Eye-Tracking Technology Applications in Educational Research

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Chapter 9
Eye-Tracking the Emergence of Attentional Anchors in a Mathematics Learning Tablet Activity

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ABSTRACT

Little is known about micro-processes by which sensorimotor interaction gives rise to conceptual development. Per embodiment theory, these micro-processes are mediated by dynamical attentional structures. Accordingly this study investigated eye-gaze behaviors during engagement in solving tablet-based bimanual manipulation tasks designed to foster proportional reasoning. Seventy-six elementary- and vocational-school students (9-15 yo) participated in individual task-based clinical interviews. Data gathered included action-logging, eye-tracking, and videography. Analyses revealed the emergence of stable eye-path gaze patterns contemporaneous with first enactments of effective manipulation and prior to verbal articulations of manipulation strategies. Characteristic gaze patterns included consistent or recurring attention to screen locations that bore non-salient stimuli or no stimuli at all yet bore invariant geometric relations to dynamical salient features. Arguably, this research validates empirically hypothetical constructs from constructivism, particularly reflective abstraction.

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INTRODUCTION

Eye-tracking is a technique for collecting data in the context of conducting empirical studies of human perception, cognition, and behavior. As its name suggests, eye-tracking is a means of determining aspects of participants' sensory perception in the visual modality—where they are looking. In turn, locations of visual perception can be used to infer foci and patterns of gaze as these are relevant to making sense of human cognition and behavior. Over the years, eye-tracking hardware has advanced to the point that the instruments are now mobile, so that gaze data can be collected not only in research laboratories but also in the field, such as in investigating the perceptual behavior of supermarket consumers.

Although eye-tracking technology has been used quite widely in cognitive psychology for many decades now, it has only quite recently been employed in the context of conducting educational research. In particular, eye tracking has been used to study how students solve problems in the domains of physics and mathematics (Hegarty & Just, 1993; Hegarty, Mayer, & Green, 1992; Landy, Jones & Goldstone, 2008; Suppes, 1990), reading and comprehension (Paulson & Henry, 2002; Rayner, Chace, Slattery & Ashby, 2006), and multimedia learning and interaction (van Gog & Jarodzka, 2013; van Gog & Scheiter, 2010). The particular contribution of eye-tracking to educational research has been by combining it with data of students' physical movements and verbal utterances. Knowing where and possibly what students are looking at as they interact in a designed environment has enhanced micro-level analyses of learning to a level that had not been possible without this multi-modal approach.

The purpose of this chapter is to show an application of eye-tracking in studying students' learning of mathematics with a tablet application. As will be revealed, a proposed novelty of the chapter is that the objects that students gazed at often were not really there, in the sense that there were not objective stimuli on the tablet interface that could account for the students' perceptual behavior. To be clear, we do not simply mean that these objects were virtual, rather, they were invented by the students as figments of their own active imagination: imaginary gestalts or assemblages that the students created spontaneously in their attempts to solve an interactive manipulation problem presented on the screen. Moreover, our data suggests that as these objects were created, the students began using information about these objects to facilitate their motor activity. At times the students would manipulate these objects, referring to them in speech and gesture as if they were actually there.

As educational researchers, these imaginary objects are important to us, because we interpret them as revealing of cognitive mechanisms at play when situated sensorimotor activity evolves into generalized conceptual understanding. In particular, we believe that these present–absent objects that our study participants are building in their mind's eye are vital for developing mathematical concepts. We submit that these figments are evidence of a goal-oriented coordination between the sensory input and the body; what cognitive-developmental psychologists call sensorimotor schemes. That is, eye-tracking appears to give us windows onto students' mental constructions. It could be that we are offering the field first glimpses into a form of human behavior that is fundamental to Jean Piaget's theory of genetic epistemology known as constructivism (Abrahamson, Shayan, Bakker, & Van der Schaaf, 2016).

Our research is situated within a larger program investigating the nature, emergence, and cultivation of embodied mathematical cognition (see next section). The particular mathematical topic targeted by our materials and activities is that of proportional relations, and more specifically student development of deep understanding for the meanings inherent in a symbolic form such as "2:3 = 4:6". We were inspired by recent ideas in psychology and educational-technology research and practice: embodied cognition. Embodied cognition refers to the idea that we think with and through our bodies (Antle, 2013; Howison, Trninic, Reinholz, & Abrahamson, 2011; Kirsh 2013). The idea of our research is based on a proposed methodology for evaluating claims from scholars of embodiment to the effect that even "abstract" concepts, such as proportion, could emerge from the interaction between our senses (perception) and body movement (action) (Lakoff & Núñez, 2000).

In addition to embodiment theory, we have been inspired by recent findings in studying motor-action skill performance in the sports. Hutto and Sánchez–García (2015) developed the construct of an attentional anchor. Attentional anchors are elements or aspects of engaging the environment that humans invent or adopt to gain sensorimotor control. The attentional anchor may also be an insubstantial object or an invisible constellation of features, and yet it bears subjective psychological reality—it is a phenomenological entity, a thing to be reckoned with and even to control so as to maximize our grip on the world. Attentional anchors are used pervasively yet often subconsciously by experts in many different physical practices, ranging from juggling to driving. For example a juggler better coordinates her actions by imagining and attending to a transparent rectangle hovering in the air. Or a table-tennis player might overlay an imaginary triangle connecting three points in space so as to guide the effective motion of the bat, a technique that Liao and Masters (2001) call a "biomechanical metaphor". Abrahamson and Sánchez–García (2015; 2016) have argued that attentional anchors are relevant to educational research on mathematics learning, because these mental constructions mediate students' reflection on quantitative aspects of situated engagement with pedagogical activities. Through discourse, students reify these tacit constructions as objects of conscious mental activity (see also Hutto, Kirchhoff, & Abrahamson, 2015).

As we elaborate below, our methodology involves studying students' eye-gaze behavior and hand-movement as they engage in task-oriented sensorimotor interactions with a tablet device. The task is designed explicitly so as to foster the development of proportional reasoning through noticing how the hands are moving, for example realizing that the hands are moving at constant yet different speeds. Eye-tracking serves our research in documenting the emergence of stable sensorimotor patterns as students' spontaneous solutions to manipulation problems, that is, documenting the emergence of attentional anchors. We also investigate whether such patterns constitute task-effective sensorimotor schemes and whether (and how) they lead to conceptual knowledge of the task. More specifically we ask the following research question: How does visual attention change during the emergence of sensorimotor schemes for enacting proportional action tasks? Our thesis is that these attentional anchors indeed exist and play critical roles in the emergence of sensorimotor schemes and hence in the development of mathematical concepts. This chapter summarizes our efforts to date that support this thesis.

We will present results of two recent studies that to our knowledge are novel in their approach, combining embodied-interaction pedagogical tasks with eye-tracking methods. We will show how each of these approaches in and of itself opened a new angle on the problem at hand, whereas their combination suggested new questions to pursue and further insights to follow.

BACKGROUND

The essence of the embodiment approach is that cognition evolves from and for the body's interactions with its environment. As a result our motor-action neural circuitry is also engaged when we think *without* external interaction (Glenberg 1997; Barsalou, 1999, 2010; Clark, 2001; Zwaan 1999; Wilson 2002; Myin & O'Regan, 2008). For example research shows that reading words that have sensory (e.g., cinnamon, salt) or motor connotations (e.g., kick, grasp) activates brain regions that are involved in smelling, tasting, and moving feet and fingers (Barrós-Loscertales *et al.* 2011; González *et al.* 2006; Hauk, Johnsrude, Pulvermüller, 2004). Likewise there is evidence that mentally processing numbers activates areas related to finger movements (Andres, Seron, & Oliver, 2007; Roux *et al.* 2003; Zago *et al.* 2001). The thesis that cognition is embodied is supported by converging philosophical arguments and theoretical models (Barsalou, 1999, 2010; Clark, 2001; Lakoff & Núñez 2000) and empirical evidence from multiple and varied domains of scholarship (see Spackman & Yanchar, 2013, for a review), including mathematical reasoning (Abrahamson 2004; Alibali *et al.,*1999; Hutto *et al.,* 2015; Landy *et al.*, 2014).

Embodiment theory bears implications for education. Within the field of mathematics education, the guiding principle is that even the most abstract mathematical idea is grounded in sensorimotor schemes. For example, straight movement forms the ground of understanding mathematical lines; collecting items grounds the idea of sets; checking one-to-one correspondence between sets is an embodied basis for understanding the concept of number (Lakoff & Nunez, 2000; Nemirovsky, 2003). Abrahamson and his collaborators found that students adopt mathematical frames of reference so as better to enact, explain, or evaluate their sensorimotor interaction; in so doing they shift into professional ways of seeing, moving, and talking (Abrahamson *et al.*, 2011, Abrahamson *et al.*, 2012; Abrahamson & Lindgren, 2014; Abrahamson & Sánchez–García, 2015; 2016).

