies for maintaining this identity across change, students learned to describe the common property of the nondescript set, that is, the constant functional relation of ratio. Yet, key to fostering a coherent conceptual network was steering students to align and coordinate their own additive and multiplicative competing visualizations of the polysemous solution action.

As this study has demonstrated, one way of steering students to visualize additive elements multiplicatively is to "confront" them (Bamberger and Ziporyn 1991, p. 55) with the non-routine request to reflect on logical relations among their additive and multiplicative solution strategies for one and the same problem. These confrontations present values added for grounding multiplicative reasoning—they encourage the generation of new meanings that may serve to integrate and consolidate a conceptual network.

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