THE EMERGENCE OF PROPORTIONAL REASONING FROM EMBODIED INTERACTION WITH A TABLET APPLICATION: AN EYE-TRACKING STUDY

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Abstract

Embodied cognition is emerging as a promising approach in educational technology. This study is based on the conjecture that a potentially powerful methodology for researching embodied mathematics design would be to gather empirical data on children's shifting visual attention as they learn to operate the technological devices so as to solve the situated problems. The aim of the study is to gain insight into the role of visual attention in the emergence of new sensorimotor schemes underlying mathematical concepts, especially proportion tasks. The research question is: How does visual attention change in the emergence of sensorimotor schemes during proportional reasoning tasks? An exploratory study was conducted amongst eight 5th- and 6th-grade students (age 10-12 years). Based on the original Mathematical Imagery Trainer for Proportion (MIT-P) designs for Wii and iPad (Abrahamson), a new tablet app has been developed. While students completed the hands-on proportion tasks, a Tobii x2-30 and an external camera were used for real-time processing and storage of both video and gaze data, resulting in integrated videos of hand- and eye movements. Data analysis revealed cross-participant variation in exploration path, progress rate, and inferences. Yet across all participants insight coincided with a shift from: (a) random finger movements accompanied by gazing at salient figural contour; to (b) new bimanual coordinations accompanied by gazing at new non-salient figural features. The study thus supports classical constructivist claims that mathematical concepts are grounded in operatory schemes. In particular, the data literally show the dynamical emergence of attentional anchors for situated problem solving as mediating the development of mathematical concepts. Thus new research methodologies stand to validate claims from embodiment.

1 INTRODUCTION

Embodied cognition, or the idea that we think *with* and *through* our bodies, is emerging as a promising approach in educational technology [1, 2, 3]. According to this theory, human cognition rises from and for the body's interactions with the physical environment, and yet the same neural circuitry is engaged "offline," that is, when we think *without* external interaction [4, 5, 6]. There is compelling evidence from various research domains supporting the view that cognition is embodied [7]. This evidence includes findings from cognitive scientists' empirical research studies of mathematical reasoning [8,9,10], literacy [11, 12, 13], problem solving [14], inference [15], and interactions with artifacts [16].

Embodiment theory bears implications for education. Within the field of mathematics education, the guiding principle is that even the most abstract mathematical idea in fact is grounded in sensorimotor schemes. This principle had previously been made in cognitive linguistics [17], but had not been evaluated empirically. Pioneering empirical research on the embodiment of mathematical concepts was conducted by Nemirovsky and his collaborators [18,19,20]. They have designed various dynamical physical interactions with technological devices so as to introduce students to new ways of feeling, moving, inhabiting the world, and thinking. Students learned new notions via a process of sensorimotor integration that was mediated by the technology that they operated to solve situated interaction problems. These research efforts were further articulated into the framework of embodied design [21]. The claim is that we can create dedicated environments in which students learn new mathematical notions, even though the environments initially have no mathematical symbols at all. Rather, students first develop appropriate sensorimotor schemes (goal-oriented, situated ways of interacting with the environment), and only then they articulate those schemes within the discipline's conceptual field [22]. To elaborate, initially the students develop the new sensorimotor schemes through solving interaction problems, just as when infants learn to walk, children learn a new motor-action skill in sports, or adults develop any vocational capacity involving manipulation. Later, frames of

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reference from the discipline, such as a grid, are introduced into the environment. The students adopt those frames so as better to enact, explain, or evaluate their interaction, yet in so doing they shift into professional ways of seeing, moving, and talking [23]. This principle has been demonstrated in a variety of techno-scientific fields [24,25].

Despite significant recent progress in the design and evaluation of instructional methodology inspired by embodiment theory, numerous questions still remain open. For example, we do not know how exactly students first determine effective sensorimotor schemes for operating the various technological devices typically used in embodied-interaction learning environments. This study is based on the conjecture that a potentially powerful methodology for researching embodied mathematics design would be to gather empirical data on children's shifting visual attention as they learn to operate the technological devices so as to solve the situated problems. When children's visual attention is tracked and then these data are combined with automatic logging of children's physical actions with the technology, we may be able to form a better idea of how new sensorimotor schemes arise and how, later, they shift with the introduction of conventional symbolic frames of reference.