The foundational research on the embodiment of mathematical concepts has inspired and mobilized much further work on the design and evaluation of learning environments. The methods in the above-cited studies by Abrahamson and colleagues consisted of qualitative analyses of videography documenting students' interactions with educational technologies and human tutors. These analyses suggest that students master the effective sensorimotor schemes for operating the technological devices of the learning environment before they can reason about the underlying mechanism or the mathematical concepts. However these studies left open questions regarding the cognitive processes that first enabled students to learn the involved sensorimotor schemes they later developed into the relevant conceptual knowledge. Insight into the micro-evolution of these sensorimotor schemes and conceptual knowledge is essential both for evaluating the embodiment thesis and for the design of effective instructional activities. Accordingly, our study aimed to improve on Abrahamson's prior methods by supplementing eye-tracking techniques. The hope was that knowing where/what students are looking at would help us determine the nature and evolution of students' sensorimotor schemes and thus conceptual learning.

Abrahamson and Sánchez–García (2015; 2016) have suggested that mathematical notions emerge from the development and subsequent articulation of an attentional anchor. Thus there is apparent promise in using the construct of attentional anchor as a means of describing the development of motor-action underlying mathematical concepts. This promise depends both on creating appropriate learning environments and analyzing learning process via the lens of embodiment theory. And yet whereas the construct of an attentional anchor has borne analytical appeal, so far no compelling empirical evidence has been put forth for the emergence of attentional anchors from sensorimotor action and through to verbal articulation of a new mathematical notion. In pursuit of theoretical frameworks that could potentially help us understand the micro-evolution of goal-oriented sensorimotor schemes we are attempting to rectify this gap in the literature. Our chapter reports on a set of studies designed to gain insight into the role of visual attention, and in particular the construction of action-oriented attentional anchors, in the emergence of new sensorimotor schemes underlying mathematical concepts.

EMERGENCE OF ATTENTIONAL ANCHORS IN MATHEMATICAL LEARNING

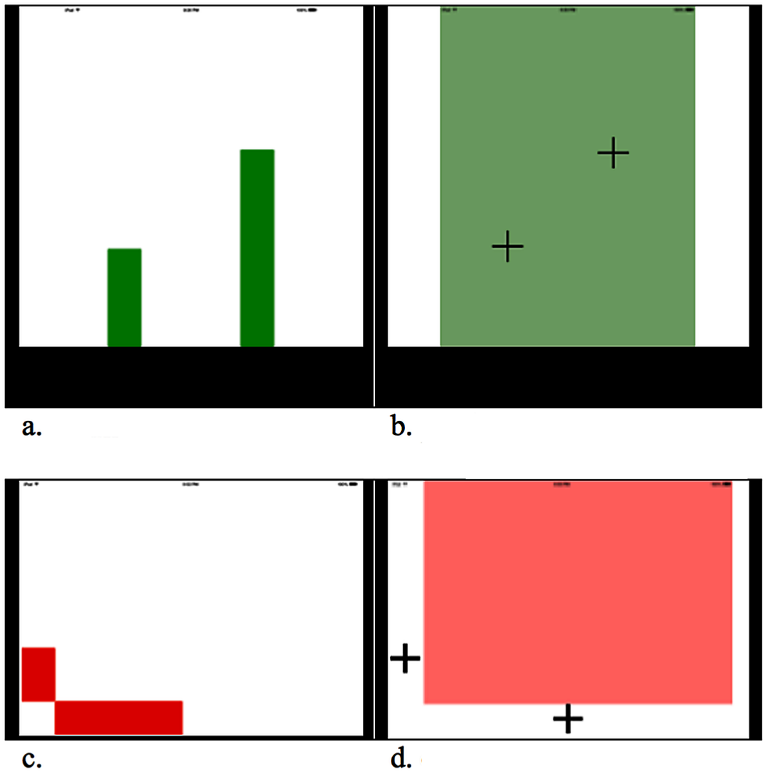
When solving problems involving proportions (e.g., 2:3 = [?]:6), it is hoped that students not only follow rote algorithms, such as cross multiplication (? = 2/3 x 6) but also understand the meaning of proportional equivalence as indicated by the '=' symbol. However, students do not necessarily experience structured opportunity to enact, visualize or conceptualize a ratio or a proportion, and so they understand proportional equivalence only as the result of a solution procedure involving arithmetic operations (Abrahamson & Lindgren, 2014). At the Embodied Design Research Laboratory (EDRL), a series of studies has sought to provide students with experiences by which they engage deeply into understanding the meaning of proportional equivalence. Proportionality, and more broadly all multiplicative concepts such as fractions, are an essential component of early curriculum, and yet they remain the failing point of many students (Davis 2003). The design for proportion developed at EDRL was oriented in particular on students' confusion between additive and multiplicative schemes. For example, a student might treat "2:3 = ?:6" additively, concluding that the missing number is 5 (because 3-2=6-5 maintains the same difference across pairs of numbers). Consequently, EDRL's efforts were focused on creating an instructional design that would support multiplicative schemes that begin from students recognizing the inappropriateness of additive schemes in particular interaction contexts that prove to be based upon proportional relations.

Design

The current project expands on EDRL's line of research on the cognition of ratio and proportion, with the earlier work providing us intellectual framing, instructional design, and unsolved research questions. Our research project extends Abrahamson's work by focusing on new possibilities for both facilitating and evaluating embodied learning of proportion. Our technology (see Figure 1) is a variant and extension of Abrahamson's Mathematical Imagery Trainer for Proportion (MIT-P), and in particular a variant on his tablet implementation of the design that represented proportions as a set of two vertical projections (Abrahamson 2012; Lee *et al.*, 2013; Negrete *et al.*, 2013). Consequently, we named our design "MIT-Ext." One way in which MIT-Ext extends MIT-P was to include an option with orthogonal bars. Moreover, MIT-Ext included embedded action-logging, which could be combined with gaze data. These recent methods for gathering sensorimotor data would complement the more traditional audio–video data that capture the student's (and teacher's) multimodal utterances (see Blikstein *et al.*(2014), for a review of multimodal learning analytics). We now present the MIT-Ext, explaining the instructional materials and activities as well as the research design employed in this study to gather empirical data from participants' interactions with the device.

The MIT-Ext is implemented in a tablet. The activities consist of bimanual interaction problems: The user must place a left-hand finger and a right-hand finger on the screen at the same time and move both fingers simultaneously in an attempt to receive a particular goal visual feedback, a green coloration either of the background or of objects they are manipulating. The manipulation is constrained. The "parallel" version (see Figure 1 a, b) enables interaction only along the vertical axis, that is, raising and/or lowering either a pair of bars (see Figure 1a) or a pair of plus signs (see Figure 1b). In the "orthogonal" version (see Figure 1 c, d), the left-hand finger still moves up/down along the vertical axis, while the right-hand finger moves right/left along the horizontal axis. Note that the "orthogonal" tasks create what should be discerned as a rudimentary Cartesian space. That is, the left hand moves along a simple y-axis, and the right hand moves along a simple x-axis. An objective of this activity is that through determining a

Figure 1. Sample screens from four modules of the MIT-Ext touch-screen tablet: a. parallel bars; b. parallel full screen with plusses at finger point; c. orthogonal bars; d. orthogonal full screen with plusses at finger point



"green" sensorimotor scheme the participants would reinvent the Cartesian space. That is, we hoped that the participants would spontaneously develop an attentional anchor that somehow would incorporate sensory information from both the left and right hand and coordinate them into a single location on the screen that captures both in the form of an ordered pair.

The activity begins with a plain white screen. Moving the fingers in designated areas changes the color of both bars in the bar version (Figure 1 a, c) and the whole screen in the full-screen version (Figure 1 b, d) along a gradient between green and red. The color turns green when the two fingers are at the exact pre-set proportional relations with respect to each other (1:2 in our studies)—thus at 1:2 heights in the parallel tasks and 1:2 distances from the origin in the orthogonal tasks—and it turns red gradually as the proportion changes. Thus to keep the bars green all the time while moving the fingers, one has to move the two fingers simultaneously at a coordinated pace relative to the pre-set proportion, for example maintaining a 1:2 proportional relation between the extent of the bars.

Note in the parallel displays that "green" motion would necessarily involve inconsistency in the vertical gap of the two fingers. That is, the interval between the hands would change relative to their elevation along the screen, increasing as both fingers go up, decreasing as they go down. The pedagogical rationale of the MIT-P, and hence the MIT-Ext, is that by moving their fingers "in the green" and recognizing both the changing spatial interval between the fingers and their different pace of motion, the core knowledge of proportionality may emerge.

