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ORIGINAL ARTICLE

Teaching with embodied learning technologies for mathematics: responsive teaching for embodied learning

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Abstract
As technologies that put the body at the center of mathematics learning enter formal and informal learning spaces, we still know little about the teaching methods educators can use to support students' learning with these specialized systems. Drawing on ethnomethodology and conversation analysis (EMCA) and the Co-Operative Action framework, we present three multimodal ways that educators can be responsive to learners' embodied ideas and help them transform sensorimotor patterns into mathematically significant perceptions. These techniques include (1) encouraging learners to use gesture to express and reflect on their ideas, (2) presenting multimodal candidate understandings to check comprehension of learners' embodied ideas, and (3) co-constructing multimodally expressed embodied ideas with learners. We demonstrate how these techniques create opportunities for learning and discuss implications for a multimodal, embodied practice of responsive teaching.

Keywords Embodied learning · Digital technology · Responsive teaching · Gesture · Ethnomethodology · Conversation analysis · Embodied cognition

1 Introduction

Over the last 20 years, the field of mathematics-education research has embraced views of learning mathematics as an enactive, embodied process (Abrahamson & Trninic, 2014; Alibali & Nathan, 2012; de Freitas & Sinclair, 2013; Nemirovsky et al., 2014; Radford, 2014). During this time, technologies deliberately incorporating the body into mathematics explorations have proliferated. Swiftly evolving forms of human–computer interaction, such as augmented reality and haptic interfaces have created a number of unique ways to enlist bodily activity in mathematics learning. However, despite this influx of embodied learning technologies, the roles mathematics educators (e.g., teachers, tutors, museum personnel, and parents) can play to support learning with these technologies are frequently underspecified. In particular, we know little about how educators can foster mathematics learning through technology-enabled kinesthetic and tactile experiences.

In this paper, we identify and characterize teaching practices for productively engaging with learners' embodied ideas when they are using embodied mathematics learning technologies. We focus on communicative strategies that rely on multimodal interactional resources such as gesture. Inspired by responsive-teaching and clinical-interviewing techniques, our analysis illuminates three strategies for eliciting and engaging with learners' multimodally-expressed, embodied ideas: (1) explicitly encouraging learners to gesture and looking for discrepancies between gesture and speech; (2) providing multimodal candidate understandings of learners' embodied ideas; and (3) co-constructing multimodal, embodied ideas together. We use the case of tutors and students working with the Mathematics Imagery Trainer for Proportions (MIT-P) to illustrate these strategies.

2 Supporting technology-enabled embodied learning experiences for mathematics

Numerous studies have established that mathematical thinking and learning emerge from and through embodied experiences in the world (Alibali & Nathan, 2012; de Freitas & Sinclair, 2013; Lakoff & Núñez, 2000; Nemirovsky et al.,

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2014; Radford, 2014). Although there is no unified theory of embodiment informing mathematics education (Alibali & Nathan, 2012; de Freitas & Sinclair, 2013), many agree that the mind is irreducible to the brain, and that all forms of cognition, including mathematical cognition, are fundamentally entangled with the human body and material environment. From this perspective, mathematics is a culturally-determined form of sense-making made possible by the body's perceptuo-motor system coupled with the physical world (Lakoff & Núñez, 2000). In response to this embodied turn, educational designers have enlisted novel technologies to engage students in dynamic physical experiences that can provide grounding and insight into mathematical ideas. Some examples include creating "walking-scale" geometrical figures with GPS trackers (Hall et al., 2014), operating a life-sized etch-a-sketch like instrument to explore parametric functions (Nemirovsky et al., 2014), and using motion tracking to explore ratio and proportion through bimanual motion (Abrahamson & Trninic, 2014).

When learners use embodied-mathematics learning technologies, many of their discoveries—for example, the patterns they notice—pertain to kinesthetic and tactile sensations they perceive through task-oriented movement (Abrahamson & Trninic, 2014; Nemirovsky et al., 2014). These new ways of moving and perceiving are *embodied ideas* that may not be readily translatable into words. To convey these experiences to others, students often express them *multimodally* using rich configurations of heterogeneous semiotic resources (C. Goodwin, 2000), notably demonstrative action with interfaces, gesture, gaze, and talk (e.g., Nemirovsky et al., 2014). Communicating these experiences provides opportunity for reflection which can lead to mathematical insight (Shvarts, 2018). However, learners' embodied experiences and their attempts to represent these experiences do not always automatically result in mathematical understandings. Instead, learners need assistance organizing their embodied experiences with the cultural forms (Saxe et al., 1996) of mathematics (e.g., definitions and sign systems).