Building on the above-cited earlier efforts of Abrahamson and his collaborators at the Embodied Design Research Laboratory (http://edrl.berkeley.edu/), a Dutch research team decided to remain within EDRL's chosen mathematical subject matter of ratio and proportion. Proportionality, and more broadly all multiplicative concepts such as fractions, are an essential component of early curriculum, and yet they remain the failing point of many students [26]. Our joint current design-research project extends Abrahamson's work by focusing on new possibilities for both facilitating and evaluating embodied learning of proportion. Our technology is a variant and extension of Abrahamson's Mathematical Imagery Trainer for Proportion (MIT-P), and in particular a variant on his tablet implementation of the design that represented proportions as a set of two vertical projections [27,28,29]. One way in which we extended MIT-P was to include an option with orthogonal bars. Moreover, in our MIT-P tablet variant we added embedded action-logging, which could be combined with gaze data. These recent methods for gathering sensorimotor data would complement the more traditional audio–video data that capture the student's (and teacher's) multimodal utterances (see Blikstein et al. [30], for a review of multimodal learning analytics).

Abrahamson and Sánchez–García [31] have suggested that mathematical notions emerge from the student's conscious articulation of an *attentional anchor*, that is, the specification of a new feature or aspect of the environments that the student selects as preferred effective means of engaging and controlling the environment. Attentional anchors are used pervasively yet often subconsciously by experts in the multifarious physical practices, ranging from juggling to driving. For example, a juggler might imagine a geometrical structure hovering in the air to help coordinate her iterated actions through time and space. Or a table-tennis (ping pong) player might construct a triangle connecting three points in space so as to guide the effective motion of his bat, a technique that Liao and Masters [32] call a "biomechanical metaphor" because a geometrical form is overlaid on the perceived world.

There is apparent promise in using the construct of attentional anchor, which was first coined to describe the development of motor-action in sport [33] as a means of describing the development of motor-action underlying mathematical concepts, and this promise depends both on creating appropriate learning environments, such as the MIT-P, and analyzing learning process via the lens of embodiment theory. And yet whereas the construct of an attentional anchor has borne *analytic* appeal, so far no compelling *empirical* evidence has been put forth for the emergence of attentional anchors from sensorimotor action through to verbal articulation of a new mathematical notion. We are attempting to rectify this gap in the literature. The current study thus aims to gain insight into the role of visual attention in the emergence of new sensorimotor schemes underlying mathematical concepts. More specifically we ask the following research question: *How does visual attention change during the emergence of sensorimotor schemes for enacting proportional action tasks?*

This paper presents preliminary findings from the pilot implementation of the study. Whereas our claims are therefore as yet tentative, they may nevertheless stimulate productive discussions, elicit constructive feedback, and perhaps inspire similar research.