As in the original MIT-P, the MIT-Ext app is designed such that an experimenter can illuminate a grid onto the screen, which quantizes the tablet space and thus draws the user's attention to the position of the bars on the screen, as well as numbers that appear along the "*y*-axis" of the grid (see Figure 2). In previous EDRL research, these supplementary virtual elements in the visual display attracted the students as potential frames of reference and action and consequently modified the students' forms of interaction. In particular, the students appropriated embedded utilities in the grid and numerals so as to enhance their enactment, evaluation, or explanation of their "green-making" bimanual coordination, yet in so doing they shifted into quantitative visualizations and solution strategies. For example, the introduction of the grid caused students working in the 1:2 setting to shift from a simultaneous qualitative strategy of modifying the interval co-relative to its elevation on the screen to a sequential quantitative strategy of alternately raising the fingers 1 and 2 units (Abrahamson *et al.,* 2011).

The two studies presented here focus on analyzing and modeling interaction dynamics between students' perceptual experience (by tracking the eye-movement), actions (by logging touch locations), and conceptual development (by applying a "thinking aloud" semi-structured clinical-interview protocol).

The eye-tracker used in the study is a Tobii™ x2-60 model. It is a portable, small device that can be used together with the Tobii Mobile Device Stand for X2, which is specially designed for studies conducted on tablets or smartphones. The stand also allows for installation of an external camera to videograph relevant interaction behaviors during the experiment. Videography is then exported to the Tobii software for the real-time processing and storage of an integrated video-and-gaze data. Figure 3 shows the setup of the eye-tracker and its stand, together with an iPad and the external scene camera.

Seventy-six volunteering students participated in two different studies. In both studies the children were guided through the activity by an interviewer–researcher using an interview protocol so as to ensure consistency in procedure across participants (Abrahamson et al., 2011). The interview protocol included the specification of task objectives, such as "Try to make the bars green, and maintain the green bars even

Figure 2. Appearance of grids and numerals: a. grid; b. grid as well as numerals along the grid's vertical axis

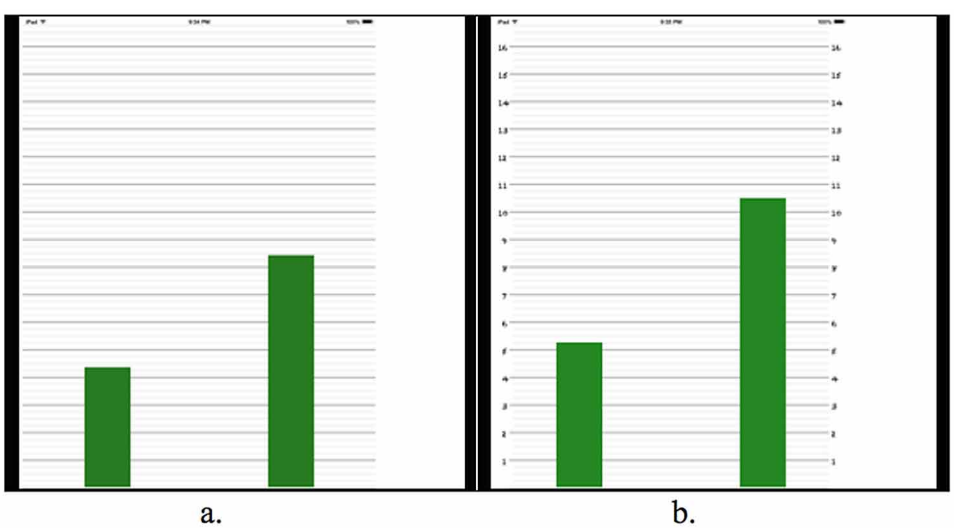


Figure 3. The eye-tracker is placed on the stand base with the camera on the top and the iPad attached in the center.
Adapted from User's manual of Tobii Mobile Device Stand for X2.

when you move your hands," and various prompts to probe them on their reasoning, such as "why do you think it turned green", "can you think of a rule?". This protocol was pilot-tested prior to data collection.

Our focus in both studies is on the role of sensorimotor experience in constructing mathematical knowledge. We predicted that in the course of solving the interaction problem the students would develop an attentional anchor as a means of coordinating their emerging sensorimotor schemes appropriate to the task objective. In both studies we investigated whether and how the perceptual processes (particularly the visual information) and sensorimotor schemes were related to the students' emerging proportional reasoning. In the remainder of the chapter, we present the main findings of each study separately and then discuss the key overall findings of this research.

STUDY 1: WORKING WITH ELEMENTARY SCHOOL STUDENTS

Method and Procedure

The participants in this study were 30 fifth- and sixth-grader students (13 male, 17 female; mean age = 11 years and 3 months), from five elementary schools located in the Netherlands. The entire procedure lasted the duration of approximately 20-30 minutes per child. The app condition used for this study was the parallel bar version, as depicted in Figure 1a. A set of interactive phases was pre-planned, and the experimenter followed these phases. The experimenter first allowed the participants a maximum of 5 minutes of exploration to work on the task and find “the greens” without asking them to express their thoughts. After a few minutes of unfacilitated play, the second phase began, in which an experimenter probed the students’ interaction by asking them to think aloud and encouraging them to reason about their actions. This phase continued either for 5 minutes or until the participants expressed some thoughts indicating that they had figured out the rule of the interaction (i.e., for moving “in green”). In exceptional cases, some participants received more time to interact with the app. At the end of this phase, and regardless of whether they had discovered any rule, participants were instructed to move both bars beginning at the base and then rising upward as far as possible, keeping them green. Next the experimenter introduced the grid onto the screen and asked the participants to play some more and see whether the grid might somehow help them find a rule. After 3-5 minutes of this activity, the numerals were revealed along the grid’s vertical axis. Finally after 5 minutes of moving the bars on a screen that had both a grid and numerals, participants were asked one last time to begin from the bottom of the screen and move the bars as far up as possible, all the while keeping them green. These consecutive interaction phases parsed the interview data for subsequent analysis and comparison. Whereas the phase durations varied slightly across individual participants, the following summarizes the above phases:

1. 5 minutes: explore.
2. 5 minutes: explore and think aloud.
3. Move the bars from the bottom toward the top and keep it green.
4. 5 minutes: explore and think aloud with the grid on the screen.
5. 5 minutes: explore and think aloud with the grid and numbers on the screen.
6. Move the bars from the bottom toward the top and keep it green.

We began our analysis by observing the processed videos that overlaid eye-tracking traces onto the tablet screen actions. These integrated videos captured a view from above of the tablet, with fingers moving on the screen surface and gaze patterns plotted on top of this view. The analysis was conducted both “top down” and “bottom up.” It was top-down, in the sense that we were searching for evidence of a particular form of behavior—gaze patterns that might be interpreted as indicating the construction of an attentional anchor, in line with our research hypotheses. Yet it was also bottom-up, in the sense that we did not know what features of eye-gaze we might find, how exactly we would model them, and what patterns we might find across participants. Also, this being an explorative study with a modest number of participants, we did not conduct formal analysis of micro-events nor use grounded theory or interaction analysis, and yet we hoped this study could lead toward the creation of a coding scheme that would inform future studies.

The total corpus of video data was split in two equal parts. Two researchers each watched half of the data. Next they shared their findings and watched the videos again and again until they reached agreement over all their observations.

The first round of analysis treated the videography by focusing primarily on the overall interaction and paying little to no attention to the eye-gaze paths. This round suggested certain commonalities across participants in terms of the types of events observed and the timing of their appearance along the protocol. Next, the overlaid eye-tracking data was studied in search of distinct visual patterns. This round enabled a refining of the event logging so as to include sub-phases: exploration, discovery, verification by grid and number, etc. Focusing on the discovery phase, both researchers logged any visual patterns that included noticeable screen locations other than the fingertips. Summative discussions of the research team led to the consolidation of the patterns reported below in the Results section.

Results of Study 1

Initial observations revealed much cross-participant variation along several dimensions of relevance to the study, including:

1. Duration of time elapsed until discovery of a first effective interaction routine, that is, a regulated motor-action scheme for keeping the screen green while moving both hands;
2. Time to complete the whole task; and
3. Pace of finger movement (fast or slow) at the initial exploration phase.

Participants also differed in their initially incorrect rules, the time it took them to show their ultimately effective eye-gaze patterns, and their lines of reasoning toward these effective solutions.

Despite these individual differences, we came to notice at least two consistent patterns across all participants. First, all participants began the task by exploring the interaction space according to no apparent rule. In this phase, their eye gaze shifted between the top of the two bars. However, at the end of the second phase, where they had to move the bars upward from the bottom, keeping them green, the participants apparently all used a similar strategy. This common strategy was enacted at about the moment where they verbally stated the multiplicative rule, that is, when they determined the height of one of the bars as double or half the other. It is then that we observed a distinct gaze pattern emerge. In this pattern, which all participants constructed themselves without explicit instruction, their gaze shifted intensively among three points on the screen that together configure an upright right triangle (see Figure 4): the top of the Left Bar, the top of the Right Bar, and halfway along the Right Bar. Note in Figure 4 how the base of this upright right triangle runs between the top of the Left Bar and its projected location on the Right Bar. It appears as though the participants were dynamically calibrating the height of the Left Bar relative to half the height of the Right Bar, and vice versa. We observed a variety of gaze paths among these three locations (see Figure 4).