Vygotsky argued that *spontaneous* interpretations of experience can be interconnected with *academic* ways of organizing those experiences through social interactions with more culturally-competent members of society (1986). However, he didn't specify exactly how. From Goodwin's perspective, learners' technology-mediated embodied experiences and their first attempts at representing these experiences comprise a *substrate* (2018) that we believe can be cultivated into robust, disciplinary understandings of mathematics through guided reflection, negotiation, and signification with more-capable others (tutors, teachers, parents, museum educators, etc.; (Abrahamson & Trninic, 2014; Flood, 2018; Shvarts & Abrahamson, 2019). When culturally-competent others and learners negotiate divergent interpretations of experiences (Wertsch, 1984), zones of proximal development (Vygotsky, 1986) emerge that make socialization—including mathematical socialization—possible. We follow Wertsch's recommendation that to understand how zones of proximal development emerge, we must focus on the fine details of adult–child interactions.

To date, there have been very few examinations of how educators can best facilitate learners' reconciliation of their *embodied* experiences with the academic cultural forms and practices of disciplinary mathematics. A key aspect of successful guidance in technology-enabled embodied-mathematics learning technologies is an educator's ability to attune to the learner's movements, idiosyncratic perception, and interpretation of the experience at any given moment (Abrahamson & Trninic, 2014; Flood, 2018; Shvarts & Abrahamson, 2019). When educators recognize the disciplinary potential in learners' ways of moving, perceiving, and expressing their embodied experiences, opportunities arise to connect learners' embodied ideas with mathematical ways of organizing those ideas. These connections are a critical part of how technology-enabled embodied experiences can ground new mathematical understandings. In order to investigate how these connections are made, we turn to responsive teaching and clinical interviewing to understand strategies educators have at their disposal for attending to and engaging with learners' ideas.

3 Guidance from responsive teaching and clinical interviewing

In mathematics and science education, a number of scholars have documented ways that educators attend and respond to learners' ideas (for a recent volume see Robertson et al., 2016). This collection of practices, known as *responsive teaching,* involves valuing and eliciting students' thinking, engaging with it, and using it to fundamentally shape the course of instruction (Ball, 1993; Ball, Lubienski, & Mewborn, 2001; Pierson, 2008; Saxe, Gearhart, & Seltzer, 1999). Responsive teaching involves (1) drawing out, responding to, and working with aspects of ideas that have potential disciplinary value or substance (Coffey et al., 2011); (2) finding ways to connect these ideas with cultural forms of science and mathematics; and (3) engaging in ongoing *proximal formative assessment* (Erickson, 2007) by continuously monitoring students' ideas to adapt instructional guidance in the moment. For example, in the famous Sean numbers tape, after eliciting student ideas, Ball recognizes that an (incorrect) rule student Sean shares about odd and even numbers provides an opportunity for the classroom to examine and evaluate mathematical definitions (Ball, 1993). Mathematics classrooms where teachers are responsive to learners' ideas lead to increased mathematical learning (Pierson, 2008; Saxe et al., 1999).

A variety of strategies educators use to be responsive to learners' ideas have been identified. These include eliciting, probing, summarizing, expanding, reformulating, reflecting on, offering interpretations of, clarifying, or highlighting parts of the thinking learners share (Jacobs & Empson, 2016; Lineback, 2015; Pierson, 2008). However, the majority of previous studies have focused on educators' *verbal* forms of responsiveness to students' *verbally* expressed ideas and written work. To date, only a small handful of studies have investigated how educators are responsive to learners' embodied ideas expressed through gesture and bodily performances. Flood et al. (2015) proposed that responsive teaching is a multimodal, embodied phenomenon, and demonstrated how instructors can use students' gestures to identify when students have productive ideas but lack words to describe them. Others have shown that repetition and reformulation of students' gestures through multimodal re-"voicing" serve a variety of production functions, including helping learners connect gestures with mathematical terminology (Alibali et al., 2019; Arzarello et al., 2009; Flood, 2018; Shein, 2012). Our present study aims to further examine multimodal, embodied dimensions of responsive teaching.

Clinical interviewing techniques have much in common with responsive teaching, and also provide methods for attending and responding to students' ideas. The central goal of the clinical interview, according to Ginsburg (1997), is to deeply understand and build off children's knowledge and thought-processes. In his famous book *Entering the Child's Mind*, Ginsburg describes a number of techniques for eliciting and engaging with children's thinking that include probing and questioning the thinking children share, helping children introspect and reflect, establishing children's competence, and offering counter-suggestions to test children's conviction in certain assertions. Perhaps unsurprisingly, clinical interviews are often sites of teaching and learning (diSessa, 2007; Ginsburg, 1997; Saxe et al., 1996). Ginsburg (1997) and diSessa (2007) both argue that by reflecting on experiences and introspecting, clinical interviews provide opportunities for interviewees to clarify their thinking and make discoveries. However, like responsive teaching strategies, clinical interviewing techniques for eliciting, responding, and engaging with children's thinking do not often address how to engage with children's *multimodally-expressed embodied* ideas.