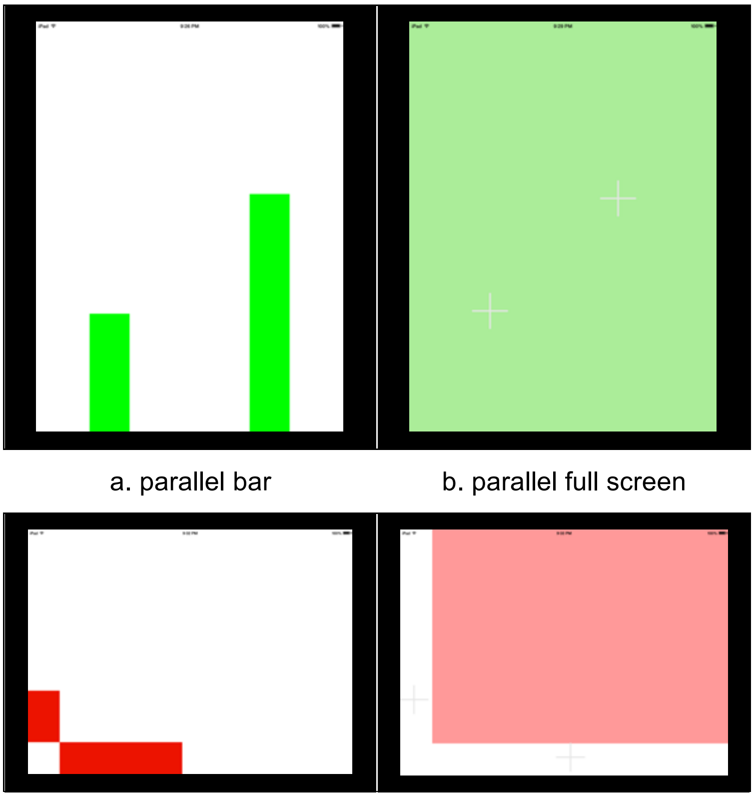
2 METHODOLOGY

2.1 Participants

The sample used in the current study consists of eight 5th- and 6th-grade students—3 female, 5 male—with age ranging from 10-12 years (*M=11 years, SD=7.48)*. All children attended primary schools in the Netherlands. Before approaching the schools, the ethical committee board of the faculty of Social Sciences at Utrecht University approved the study. Additionally, informed consent was obtained from all legal guardians before testing each student. Participation was voluntary, and no compensation was offered.

2.2 Materials and Instruments

Based on the original MIT-P design, an app was developed to implement the same task of learning proportion through finger movement on a touch-based device (see Fig. 1). The app at its current state is designed for Mac OS and can be installed on Apple iPads. Compared to the original, Utrecht MIT-P tablet has several variants, differing on the direction of the movement (parallel vs. orthogonal) as well as figural and color properties of the visual feedback (coloring full screen vs coloring the bars).

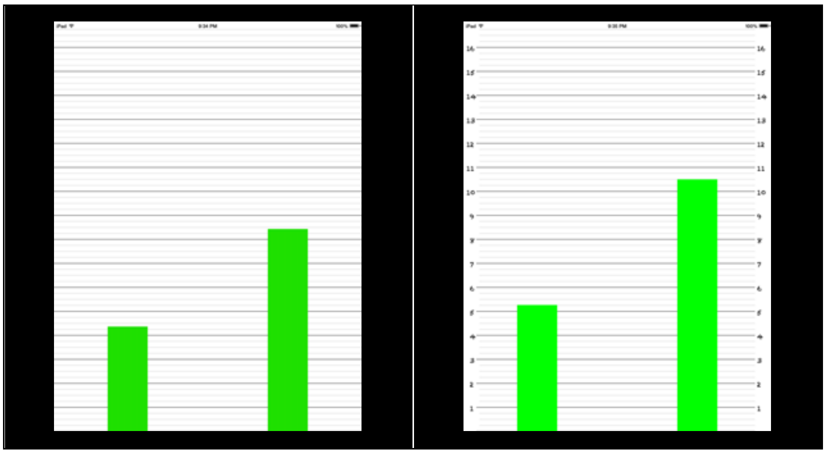


a. parallel bar b. parallel full screen

c. orthogonal bar d. orthogonal full screen

Fig.1 Sample screens from four modules of the Utrecht MIT-P touch-screen tablet.

In this study, a variant in parallel mode with colored bars is used. The program starts with a plain white screen. Moving the fingers in parallel motion may change the color of both bars along a gradient between green and red. The color turns green when the two bars are at the exact pre-set proportional height with respect to each other, and it turns red gradually as the proportion changes. To keep the bars green all the time while moving the fingers, one has to move the two fingers at a pace relative to the pre-set proportion. This is one way that the core knowledge of proportionality is eventually captured—by moving the fingers with a proportional pace and feeling the changing difference between the pace of the two fingers (see Abrahamson et al. [34], for a variety of strategies for proportoinal bimanual motion). As in the original MIT-P, the Utrecht MIT-P app is designed such that an experimenter can show a grid on the screen to draw the user's attention to the position of the bars on the screen, as well as numbers that appear next to the grid (see Fig. 2).



a. appearance of the grid b. appearance of the numerals along the grid's vertical axis

Fig. 2 Appearance of grids and numerals.