Figure 4. Overview of the triangular focal pathway apparent in our sample. Circles represent attentional foci, lines represent gaze-patterns. Triangle A was most prevalent across participants.

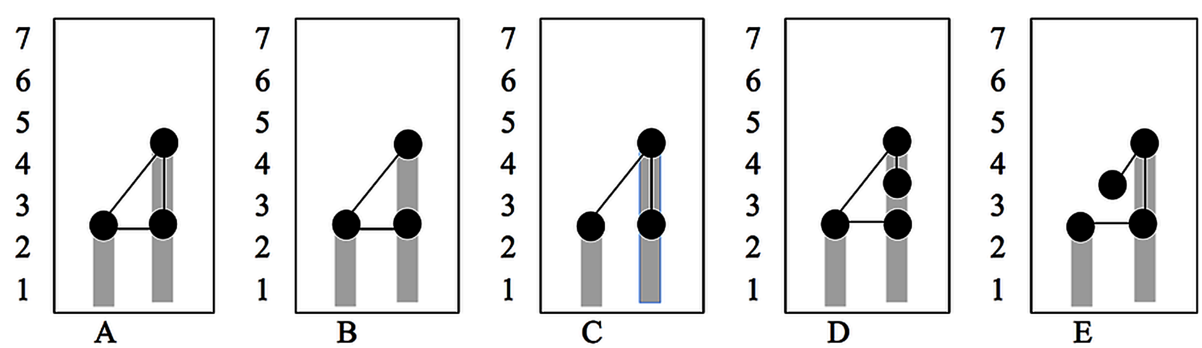


Figure 4 shows a schematic overview of the triangular focal pathway apparent in the data. The black circles represent the focal points that were most frequently and persistently gazed at, and the lines show the most frequent gaze shifts among these points. We observed these patterns most often when the participants had explored the space and were attempting to "keep the green" while moving their hands. We also observed a gradual increase in the participants' facility in applying their new strategy of using the focal pathway. The strategy first emerged around the end of Phase 1 and beginning of Phase 3, when they were asked to move in the green. We then looked at measures of time duration until participants' initial statement of the rule as well as the measures of time duration until they began forming the triangular pathway. We found that the participants looked at the relevant focal points shown on Figure 4 well before they expressed the multiplicative rule (delta M = 365 seconds, SD = 283). That is, the "Aha" moment always followed *after* the enactment of a new hand–eye coordination.

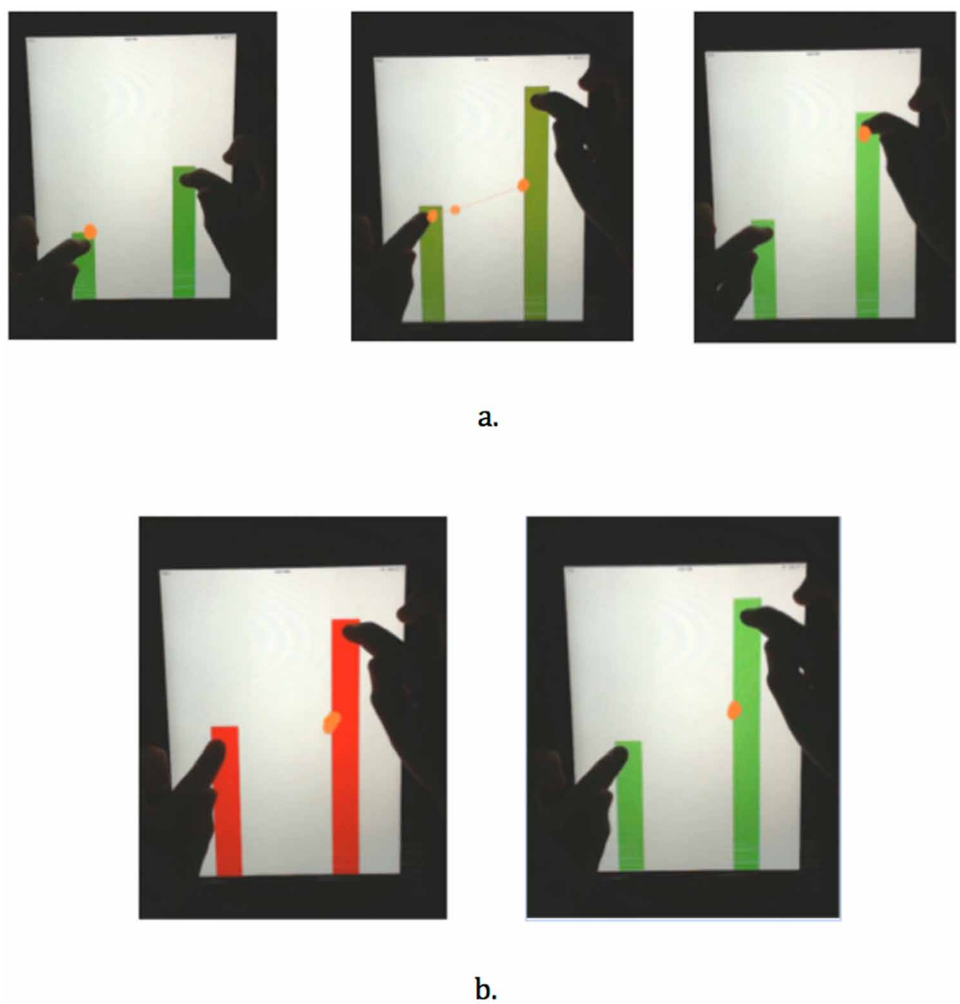
Thus, despite great variation in their individual exploration, all students eventually developed a stable gaze path, all the pathways were of the triangular forms, these perceptual constructions coincided with improvement in the efficacy of the motor actions (more green), and these perceptual and motor changes preceded verbal reports describing what amounts to a new sensorimotor scheme. In order to exemplify this juxtaposition between the variety in individual learning paths and the uniformity in their ultimate achievements, we now present a set of three brief cases of individual participants.

Participant-1 (female, age 10 years and 9 months) progressed through the task with slow hand movements, focusing most of the time on the Right Bar. A first gaze pattern shifted rapidly between the respective tops of both the Left and Right Bars. Then at 05:08, at a moment when both bars were green, she first looked halfway up the Right Bar. In the subsequent several minutes, she frequently repeated this gaze pattern: top left, mid right, top right (Figure 5a). Before arriving at an effective strategy, this participant reported on numerous differing yet ultimately ineffective action theories (e.g., "The bars turn green when they move in the same pace"). These early theories were subsequently falsified through further exploration. Yet at 14:33, with the grid already on the screen, she found an effective rule, stating, "The smallest is half of the other," pointing to the location upon the Right Bar that is as high as the Left Bar.

Participant-2 (female, age 10 years and 9 months) also began with much exploring and also focused mainly on the bars' respective tops. This participant initially found five 'greens' and demonstrated each of them to the experimenter. When the experimenter then asked her to replicate all five, she spontaneously initiated the pattern of shifting her gaze between the top of the Left Bar and halfway up the Right Bar. Around 05:24 she seemed to be reflecting on a possible principle governing her findings, while repeating that same eye-gaze path. Then at 05:34 she began manifesting the triangular focal pathway: top left, top right, mid right. Only at 07:55 did she articulate, "Yeah, one seems to be half of the other." Stating this, her eyes shifted between the top of the Left Bar and the middle of the Right Bar and then completing the triangle.

Participant-3 (female age 11 years and 10 months) demonstrated good finger coordination from early on. From 05:22 onwards she focused almost exclusively on our canonical triadic locations. At 06:48, over a minute later, the "Aha" moment arrived: focusing more frequently and more intensely at the Right Bar's midpoint (Figure 5b), she said, "When you keep the left bar a little halfway [of] the right bar, then they both stay green, and if you make the left bigger than the half [of the right, then] they turn red".

Figure 5. a. Three consecutive frames of a time-sequenced triangular attentional anchor—top-left, middle right, top right—for keeping the bars green while moving the hands. The middle frame shows where the eye gaze (on the right bar) anticipates the next height of the left bar (in a subsequent frame not show here). b. Two consecutive frames of gaze patterns before and after finding the next green. Orange dot indicates the fixation of the eye gaze, expecting the half and turning the red to green.