4 Research questions

In this study, our goal was to understand how educators can be responsive to learners' multimodally-expressed embodied ideas and how this responsiveness can facilitate mathematical discovery in technology-enabled embodied learning environments. Inspired by Ginsburg, we wondered, what does it look like to enter the child's *embodied* mind? More specifically, we asked: *How can instructors attend and respond to learners' embodied ideas to support mathematics learning with embodied technologies?* Through careful analysis of interactions between tutors and learners working with an embodied mathematics learning technology, we examine how previously identified *verbal* responsive teaching and clinical interviewing strategies like (1) eliciting and probing ideas, (2) summarizing and offering interpretations of ideas, and (3) building from and elaborating ideas can be adapted for engaging with learners' multimodally-expressed embodied contributions.

We draw on ethnomethodology and conversation analysis (EMCA), which provide a methodological and theoretical approach for understanding social order as a contingent, local production of the interactional work of participants (Mondada, 2019). EMCA attempts to understand how social realities are co-accomplished moment-by-moment, by investigating the fine details of participants' efforts to build, monitor, maintain, and repair intersubjective meanings (Schegloff, 1991). Participants use a variety of practical methods and resources to publicly display their interpretations of what they think is going on in order to coordinate their activity together (Garfinkel, 2002). Instead of applying exogenous categories, EMCA takes a particular interest in how participants themselves make sense of and organize these activities [e.g., having an argument, closing a phone call (C. Goodwin & Heritage, 1990)]. Claims are grounded in the analyses of situations that participants make available to one another (ibid).

We are also informed by Goodwin's Co-Operative Action framework (2018) which elaborates EMCA by focusing attention on the ways participants take up and transform each others' multimodal contributions to build meaning together. Each utterance a participant contributes is a *substrate* that can be broken down, reused, and reshaped in the process of co-constructing new ideas from old ones (ibid). The Co-Operative Action framework is especially useful for understanding the interactional work tutors and students use to build on embodied discoveries and transform them into mathematical meanings together. Combined, EMCA and the Co-Operative Action framework help us understand how tutors and learners negotiate what is there to be experienced and what is meaningful about those experiences.

Both EMCA and Co-Operative Action advocate for paying attention to *all* of the methods and resources participants recruit and treat as meaningful, including gesture, gaze, demonstrative action with objects, body position, prosody, talk, and many others (C. Goodwin, 2018; Mondada, 2019). Accordingly, meaning-making is considered to be a process distributed across individuals, their bodies, and the physical environments they are embedded in. As a result, the combination of these approaches is well suited for studying embodied mathematics learning as an interactional achievement mediated by both social and environmental factors.

5 Methods

5.1 Study context

Our data come from a video corpus of 70-min task-based tutorial interviews featuring 23 Grade 4–6 students and four adult tutors (university mathematics education design researchers). Learners worked with a technology-enabled embodied learning device for mathematics, called the Mathematics Imagery Trainer for Proportions (MIT-P; Abrahamson et al., 2012) at an urban California school. The MIT-P provides an interactive experience for users to explore ideas about ratio and proportion through bimanual motion. To operate the device, users lift and lower two independent, hand-held remotes that move cursors vertically on a computer screen (see Fig. 1). The device generates green feedback when the cursor heights (measured from the bottom of the screen) embody a set, concealed ratio (e.g., 1:2 in Fig. 1). When the cursor heights do not fulfill the ratio, the screen turns red. Students are asked to find strategies to make the screen green when given various ratios (e.g., 1:2, 2:3, 3:4), and are given cursors, a grid, and numbers.

The MIT-P was designed to respond to students' well-documented conceptual and procedural difficulties with rational numbers that often persevere through middle school and beyond (Clark et al., 2003; Lamon, 2007). Overcoming ingrained additive forms of reasoning and embracing multiplicative forms presents a challenge (Karplus et al., 1983; Van Dooren et al., 2010). The MIT-P's bimanual movement is polysemous, creating opportunities for students to explore and integrate additive and multiplicative schemes and, more broadly, to assimilate conceptually complementary models of proportionality through a variety of entry points. For example, guided activities occasion opportunities for students to experience alternative conceptualizations of the multiplicative factors inherent to proportional progression: either as *scalar* (in 2:3 = 4:6, 2 doubles to 4, so 3 doubles to 6) or as *functional* (the 2 expands one-and-a-half times into 3, so 4 should likewise expand one-and-a-half times into 6; Vergnaud, 1994). Learners may also discover co-variation, realizing that the absolute difference between ratio numbers (i.e., between *a* and *b* in *a:b*) in a proportional progression increases as the numbers increase (a delta of 1 in 2:3, 2 in 4:6, 3 in 6:9, etc.). By enacting the two parallel, vertical, kinetic trajectories through space, students can also experience proportional progression (i.e., the iteration of an *a:b* ratio pair) as two rates or speeds (e.g., the 2:3 ratio progresses monotonously upward at 2 units-per-step [left hand] per 3 units-per-steps [right hand], where the latter motion is experienced as faster).