The eye-tracker used in the study is a Tobii x2-30 model. It is a portable, small device that can be used together with the Tobii Mobile Device Stand for X2, which is specially designed for studies conducted on tablets or smartphones. It also allows for installation of an external camera to capture the scene during the experiment and export it to the Tobii software. Fig. 3 shows the setup of the eye-tracker on the bottom part of the stand, together with an iPad and the external scene camera.

Fig. 3 Eye-tracker is placed on the bottom part of the stand, and the iPad is attached in the centre.

The eye-tracker and external camera were connected to a laptop, and the Tobii studio software was installed upon this laptop for the real-time processing and storage of the video- and gaze data.

2.3 Procedures

Students conducted the experiment one at a time. Pre- and post-intervention measurements of the children's proportion-related skills were obtained. The intentional use of PPON test items (Dutch: 'Periodieke Peiling van het Onderwijsniveau') ensured that the tests measured proportion skills in a way common to how proportion is taught in Dutch primary education. To detect learning effects, test items were slightly above the child's current knowledge level. The whole procedure lasted the duration of approximately 30-40 minutes per child. The proportion set for the task was 1:2, so that the right bar had to be always twice as tall as the left bar in order for them to stay green.

At the beginning of each session, students saw a brief text on the screen together with images explaining how to interact with the app: “You have to move the bars up and down and find the green bars and try to keep the bars green while moving them.” An instruction strategy was developed to guide and encourage participants to reflect and report on their actions and thoughts. For example, the experimenter repeated sentences, such as “Try to make the bars green and maintain the green bars even when you move your hands,” “Could you tell me what you are doing right now?” This strategy was pilot-tested before data collection commenced (on common interaction strategies for task-based semi-structured tutorial interviews, see Ginsburg [35]).

A set of interactive phases was pre-planned, and the experimenter followed these phases. The experimenter first allowed the student to work on the task and find “the greens” for a maximum of 5 minutes. This was an exploration phase without asking them to express their thoughts. After a few minutes of random play, the second phase started in which participants’ interaction was probed for more conversation and thinking aloud about what they were doing. Students then spent more time playing, and they were encouraged to talk about their actions for up to 5 additional minutes or until they expressed some thoughts indicating that they have figured out the rule of the interaction (i.e., for moving “in green”). In exceptional cases, some students received more time to interact with the app. At the end of this phase, regardless of their discovery of any rules, students were instructed to move the bars all the way up from the bottom to the top and keep them green. In the next interview phase, the experimenter introduced the grid onto the screen and asked the participant to play some more and find out what the rule is. Another 3-5 minutes spent moving the bars, number, too, were shown next to the grid. Finally after 5 minutes of moving the bars on a screen that had both a grid and numbers, participants were asked one last time to begin from the bottom of the screen and move the bars all the way up while trying to keep them green as much as possible. These consecutive interaction phases parsed the interview data for subsequent analysis and comparison. While the phase durations varied slightly across participants, the following summarizes the above phases:

1. 5 minutes: explore
2. 5 minutes: explore and think aloud
3. Move the bars from the bottom to the top and keep it green
4. 5 minutes: explore and think aloud with the grid on the screen
5. 5 minutes: explore and think aloud with the grid and numbers on the screen
6. Move the bars from the bottom to the top and keep it green

2.4 Data Analysis

Data reported in this paper were analysed by video observations. Captured videos from the scene, with patterns of gaze-data mapped on them, were exported using the Tobii studio software. The result was a real-time hand-movement view from the above with eye-movement patterns mapped on it (Fig. 4).

Two researchers watched the videos several times in search of pivotal action, speech, or gaze events and patterns as well as interactions preceding the enactment of those events or patterns. A pivotal speech event was the report of a discovery. A pivotal gaze event was a change in eye-movement pattern, for instance from rapid switches to focused visual attention. It was also noted in which phase of the interaction these event occurred. These observations were first conducted independently by the two researchers. Next, they shared their findings and watched the video again and again until they reached agreement over all their observations.