Conclusion of Study 1

We interpret the triangular focal pathway configuration as an attentional anchor and submit that accomplishing this sensorimotor (eye–hand) coordination mediated students' effective enactment of strategies by which they kept both bars green (cf. Shayan et al., 2015; see also Duijzer, 2015). The pervasiveness of this particular attentional anchor across all study participants suggests a robust phenomenon. Further the numerous individual variations on this configuration (see Figure 1), indicates that individual participants tacitly customized this attentional anchor (see Renshaw, Davids, & Savelsbergh (2010), on the importance of enabling subjective solutions to motor-action problems). We laid out a learning trajectory that suggests the relevance of attentional anchors to mathematics learning. As such, we contributed to our research assumption that eye-tracking methodology may contribute to both theory and practice of mathematics education.

STUDY 2: WORKING WITH PRE-VOCATIONAL-SCHOOL STUDENTS

Method and Procedure

In this study we took our experiments to two pre-vocational schools in the Netherlands. When Dutch students graduate from elementary school, they are steered toward a 4-year pre-vocational education program. About 60 percent of Dutch students take this route. Working with vocational-education students was instrumental to our research design, because we could gather additional empirical data from a group of older students. At the same time, this second study would potentially bear wider implications for the mathematics-education community. Vocational-education students are typically those who experienced difficulty with the standard elementary-school curriculum or, in any case, achieved lower grades in mathematics as compared with students who continued in the academic track (CITO, 2013). Our study could lend us some insight onto the utility of engaging vocational-education students in embodied-design activities.

A total of 46 volunteering students (29 male, 17 female) participated in this study. They were between 12-15 years of age (mean age = 13 years and 5 months). Our research question for Study 2 was the same as in Study 1, that is, whether and how participating in the interaction activity and developing particular sensorimotor schemes and visual patterns is related to effective manipulation of the device and the emergence of proportional reasoning. However in this study we also wanted to know whether differences in visual input and interaction would lead to different gaze patterns and variations in the emergence of attentional anchor.

Study 2 tasks both subsumed and extended that of Study 1. Here we used a 2x2 design to test all four variants (Figure 1). We randomly assigned the totality of Study 2 participants into two groups. Group A worked only on the two Parallel activities, completing first the "plusses" configuration (i.e. cursors; see Figure 1b) and then the "bars" (see Figure 1a). Group B worked only on the Orthogonal configuration, similarly completing first the "plusses" (see Figure 1d) and then the "bars" (see Figure 1c). In both groups and for both activities the proportion was set at 1:2. Within their respective activities, the procedure was identical to Study 1. Across Groups A and B individual interviews lasted 40 minutes on average, with the first task typically lasting longer than the second. Our research team used for Study 2 the identical analysis process as for Study 1 (please refer, above). We also transcribed all videos for further strategy and conversation analysis. We include one case study here as an example of:

1. The emergence of attentional anchors mediating the solution of complex manipulation tasks;
2. Students' challenges in analyzing, representing, and explaining their tacit sensorimotor schemes in mathematical forms; and
3. The apparent pedagogical utility of challenging students to adopt mathematical visualizations of embodied tasks.

Results of Study 2

As in Study 1, we witnessed much variation in participants' behaviors during the early exploration phase. Yet as in Study 1, we also found similarities across participants. In each group and for each task we found that an effective attentional anchor emerged very early in the task—the attentional anchor persisted through to a verbal statement of the rule. We will present the results separately per study group and per task. As compared to Study 1, we observed in Study 2 greater variation in visual patterns, possibly due to inter-group difference in screen orientation (landscape for orthogonal vs. portrait for parallel). Interestingly, the orthogonal group showed greater variation than the landscape group.

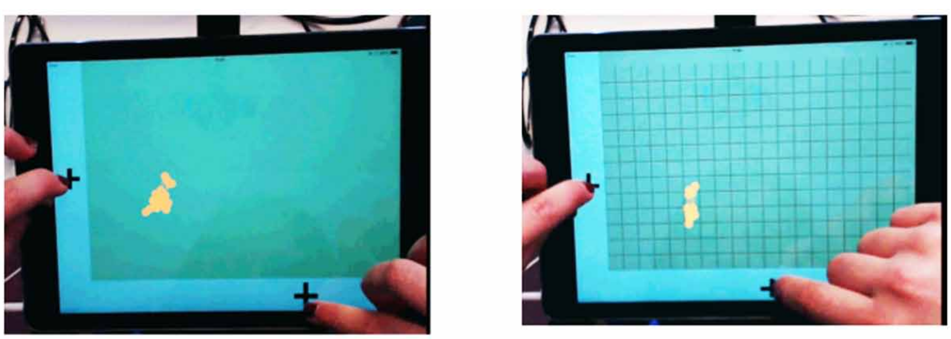
In the course of conducting analyses of Study 2 data, the research team formulated the following set of heuristic principles for determining whether or not a visual gaze pattern should be coded as an attentional anchor. The eye-gaze pattern should:

1. Manifest at or around the time when a verbal rule is stated;
2. Repeat consecutively more than twice; and
3. Include focal points other than the fingertips.

In what follows, below, we offer our categories of attentional anchors observed in Study 2. Similar to Study 1, these categories are based on data gathered during the time span that begins when individual participants first formulated an attentional anchor and ending when they articulated it to the experimenter. As will be discussed in details, all subjects began to show these eye-gaze patterns towards the end of the Phase 1 of the protocol (exploration phase). Thus by the time the first attentional anchors were formed, participants had already found the first greens and were trying to maintain green while reasoning about their actions.

By way of illustration of our gaze-overlaid video data, Figure 6 shows snapshots from moments in which a participant is working the plusses prior (Figure 6a) and after (Figure 6b) the grid has been introduced onto the screen. Note the orange "stain" that captures the student's eye-gaze location.

Figure 6. Snapshots of the emerging gaze patterns while moving the plusses prior (a) and after (b) the appearance of the grid



Parallel Task Results

1. Parallel Plus Task

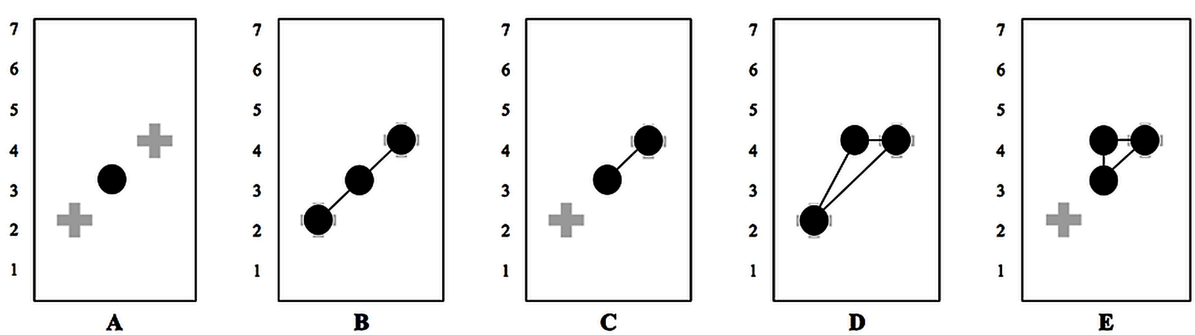
Similar to Study 1 we found a variety of visual patterns (see Figure 7). Moreover, we found within-pattern variation in gaze path. For example, the diagonal path in Figure 7b was formed by gaze shifts among multiple two- and three-point possible paths between **L**eft, **M**iddle, **R**ight focal points, with repetitions (e.g., RM, LM, RLM, RML, MLM, MRM, MLR, MRL, RMLM, LMRM, ...). Across all 14 participants, the attentional anchor was formed prior to the verbal articulation of one being half the other (range of 255 – 600 sec.; M=438; SD=114.7).

At first glance, the attentional anchors found in the Study 1 Parallel Bars condition appear very different from those found in the Study 2 Parallel Plus. In the case of the Bars, the attentional anchor included only elements of the virtual objects on the screen: the bars' contours (at the fingertips) and the bar itself (its midpoint), whereas the Plusses yielded an attentional anchor that included a point "hovering in space." However these differences might be explained in terms of the participants' consistent drive to figure out a gaze-path pattern that might support effective interaction. When the interaction space includes perceptual features that are relevant to the interaction, the user latches onto them, as in the case of attending to an otherwise indistinct point upon the bar itself. But when there are no features in space other than the objects being manipulated, the user must "invent" a point that is unattached to any object, as in the case of the plusses and, interestingly, as in the case of juggling (recall the imaginary rectangle).

2. Parallel Bar Task

The Parallel Bar task was implemented immediately after the Parallel Plus task. As might be predicted, the participants demonstrated greater initial orientation and motor facility in the second task, and consequently shorter time durations until they first "made a green." Similar to Study 1 findings, the participants developed the attentional anchor patterns represented in Figure 4, and they did so prior to articulating a rule verbally.