Teaching in these tutorials consists of guiding students towards discovering different strategies for turning the screen green, reflecting on these strategies, and mathematically signifying and generalizing embodied experiences. With the tutors' guidance, learners develop sophisticated quantitative methods for describing the patterns and forms of movement they perceive. Interviews were semi-structured and the specific questions and guidance tutors offered was responsive to (i.e., dependent on) the strategies and patterns students shared. For more details on tutorial protocol, please see Abrahamson et al. (2014). Overall, the tutorial approach employed many verbal practices from responsive teaching and clinical interviewing, as well as many novel embodied responsive practices that have not yet been examined closely.

5.2 Data analysis

Our interest in responsiveness to students' multimodally-expressed embodied ideas emerged out of a broader project to characterize the range of MIT-P tutorial practices to inform the design of a virtual pedagogical agent. To identify and characterize ways tutors were responsive to learners' embodied ideas, we reviewed the MIT-P task-based tutorial sessions and looked for forms of engagement inspired by practices from responsive teaching and clinical interviewing, including eliciting, probing, summarizing, interpreting, expanding, reformulating, and highlighting learners' ideas, as well as encouraging learners to reflect on, clarify, justify, or elaborate their contributions. We then created a collection of cases where: (1) learners used multimodal semiotic resources (especially gesture and/or demonstrative action with the device) to express an embodied idea; and where (2) tutors engaged with students' multimodally-expressed embodied ideas either verbally or with other

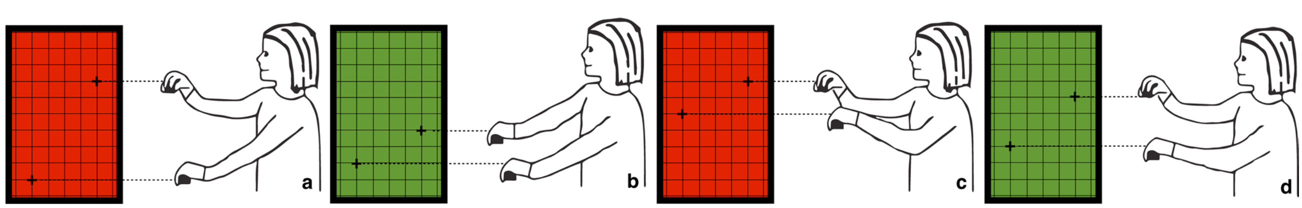

Fig. 1 When the Mathematics Imagery Trainer for Proportion (MIT-P) is set to a 1:2 ratio, the screen is green when the right hand remote is twice as high as the left hand remote (**b**, **d**); otherwise it is red (**a**, **c**) (color figure online)

multimodal semiotic resources. Our unit of analysis consisted of these interactional exchanges embedded in longer sequences, where students and tutors attempted to negotiate the significance of a particular discovery. We examined these sequences to understand how educators can be responsive to learners' embodied ideas and how this responsiveness can facilitate mathematical discovery with technology-enabled embodied learning environments.

From our collection of cases, three previously undescribed practices for engaging with learners' embodied ideas emerged. We selected examples of each of these three forms for their capacity: (1) to clearly illustrate the practices; and (2) to demonstrate the pedagogical utility of these practices. Each example interaction was subjected to microanalysis (Erickson, 1992) using the software ELAN (Lausberg & Sloetjes, 2009). ELAN makes it possible to separate multimodal utterances into parallel streams of co-occurring gesture, talk, gaze, and other resources to examine how modalities are coordinated and overlap with one another. Using ELAN, each interactional exchange was transcribed using the notational system developed for conversation analysis by Jefferson (2004). A key to the transcript annotations appears in Appendix 1.

6 Findings

6.1 Eliciting and probing contributions: Encouraging students' multimodal exploration of ideas and responding to gesture-speech mismatches

In responsive teaching and clinical interviewing, teachers and interviewers often look for ways to provide opportunities for learners to share and reflect on their reasoning (Ginsburg, 1997; Robertson et al., 2016). These opportunities make it possible for learners to clarify and elaborate their own ideas, as well as for educators to build on these ideas. In the case of embodied learning technologies, learners can use the technology to demonstrate a strategy, but sharing the strategy "offline," without feedback from the technological environment, can present important opportunities to reify and signify observations with culturally meaningful forms. Students often use gesture and speech to share these observations, and it can be a challenge to capture descriptions of bodily motion and dynamic spatial information in words. Both in and outside of embodied learning environments, non-verbal aspects of learners' explanations can contain discrepant information when compared to verbal aspects (e.g., Alibali & Goldin-Meadow, 1993). Thus, a key approach for being responsive to learners' embodied ideas involves finding ways to elicit their ideas in modalities beyond speech and being on the lookout for ways gesture is non-redundant to or mismatched with speech.