3 RESULTS

Data analysis revealed cross-participant variation along several important dimensions, including: duration of time elapsed until they discovered a first correct interaction principle for the task; time to complete the task; and pace of finger movement (fast or slow) at the initial exploration phase. Participants also followed different paths of discovery and deduction, and they varied in their initially incorrect rules.

Despite these individual differences, we can indicate at least two consistent patterns across all students. First, they all started the task with purely random movements of the two fingers and constant gaze shifts between the top of the two bars. However at the end of the second phase, where they had to move the bars all the way up from the bottom to the top and keep them green, participants use an apparently similar strategy. Most participants utilized an attentional anchor, *which they themselves constructed without explicit instruction,* to manage a perceptual-motor (eye–hand) coordination and thus keep the bars green while moving the fingers across the screen. During the exploration phase, participants tended to shift their gaze between the right bar and the left bar following their hand movement. However, while moving the bars all the way up the screen they attended to a more focused point or set of points. Some of the students focused on the higher bar, on or around the point that matches the height of the lower point (Fig 4). Some others shifted their gaze between the top of the left bar, top of the right bar and length of the right bar on the left bar (Fig 5). Either way, participants appeared to be forming an attentional anchor that helped them succeed in making the bars green and keeping them green while moving their fingers. Second, the moment of discovery, what we call the "Aha" moment, usually followed this new hand-eye coordination. Students looked at the relevant height (e.g., half way up the tall bar) well before they expressed that one bar should be half as tall as the other. Following are supportive examples of pivotal events across participants.

Fig. 4 Two consecutive frames of gaze patterns before and after finding the next green. Orange dot indicates the fixation of the eye gaze, expecting the half and turning the red to green.

Participant-1 (total duration: 31 minutes and 18 seconds) focused from 05:22 onwards almost exclusively on the top of the left bar, top of the right bar, and halfway up the right bar, making a triangle of focus (see Liao & Masters, 2001, on biometaphors for motor-action skill learning). At 06:48, a little more than one minute later, she found and expressed a rule: while moving her fingers and focusing more intensely at halfway along the right bar, she said, "When you keep the left bar a little halfway [of] the right bar, then they both stay green, and if you make the left bigger than the half [of the right, then] they turn red", Fig. 4 shows two consecutive frames of the video at this moment.

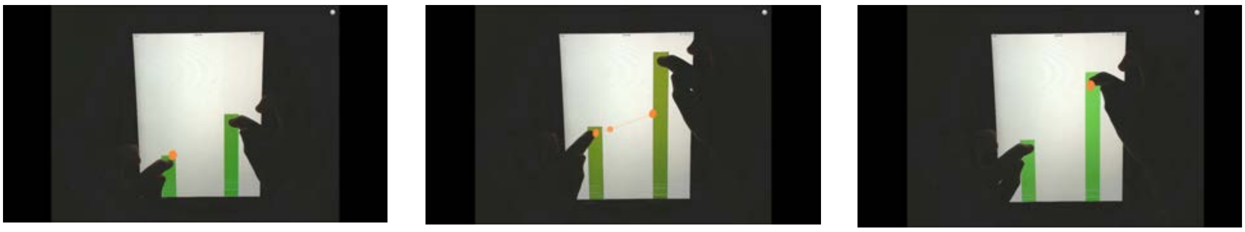


Fig. 5. Three consecutive frames of forming a triangle of top-left, middle right, top right, while trying to keep the bars green. The middle frame shows where the eye gaze is anticipating the next position.

Participant-2 (total duration: 29 minutes and 39 seconds) largely showed the same progression. At 05:28 he began looking halfway up the right bar, and this behavior peaked at 08:42 with focused eye-movements along the same triangle (top left, half-way right, top-right; see Fig. 5). This is almost three minutes later than the Participant-1. And yet as opposed to the Participant-1, Participant-2 did not mention the rule, or did not seem to realize it at the time. His focused eye-movements were followed by a relative long period of exploring, when at 11.45 he again focusing at halfway up the right bar. Still, he did mention the rule as such. At 12:44, after the grid had appeared on the screen, he stated, "The left must always move a half and the right must always move a whole" (.5 per 1) Recall that Partipant-1 articulated a rule for the "static" relation between the two bars' heights. Participant-2's rule is dynamical—it speaks to the relative *changes* in elevation required to switch from one green state to another. At one moment, Participant-2 apparently attempted unsuccessfully to enact the "static" rule.