Figure 7. Overview of the patterns-of-focus apparent in parallel-plus task; circles connected by lines are representative of the gaze-patterns. Gaze pattern B was most prevalent among participants.



Orthogonal Task Results

The orthogonal tasks expand on previous MIT-P tasks that were parallel only (but see Lee *et al.*, 2013 for a cognitive domain analysis of the Cartesian expansion to the MIT-P design). As such, research on interactions with the orthogonal tasks was comparatively more exploratory. Here, too, we predicted the emergence of attentional anchors as the participants' tacit means of acting and reflecting. Fourteen students participated and completed first the Orthogonal Plus task (See Figure 1d) and then the Orthogonal Bar task (see Figure 1c). Following initial desultory exploration, they, too, quickly formed attentional anchors and later stated the rule. We witnessed an even greater diversity of emergent gaze-path patterns as compared to the Parallel conditions. Following, we report on these patterns.

1. Orthogonal Plus Task

In comparison to Parallel tasks, Orthogonal tasks appear to yield a greater proliferation of diverse eye-path patterns. Also, orthogonal tasks plotted on the iPad's landscape mode privilege the right hand with greater linear space to move along. Indeed, participants moved the left finger in shorter steps compared to the right finger. This spatial setting and the participants' reflexive scaling response in turn unintentionally facilitated both a quicker finding of 1:2 pairs (because the right hand was already further along its axis as compared to the left hand) and a quicker emergence of attentional anchors as compared to the parallel condition.

These landscape mode and orthogonal task also created an asymmetric gaze space by privileging the bottom-left corner as a starting point. As the participants glided both fingers away from the origin, we observed an intriguing shift in their eye-gaze path patterns. When their fingers were still in the bottom-left quadrant of the screen, participants tended to gaze on the screen at a point upon an imaginary line running between their fingers. Yet as the fingers moved farther along their respective axes, the patterns across participants started to differ quite remarkably. Figure 8 displays gaze-path patterns during moments of activity where the hands were farthest along their respective axes. At those moments, the gaze-path patterns were many and varied, and yet they always fell within a triangular region configured by the screen's top-left, top-right, and bottom-right corners (see dashed-line triangles in all Figure 8 images). Moreover, all of these varied gaze-path patterns have at least one focal point lying directly on the triangular region (or tangent to it, in the case of Image E).

Once again, we view our findings as supporting our prediction that attentional anchors would emerge in the course of manipulating the objects, and once again we view a mixture of eye-gaze locations that bear distinct stimuli (the plusses and fingertip) and unique geographical features (screen corners) as well as objectively indistinct locations "hovering" in visual space (see also Figures 9-10).

2. Orthogonal Bar Task

Similar to our findings from the Orthogonal Plus task, in the Orthogonal Bar task we again observed by-and-large two groups of gaze-path pattern types: near the origin the gaze fell on points along a line running between the fingertips, and far from the origin the gaze-path patterns became increasingly diverse (see Figure 11). However these patterns did not fall in the triangular region that we had discerned in the Bars condition. Rather, participants focused on different locations on and around the two bars. Once again, it appears as though the presence of task-relevant features (the bar contours) attracts visual attention.

Figure 8. Overview of visual patterns formed across participants as were successfully solving the first task of the Orthogonal condition. All images show eye-path patterns at moments during the activity when participants' hands were farthest along their respective axes. Circles represent the most frequently occurring focal points, and lines represent bidirectional gaze shifts between the points.

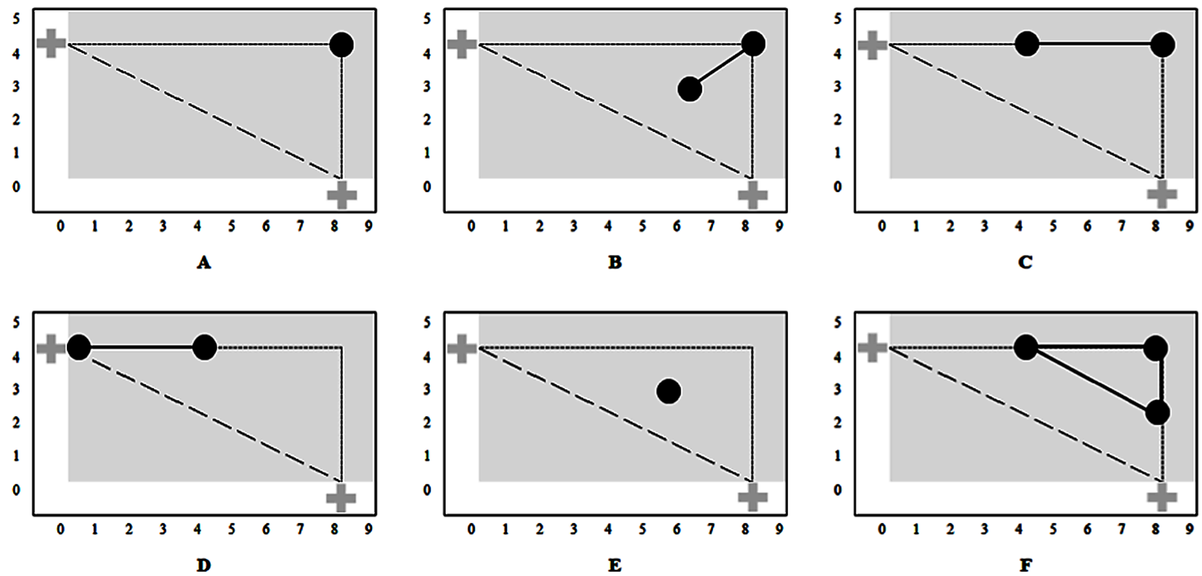
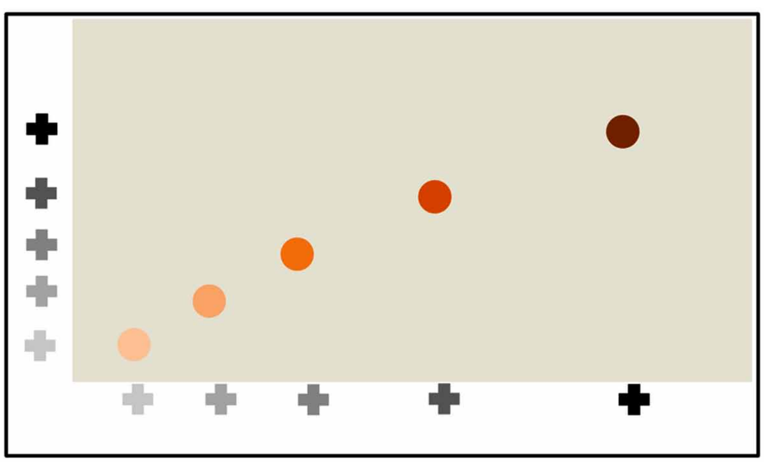


Figure 9. Dynamical visualization of consecutive screenshots from gaze-overlaid screen actions (see Figure 8a)



These patterns in and of themselves—irrespective of the participants' simultaneous verbal utterance—suggest that the participants are attempting to determine some point along the horizontal line that relates in particular ways to other information on the screen, such as other points or intervals. For example, Figure 11 b, d, e, and f suggest a rotational projection between the vertical bar and midway along the horizontal bar, and Figure 11 c, e, and f suggest attention to the right-hand half of the horizontal bar.

Figure 10. Dynamical visualization of consecutive screenshots from gaze-overlaid screen actions (see Figure 8e)