We demonstrate the pedagogical potential of encouraging learners to gesture and attending to gesture-speech mismatches in Excerpt 1 (The full excerpt appears in Appendix 2). Ben, a middle school student, is working with the MIT-P, guided by two tutors, Don and Dor. Currently, the MIT-P is set to a 1:2 height ratio and Ben has figured out how to move his hands to keep the screen green. Just before the excerpt begins, Ben shares a theory for producing green feedback that is difficult to interpret: He says, "my right hand is sort of the pinpoint sort of thing so, and then to keep it green you have to even them out I would say. You have to make the right hand go higher um than the left hand." As he says "pinpoint sort of thing," Ben, still holding the remotes, waves his right hand in the air, and as he says "even them out" he waves each hand up and down. It is not immediately apparent what "sort of the pinpoint sort of thing" or "even them out" could mean from Ben's speech or gestures. He hedges with "sort of" and "sort of thing," displaying that his words only loosely capture what he means. Speakers often use hedging language to display an *epistemic stance*—the level of certainty a speaker has with respect to a spoken proposition (Ochs, 1996). Ben's vague hand-waving gestures also contribute to this display of uncertainty.

In Excerpt 1.2, Dor, the tutor, is responsive to Ben's idea and his display of uncertainty. He encourages Ben to use his hands, stretched out flat without the remotes, to elaborate on this discovery (Fig. 2). Ben raises his flat right hand about twice as fast as his left hand, so that his right hand continuously remains about twice as high as his left hand (Fig. 3). Ben's gestures precisely and accurately capture the movement that produces green feedback. He again says that the right hand has to move faster than the left. As he continues, he trails off as he lowers his hands in the same height ratio (Fig. 4). Ben's gaze at his own gestures as well as his trailing

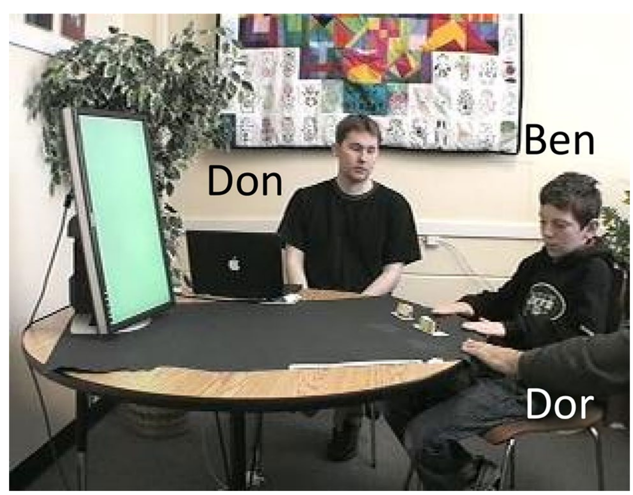

Fig. 2 Dor encourages Ben to use flat hands to explain his idea

off speech suggest he is trying to make sense of what is going on with his gestures (Crowder, 1996) by observing his own movements. He displays to the tutors that he is "doing thinking."

After a pause, Ben raises his hands again and describes his hands as "even apaced" (1.2 20). He looks to Dor, possibly to gauge his reaction. Then Ben again trails off and starts spinning his right hand in the air in silence for three seconds (Fig. 5). Ben's hand-spin displays his engagement in the activity of searching for the right words (M. H. Goodwin & C. Goodwin, 1986). Visible displays of word searches often invite listeners to participate by supplying the missing terms (ibid). However, Dor and Don respond with silence, making space for Ben to continue his explanation independently. When listeners withhold assessments, speakers tend to elaborate.