When the numbers were added to the grid, he stated, "When this one [left] moves up one, then this one [right] should move up two" (1 per 2). When asked to explain the implications of his "dynamical" rule for the distance between the two bars, Participant-2 mentioned that the distance increases as you move up. When asked to support this statement, he answered, "because the left goes up one and the right goes up two."

Participant-3 (total duration: 28 minutes and 28 seconds) progressed more gradually through the task, with slow hand movements, focusing most of the time on the right bar. This participant also showed gaze patterns in which she focused on the top of the left and the top of the right bar, with a lot of shifting in between. Then at 05:08, when both bars were green, she first looked halfway up the right bar. In the subsequent several minutes, she frequently repeated this gaze pattern: top left, top right, and half up right. With respect to finding the rule, this participant deployed numerous different theories (e.g., "The bars turn green when they move in the same pace"). At 14:33 (the grid has already been added onto the screen) she found an effective rule, stating, "The smallest is half of the other." In so doing, this participant manifested the same eye-movement patterns as the first two participants. She also indicated with her fingers the point upon the right bar that is as high as the left bar.

Participant-4 (total duration: 38 minutes and 3 seconds) began with much exploring (e.g., gaze shifts), mainly focusing on the top of the left bar and the top of the right bar. This participant found many 'greens' and demonstrated this achievement to the experimenter. When the experimenter asked her to replicate all her earlier greens (five times total), she shifted gazes between halfway up the right bar and the top of the left bar. Around 05:24 she seemed to be reflecting on a possible principle governing her findings, while again focusing on the top of the left bar and halfway up the right bar, constantly shifting between them. But she did not as yet feel a need to articulate her thoughts. Then at 05:34 she began manifesting the same eye-movements as the other three participants (forming a triangle between the two bars: top left, top right, halfway right). At that moment (07:55) she found the rule: "Yeah, one seems to be half of the other." Stating this, her eyes shifted between the top of the left bar and the middle of the right bar, again forming a triangle.

4 CONCLUSION

Students' development of new ways of moving anticipate their articulation of mathematical concepts, and technologically enabled embodied-interaction learning environments can facilitate such learning. This study both corroborates earlier results from the Abrahamson team at Berkeley [34] and elaborates on their findings. As in the earlier studies, it was found here that children are able to discover forms of interacting with virtual objects that undergird mathematical concepts. Also as in those studies, it was found that the children are able to connect deeply among different yet related mathematical notions through reflecting on their action strategies, even before they engage in the enactment of standard algorithmic procedures using symbolical notation. This study innovates in the empirical validation of the phenomenological construct "attentional anchor" that was first proposed by Hutto and Sánchez–García [31] and then elaborated by Abrahamson and Sánchez–García [31]. Namely, mathematics students engaged in embodied-interaction problem-solving tasks discover in the working environment features, or constellations of features, that facilitate the formation of effective motor-action schemes. These operatory schemes engage the attentional anchor, even as all the while the person monitors for the effectiveness of this coordinated action with respect to the task. In a sense, the attentional anchor—even when it is an imagined, immaterial, psychologically constructed gestalt imposed upon the world—becomes a new artifact mediating the child's control of the environment. Out of thin air, an attentional anchor is crafted that serves both an effective instrument and, subsequently, as the psychological basis of mathematical knowledge. As such, the child is an active agent in the interactive re-invention of cultural–historical practice [36]. The study thus demonstrated learners' agency in forming artifacts [37].

It is striking for researchers to witness firsthand the birth of new artifacts through interaction. The combined methodology of videography and eye-tracking proved instrumental in enabling the researchers to document, analyze, and interpret student thinking as it was changing. In particular, we were able to demonstrate empirically the emergence of attentional anchors en route to the quantitative articulation of action plans that anticipate a challenging mathematical concept. Future research should continue to utilize multimodal learning analytics to evaluate, refine, or generate theory of learning.

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