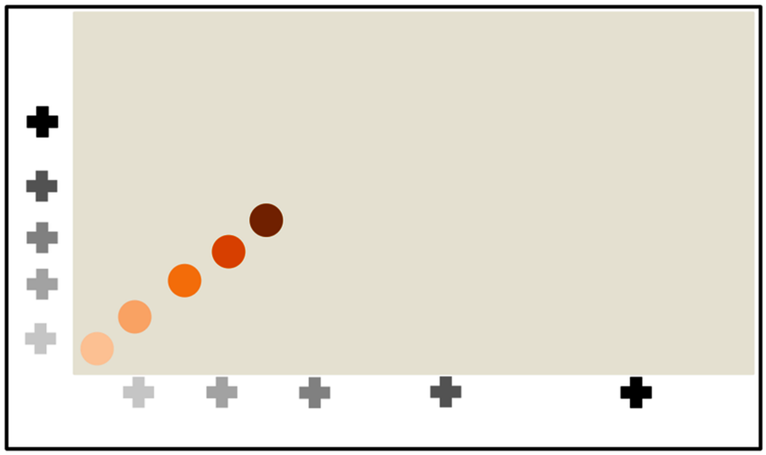
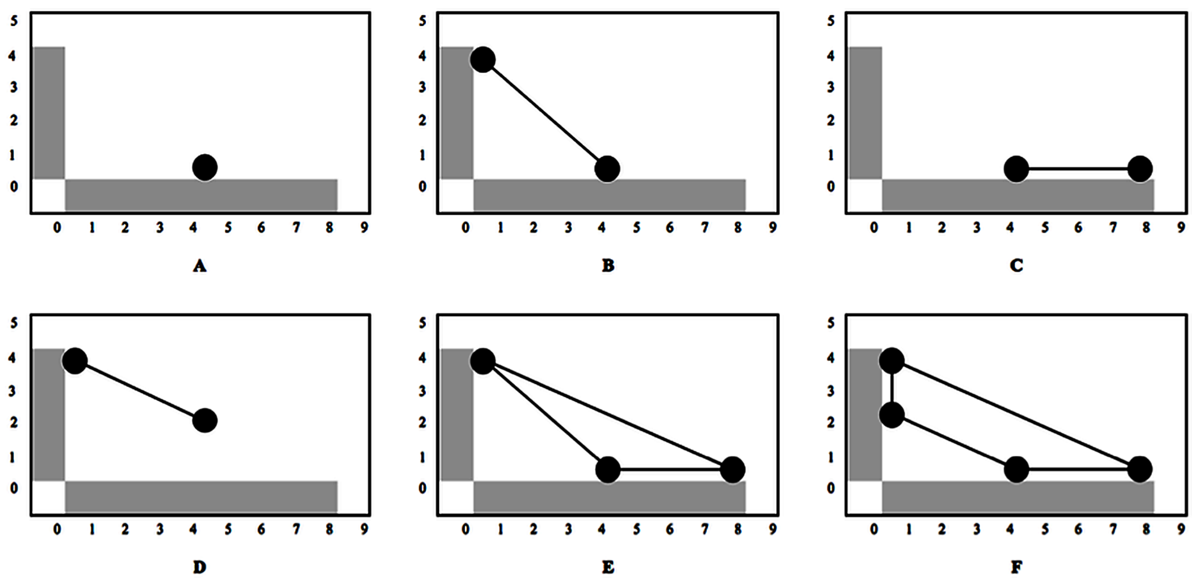


Figure 11. Overview of various visual patterns formed in the second task of Orthogonal condition when the hands were at the far ends; circles connected by lines are representative of the gaze-patterns.



By way of illustration, we again produce sample gaze-overlaid video screenshots. Figure 12 is the screen shot of a pattern depicted in 11A. Where the attentional anchor was a single point, we speculate that other points of interest were sufficiently near so as to be attended without shifting the gaze. These uni-point attentional anchors might also suggest a special role for those particular locations, such as a point of symmetry.

Despite numerous contextual variations between the Orthogonal and Parallel Bars tasks, across the two tasks gaze-path patterns demonstrated unique attention to the edges of the bars (the fingertips) as

Figure 12. Three consecutive frames of moving fingers in the Orthogonal Bar task; an apparent attentional anchor was formed on the midpoint of the horizontal bar. This location was maintained dynamically through to the extreme points.

well as the middle point of the longer bar. For example, Figure 11e might be viewed as the rotated version of the upright right triangle observed in Study 1.

3. Case Study

Lars (pseudonym) is a 14-year-old male participant. A pre-vocational-track student, he studies in the least theoretical level (basis). Lars begins the interaction with quick movements, switching his gaze between his fingertips. After two minutes his fingers slow down. He then succeeds in making the screen green. Still shifting gaze, he finds another green pair of finger locations, then another.

At 3:00, Lars' fingers are at the origin. He moves his fingers smoothly, generally keeping the screen green. By 3:15 an attentional anchor is formed on the forth corner of the rectangle (see Figure 8a). At 3:20 Lars stops halfway along the axes. Maintaining the same focal point, and all the while keeping the screen green, Lars slides his fingers back to the origin corner. After a break of several seconds, Lars resumes from the corner. This time he does not quite keep the screen green, possibly because he is attempting to move his fingers faster than before. He stops abruptly.

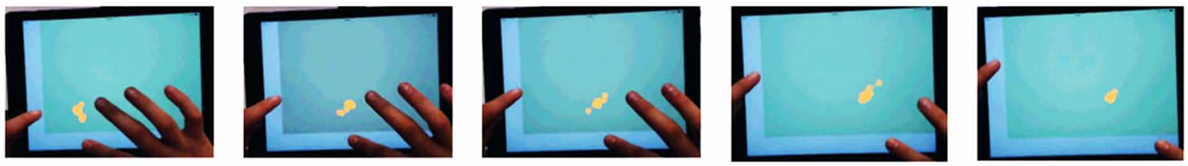
Lars: *In the corner you have to move them together, they have to be the same, so that… let's say that the angle has to be the same. (Sliding his index finger across the iPad screen, Lars depicts an imagined line connecting between points on the y- and x- axes where his two fingers had just been.) …so that they stay on one line.*

Exp.: *So you have to keep them in one line, uhum, ok. So then how do you move your fingers?*

5:30

Lars: *You put them on these two sides and move them on one line. They have to stay on one line. (Lars gestures on the screen. He forms a right triangle with the left finger, right finger, and the bottom-left corner. Lifting his fingers off the screen, Lars "expands" this triangle. His hand becomes the line between where his fingertips had been, and he moves that hand up and to the right, keeping it parallel.) Yes, because if you put one farther than the other one [it] becomes red again. (He returns his fingers onto the screen and moves them farther). Yes, with one I push harder than the other. I try let's say to keep it as good as possible, so you push with more pressure. (Lars moves both fingers, apparently applying greater pressure with his right finger along the horizontal line).*

Figure 13. Lars, a 14 years-old vocational-education student, uses an emergent attentional anchor to guide proportional bimanual coordination; he is keeping parallel the imaginary line between his fingertips.



Lars has determined an appropriate action plan of keeping constant the angle between his finger-to-finger line and the screen base. However his implementation of this action plan is not normative. Whereas he appears to have realized that the left-hand and right-hand fingers should move differently each along its respective axis, he currently assumes that this difference lies not in the dimension of speed but effort, as though greater effort in the right-hand motion would implement the plan of maintaining a constant angle. Applying greater downward force in the right hand, he plods along, not quite keeping the screen green.

At 5:50, after a brief rest, he attempts again, this time moving in green. A focal gaze point appears. It is not quite halfway between the left and right fingertips, but always up and to the right of that point, as though anticipating the prospective mid point; see Figure 13

Lars: *Yes, because with this one (y-axis) you should go down, and up, and this one (x-axis) stay on one line. Because you should for example really move at the same time and sometimes not, that is a bit odd because if you try to move at the same time and stay on one line, it becomes red, and so [then] you try to be a bit closer.*

Lars's strategy is to move his left hand up and down along its axis, with the right hand keeping parallel the imaginary diagonal between them. And yet he still believes that the hands should move at the same pace. From an educational perspective, this is a fascinating moment that constructivists view as a cognitive conflict that may instigate conceptual change. Lars is in transition. From an embodied perspective, we are intrigued that once again we witness an effective enacted solution that the learner cannot as yet articulate in mathematical forms. His mathematical knowledge seems to act as a bottleneck on his meaning making.

At 8:00 the grid lines appear. Again, Lars sets off from the corner, and we witness the same anchor points as before (Figure 13). The experimenter asks Lars whether he found the lines useful.

Lars: *Yes, I am now looking at these little blocks.*
Exp.: *Can you use them for example to find the distance between your fingers?*
Lars: *Yes.*
Exp.: *Do you have any idea now why the screen turns green sometimes and red some other times?*
Lars: *No.*
Exp: *Nothing? You can not tell anything about where your fingers are.*
9:42
Lars: *Maybe a line (gestures the line connecting the fingertips….On this (gestures to x-axis) I need to have more blocks further.*

Lars apparently realized that the hands' motion along their respective axes can be quantized with the grid and that, furthermore, the right-hand quanta are greater than the left-hand quanta.

Exp.: *Okay, how many more?*

Lars: *(looking at the y-axis) Four blocks on (y-axis), then I am at a 7.. It's about half. (His left-hand index is at 4, and his right finger is passed 7, near 8. (His gaze darts between the fingers. He continues moving, trying to count more blocks and showing how he still has to "go more blocks" on the x-axis than y-axis).*

Exp.: *Let's bring the numbers for you. (numerals appear along y-axis). Ok, so where was it that you saw a half?*

Lars: *(points to 7 on x-axis). Like look now: 1 on here (y-axis) and 2 on here (x-axis); here is 2 on here (y-axis) and 4 here (x-axis); now 6 here (x-axis) and 3 (y-axis); now 8 (x-axis) and 4 (y-axis); now 10 (x-axis) and 5(y-axis); now 18 (x-axis) and… (places right finger on 7 then 8 then 9; screen turns green) 9. It's the half.*

The appearance of the grid supported Lars to determine a quantitative rule governing the locations of the left and right hands. That said, it is not clear whether with the articulation of this multiplicative rule Lars ever revisited the earlier question regarding the hands' respective increments along the two axes (i.e. that the right hand proceeds two blocks for every 1 block the left hand proceeds).