Excerpt 1.2 Ben elaborates his original idea

```
Ben:  Mmmm [I would start the right hand=
           [((Fig. 3: Ben slowly raises his hands so that his right hand
             continuously remains about twice as high as his left hand))
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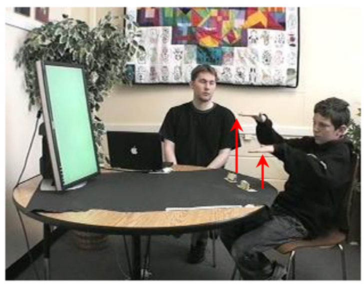

Fig. 3

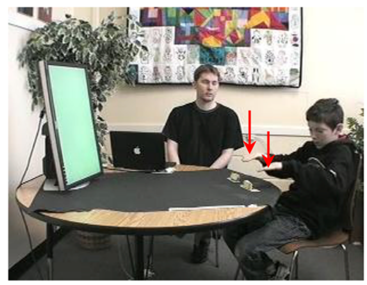

Fig. 4

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Dor: [Mhmm
Ben: [=going a little faster than the- um left hand
Dor: Uhuh
Ben: Then you [keep them (1.8)
              [((Fig. 4: Ben lowers his hands so that his right hand continuously
                 remains about twice as high as his left hand))
        [going even a-paced
        [((Ben raises his right and left hand so that his right hand stays
           higher than left))
        you have to keep them in- [(3.0)     tch        um- same pa::ce basically
                                  [((Fig. 5: Ben spins his right hand in the air))
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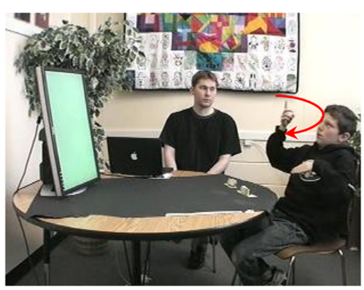

Fig. 5

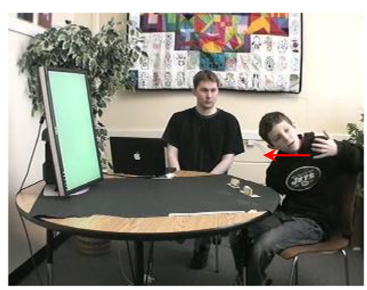

Fig. 6

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       [sort of
       [((Fig. 6: Ben moves his hand, torso, and head side to side))
Dor: Uhuh
Ben: Yeah you have to keep them [going the same pace
                                [((Ben raises his hands so that his right
                                   hand rises above his left hand))
       [so it's really complicated
       [((Ben lowers his hands so that his right hand continuously remains higher
          than his left hand))
       [when you come back down so::o (1.6)
       [((Ben picks up the remotes and raises them so that the screen stays
          green, then lowers them))
```

Without help, Ben replaces "even apaced" with "same pace," (1.2 23) but he shifts his body and waves his hand side-to-side (Fig. 6), using multimodal resources to display his uncertainty about the description. Dor responds with a continuer, "uhuh" (1.2 27). Continuers from listeners demonstrate to a speaker that the listener expects that an "extended unit of talk" is in progress and incomplete, and that the original speaker will keep talking (Schegloff, 1982). After Dor's uhuh, Ben restates his idea, this time without visible hesitation or disfluency (1.2 28).

Notably, Ben's physical demonstration in gesture (Figs. 3 and 4) and his verbal description are seemingly at odds from a mathematical point of view. Ben's hands are not moving at the same pace at equal speeds, despite his verbal description to this effect. By attending to Ben's gesture, we can tell that Ben must not mean "same pace" the way adults would, meaning both hands going at the same speeds. Thus, from an adult's point of view, Ben's gestures and speech appear contradictory or mismatched.

In Excerpt 1.3, Don is responsive to the mismatch in Ben's gesture and verbal description. He probes, signaling a problem in his understanding (Schegloff, 1991) and initiates a repair from Ben (1.3 37). This move displays an evaluation of Ben's idea—that it doesn't make sense—and marks it as in need of clarification. As a result, Ben again expands on his idea, co-opting and modifying the flat palm gesture Dor originally suggested. Using this modified flat palm gesture, he evokes an analogy to help describe what he meant by "same pace." He says you would have to keep the remotes going at the same pace, like cars, (1.3 38–39) and then demonstrates by moving his right hand horizontally in front of his left hand, with the right hand moving faster (Fig. 7). He calls this the "same speed limit" (1.3 48) which again seems to be another mismatch from the tutors' point of view. However, he then elaborates, saying that one remote would be going twenty and one would be going fifty (1.3 50–53). He then performs the gesture diagonally at 45 degrees, visually blending the horizontal-car situation with the vertical-MIT-P-remotes situation (Fig. 8).

As Ben describes this car analogy, he looks at Don to track Don's understanding. His speech contains fewer long pauses than before. This suggests that he is now using this analogy to illustrate for the tutors the "same pace" idea that he already came up with (Crowder, 1996): that the hands each move at their respective constant velocities, which in Ben's words is the "same pace." Ben's multimodal analogy with gesture makes it clear that by "same pace" Ben does not mean that both hands move at the same speed. Had the tutors corrected Ben's original notion of "evening them out" or his later idea of "same pace," they would have denied him the opportunity to explore this productive analogy.

Excerpt 1.3 Ben elaborates his idea further with an analogy

```
37  Don: What're you saying=
38  Ben: =You see if you keep them in the same like pace like
39       [same like for a car
40       [((Ben spins his right hand in the air))
41       like [if you wanted to do this with a car
42            [((Fig. 7: Ben moves his flattened right hand forward out in front of him
43              horizontally with his left hand trailing it, and repeats the
44              motion))
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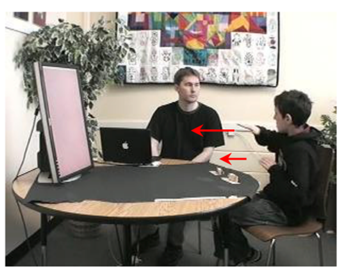

Fig. 7

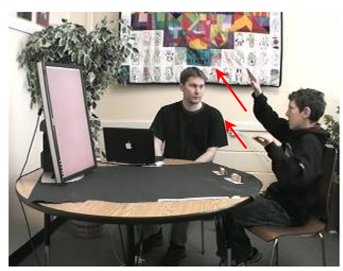

Fig. 8

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45  Ben: [It would sort of be
46       [((Ben moves his right hand forward horizontally in front of him in a
47         pinch shape))
48         [the same (.) speed limit
49  Don:   [Mhhm
50  Ben: [Like this one is going twenty
51       [((Ben points to the left remote and then the right remote))
52  Don: Okay
53  Ben: [And this one is going fifty and they have to keep on going s::o
54       [((Fig. 8: Ben moves his flattened right hand forwards and up 45 degrees from
55         his body with his left hand following, and then moves them both down))
56  Don: Okay
```

Overall, encouraging Ben to gesture and attending to his gesture-speech mismatch provided an opportunity for him to come up with a new, important mathematical idea and analogy for it: (1) both cursors are moving at a constant speed; and (2) both are moving at two *different* constant speeds relative to each other. He correctly links the case of the remotes to cars traveling at different constant speeds. Ben drew spontaneously on an experiential resource that enabled him to conceptualize his embodied experience of the intensive quantity speed as "smooth" motion, prospectively grounding rate, which is often perceived as piecemeal *a/b* "chunky" iterations of articulated quotas (Castillo-Garsow et al., 2013; Thompson, 1994).

Ben's idea emerged from his exploration and reflection on his own gestured movements. These gestures, *elicited* by the tutors, became a *substrate* for Ben to build from. In addition, by *probing*, the tutors were able to make sense of the apparent mismatch between Ben's speech ("same pace") and gesture, so that Ben's idea could be fully understood. Encouraging learners to gesture when they have shared ambiguous or difficult-to-interpret information in speech can help educators gain a better understanding of the ideas learners are trying to convey. When instructors have a better sense of a student's ideas, they can more effectively perform proximal formative assessment (Erickson, 2007) and adjust support in-the-moment to best suit the student's needs. Overall, being encouraged to "explain an idea in your own *hands*" provides productive opportunities for reflection on embodied ideas: through this reflection, learners are able to reformulate and elaborate their initial utterances in ways that demonstrate new clarity or specificity, and sometimes they are able to make new discoveries/realizations like Ben. Thus, encouraging learners to gesture and attending to gesture-speech mismatches is a powerful technique for responsive teaching with embodied mathematics learning technologies.

6.2 Summarizing and interpreting contributions: multimodal candidate understandings of students' ideas through gesture

An essential mechanism of responsive teaching and clinical interviewing is understanding and clarifying learners' ideas by offering summaries or interpretations of them. Before an educator can effectively take up or build on an idea, they must appreciate what a learner is trying to convey. In particular, educators must seek out and identify disciplinary substance in learners' ideas to determine how this idea might serve as a seed or bridge towards a more robust disciplinary understanding. However, appreciating what children mean is often challenging. As we saw in the previous example, learners may use words in unconventional ways (Ginsburg, 1997, also warns of this) or present apparent mismatches in gestures and their speech. Understanding *embodied* ideas can also be especially challenging because kinesthetic and tactile experiences are often difficult for learners to express in words. Learners frequently rely on ambiguous indexical language (e.g., this, that, here etc.) and gesture.

In conversation, there are a number of methods participants use to detect and overcome problems in hearing and understanding (Schegloff, 1991). One technique is giving *candidate understandings* where participants say something to check their comprehension of the meaning of what was said before (Heritage, 1984). In response to an utterance, a participant provides a *demonstration* of their understanding of what was meant for the first speaker to confirm or reject (ibid). Consider the following example adapted from Heritage (1984):

```
  Speaker 1: If you wanna come on over early c'mon over.
->Speaker 2: Ah for dinner you mean?
  Speaker 1: No not for dinner.
```

The line marked with the arrow is a candidate understanding from Speaker 2. Speaker 2 interprets Speaker 1 as offering a dinner invitation. However, this displayed interpretation is rejected by Speaker 1. A second speaker's utterance becomes a candidate understanding when the original speaker treats the second speaker's response *as* a candidate by ratifying or rejecting it. In conversation analysis, candidate understandings are mostly studied as a verbal phenomenon, however, they can also draw on other modalities such as gesture.

In Excerpt 2 (Appendix 2), we present an example of how tutors can use *multimodal* candidate understandings to check their interpretations of learners' multimodally-expressed embodied ideas in technology-enabled embodied learning environments. In the example, Dev, a tutor working with a student named Tal, uses gesture to check in several times about a discovery Tal had made. By checking his understanding of Tal's idea, Dev is able to propose ways of exploring it further that lead Tal to revise her initially inaccurate observation.