Lars turned to the second task (Orthogonal Plus, without grid or numerals). He places his fingers at about a quarter of the way up along the vertical bar and half along the horizontal bar. He immediately finds a green. He then moves his fingers smoothly along their axes, keeping the bars green. His gaze is initially focused on the top of vertical bar. 12 seconds later, his gaze skips along a path to the middle of the horizontal line, the tip of the horizontal line, the top of the vertical line, and the middle of the horizontal line. The following occured after 48 seconds.

Lars: *Maybe this (points with a left finger on vertical bar) is half of this (points to the horizontal bar, while rotating his left thumb from the top of the vertical bar onto its extent along the horizontal bar.*

The experimenter asks Lars to move his fingers one more time from the start point to the end. Moving the bars smoothly in green, Lars's gaze shifts between the top of the vertical axis, the middle of the horizontal axis, and the end of horizontal axis. "Yes," says Lars, "it is the half."

CONCLUSION

These are exciting days for cognitive science. A new intellectual paradigm—a confluence of embodied, embedded, extended, and enactive perspectives on epistemology, cognition, and interaction—is questioning the field's fundamental assumptions while opening up new vistas of research (Kiverstein & Clark, 2009; Spackman & Yanchar, 2013; Menary 2015; Beilock 2015).

Educational researchers, too, are considering the implications of these radical conjectures for the theory and practice of fostering teaching and learning (Nemirovsky 2003; Lindgren & Johnson-Glenberg 2013; Pouw *et al.*, 2014). In particular, some educational researchers are turning to the literature on motor-action skill acquisition, such as dynamical-systems theory (Whiting 1993), kinesiology (Bernstein

1996; Newell & Ranganathan 2010), ecological psychology (Gibson 1977), and sports science (Chow *et al.* 2007), so as to illuminate the role of sensorimotor activity in the teaching and learning of disciplinary content, such as mathematical concepts (Abrahamson & Trninic, 2015; Abrahamson & Sánchez-García, 2016). The field's collective efforts are yielding interdisciplinary collaborations. Hutto and colleagues (Hutto, Kirchhoff & Abrahamson, 2015) are one example where philosophers of cognitive science partner with mathematics-education researchers in an attempt to ground epistemological conjectures in the empiricism of design research. It is to this line of investigation that this chapter has sought to contribute through the potential of eye-tracking technology.

The eye-tracking studies reported in this chapter sought to bring forth empirical data as evidence supporting the theorization of a hypothetical construct put forth in sports phenomena, namely an attentional anchor (Hutto & Sánchez-García, 2015). An attentional anchor is an emergent systemic relation between a goal-oriented sensorimotor agent and features of its ecology. As the agent attempts to accomplish routine tasks, its behaviors are predicated on, and regulated by, its perception of affordances for action. In the case of relatively complex tasks, such as those that demand new forms of bimanual motor-action coordination, the agent may gravitate toward behavioral routines that engage new, higher-order arrangements of features in the ecology. For example, in manipulating two objects simultaneously, the agent may find it advantageous to attend to the imaginary line inscribed between the two objects; the agent then orients on that line and manipulates it—the line is a *bona fide* phenomenological entity, even if a bystander may be completely oblivious to it. In turn, the attentional anchor may rise to the agent's consciousness through reflection. In the case of pedagogical intervention, reflection on attentional anchors can bring forth conceptual insights, such as understand the mathematical underpinnings of the emergent interaction routine that includes an attentional anchor.

We have furnished a combination of action-logging, eye-tracking, and clinical-interview data that appear to lend credence to the notion of an attentional anchor. In particular, we have demonstrated that emergent perceptual routines involving dynamically stable eye-gaze scan pathways appear to facilitate the agent's coordinated simultaneous manipulation of two independent objects. Although our research design was not geared to provide evidence for mathematical learning as such, our participants' struggles to reconcile cognitive conflicts and their quantitatively articulated inference lend support to, if not corroborate, earlier claims to didactical efficacy of embodied-interaction design (Nemirovsky *et al.*, 1998; Abrahamson & Lindgren, 2014; Abrahamson *et al.*, 2012). In particular, our Mathematical Imagery Trainer Extension, which is a variant on the Mathematical Imagery Trainer for Proportion (Howison *et al.*, 2011), enabled us to capture, document, and interpret embodied interaction as instrumental in the development of sensorimotor routines that were followed by mathematical insight.

By way of summary, we believe that we have put forth some empirical data as indications for the construct of an attentional anchor, both as an ontological innovation per se and as a genuinely effective means of accomplishing situated sensorimotor tasks. Moreover, we have laid out a learning trajectory that suggests the relevance of attentional anchors to mathematics learning.

A limitation of our research is that our data so far only speak for a correlational, not causal, relation between the attentional anchor and mathematical insight. In other words we have not provided conclusive evidence that would support a claim for the absolute necessity of attentional anchors as pre-requisite for learning about proportions. Even if we conceptualize mathematical reasoning as intrinsically situated and distributed over material implements, including the body, historical artifacts, cultural routines, and social interaction (Hutchins, 2010), further research would hone this intellectual program to pin down the cognitive function of sensorimotor routines. And yet it is heartening to speculate that we now have methodological tools to evaluate seminal theory of cognitive development, and in particular Piaget's hypothetical construct of reflective abstraction. It could be that attentional anchors are the very coordinations that Piaget posited as the pivotal transition into higher-order cognitive routines (Piaget, 1968; Abrahamson *et al.,* 2016).

IMPLICATIONS

As we consider implications of our research for pedagogical design and policy, we note that under the different task conditions of Parallel vs. Orthogonal orientation of the bimanual motion we witnessed a marked difference in time duration from the launching or exploration until the development of an attentional anchor. The differences in looking patterns and response times across conditions suggest that there are shortcuts to proportional reasoning in Orthogonal conditions compared to the Parallel condition. We conceptualize these findings as bearing on central debates of educational theory and practice. In the Parallel condition students tend first to focus on the vertical spatial interval between the two bars. They struggle to realize how this interval changes correlative to the bars' heights, and this struggle appears important in enabling the students to develop proportional reasoning as a grounded variant of additive reasoning (Abrahamson *et al.,* 2012). The Orthogonal condition is mathematically isomorphic to the parallel condition with respect to the proportional coordination of quantities, and yet it appears to present a different learning sequence, one that partially circumnavigates this struggle with the spatial interval. Future research might thus consider the hypothesis that shortcuts to motor-action coordination do not necessarily constitute best pathways to conceptual development.

More generally, our research is sensitizing our pedagogical engineering process by highlighting the cognitive consequences of nuanced design decision. Both the Parallel and Orthogonal conditions ultimately create for students opportunities to struggle with the core conceptual challenge of proportion, namely that the arithmetic relations among the numbers in a proportional relation are multiplicative not additive and that, therefore, measured differences among corresponding quantities are not equivalent. For example in the proposition 2:3 = 4:6 the differences within pairs are 1 and 2 respectively (or 2 and 3 across the equation sign). However we regard this relation, we are required to perceive 2:3 as "the same" as 4:6, even though "the differences are different" (Abrahamson, 2002). The Parallel condition orients users on the within-pair interval, and so they must cope with the changing of the 1 into a 2. The Orthogonal condition orients them on the between-pair differences, and so they must cope with inequality of 2 and 3. Thus both conditions instigate cognitive conflict as children initially bring to bear default additive frames of reasoning but are then thwarted by a multiplicative system into acknowledging, reconsidering, and adapting their tacit assumptions.

But just how students come to acknowledge, reconsider, and adapt their tacit assumptions often remains obscure to scientists of learning. And yet the research design adopted in our studies enables us to form new conjectures with respect to these obscure processes. For example, Lars, our case study, did not struggle with the interval between the quantities per se but rather with the comparison of correlated increments: he assumed his hands should move at the same pace, even as he realized that his right hand had somehow to move more intensively than the left hand. Initially he implemented this pre-figurative notion of enhanced intensiveness by applying greater physical force with the right hand. Only once the grid was introduced and utilized as a frame of reference could Lars re-express his pre-figurative feeling of intensiveness in the form of enhanced lateral motion, whereby the right hands advanced two squares for each of the left hand's single squares.

We therefore are now in the position to ask new and more general research questions with respect to the relation between technology-enabled learning activities and their cognitive outcomes for mathematical content learning. These questions respecting potential tradeoffs in conceptual development processes are of immense importance for the articulation of evidence-based general heuristics for the design of interactive mathematics learning environments.

We are particularly excited by the new vistas on human learning that eye-tracking has opened to us. In more than one sense, eye-tracking is enabling us to make visible that which is invisible.

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