Tal makes her discovery while she operates the left remote and Dev operates the right remote (Excerpt 2.1). She exclaims that "it's about the same distance" (2.1 01) using an ambiguous indexical ("it's") and pointing between the remotes they each hold. Dev, who was not looking at Tal's gesture, displays a problem understanding her, and asks her to clarify (2.1 03). To repair her response, Tal uses gesture to trace an imaginary diagonal line between the remotes (Fig. 9), providing another indexical description ("this to this," 2.1 07). She appears to be directing Dev's attention to the spatial interval between the two remotes. She elaborates, calling the interval a "diagonal li-," but her speech trails off, and she opens her hand and waves it back and forth (Fig. 10). Tal's indexical and incomplete language, even when coupled with the gesture, are ambiguous and difficult to interpret. In addition, her hand-waving gesture and trailing-off speech serve as a hedge that displays uncertainty with the discovery and its description. She may even be trying to signal a willingness to abandon the idea.

Excerpt 2.1

```
Tal: [I- it's about the same distance
     [((Tal points between the remotes she and Dev hold))
Dev: What's the- What's about the same distance
Tal: [Fr-
     [((Tal points between the remote Dev holds and the remote she is
       holding twice, tracing a diagonal line))
```

Fig. 9

Fig. 10

```
     um, [this to this:
         [((Fig. 9: Tal again points between the remote Dev holds and the remote she
           is holding, tracing a diagonal line))
((Tal points again between the remote Dev holds and the remote she is
  holding, tracing a diagonal line))
     it's a- diagonal [li-
                      [((Fig. 10: Tal opens her hand and waves it back and
                        forth))
Dev: Oh so sort of [from here to here↑
                   [((Fig. 11: Dev points on the screen between the two
                     cursors))
```

Fig. 11

```
Tal: Yeah and notice like if you move one of them=
Dev: =Mhmm
Tal: The- the color changes↑
Dev: Mhmm so if you keep yours still [and I move mine
                                     [((Dev moves his cursor up and down and
                                       the screen turns red))
     [the °color changes°
Tal: [Yeah
(1.0)
Tal: Oh! And that's why [it was flashing when I was moving it up and down
                        [((Tal moves her cursor up and down the screen making
                          it flash red))
     because like I had touched (.) the sp- [like the space=
                                            [((Tal waves her hand in the air
                                              between the remotes))
Dev: =Mhmm right
```

the two cursors on the screen that corresponds to the interval between the two remotes (Fig. 11) while verbally describing it indexically ("here to here" 2.1 15). This allows Dev

Dev is responsive to Tal's multimodally-expressed embodied idea and her display of uncertainty. He takes up Tal's idea by multimodally demonstrating his interpretation of it—a candidate understanding—using his own gesture and mirroring Tal's language: He traces a diagonal between to avoid introducing his own adult perspective on how to describe this phenomenon, leaving it open for Tal to specify. He also creates an opportunity for Tal to see and reflect on her own embodied idea in the context of the cursors on the screen. In addition, Dev also mirrors Tal's hedging language,

Fig. 12 Dev traces a line between his remote and Tal's remote

using "sort of," signaling that he is not an authority on this phenomenon, and giving Tal agency and space to elaborate.

Tal immediately ratifies Dev's multimodal candidate understanding of her discovery ("Yeah," 2.1 18). However, Tal's hypothesis that the space between the remotes/cursors stays the same doesn't hold up empirically, since the distance varies depending on the pair of numbers (e.g., 1:2 vs. 5:10). However, as their conversation proceeds, Dev continues to focus Tal's attention on exploring this incorrect but productive idea. During this time, whenever Tal refers to this interval, Dev uses gesture to provide a multimodal candidate understanding, and asks her if he understands what she means. He uses gesture to provide his interpretation, pointing between the remotes (Fig. 12), and later pointing between his hands (Fig. 13a, b). Each time, he is (1) topicalizing the interval Tal describes and marking it as something important and worth pursuing; (2) presenting an opportunity for Tal to clarify or correct his interpretation of her idea; and (3) providing Tal with an opportunity to reflect on her embodied idea by viewing it in the third person.

By continuing to pursue Tal's "same distance" idea, Dev and Dor recognize an opportunity to help Tal advance her understanding by engaging her in an experiment (having her observe the spatial interval for a number of different cursor height locations that turn the screen green) and giving her tools (the grid and numbers) to quantitatively describe what she observes. This collaboratively produced experiment resembles a physically enacted form of what Ginsburg calls providing countersuggestions. It tests Tal's conviction to her initial hypothesis in light of the new empirical evidence.

As a result, Tal makes an important discovery: when the screen is green, the distance between the cursors increases *as* the cursors advance higher (Appendix 2. 2.3 105). By continuing to discuss and explore this distance with the tutors, Tal develops her initial incorrect observation into a more robust quantitative description of the device's behavior. With this discovery, Tal has come up with an important mathematical idea: covariation. She realizes that the distance she had been attending to between the cursors *co-varies* with cursor height. Specifically, the distance between the cursors grows larger the higher the cursors are on the screen. Piaget et al. (1968) inferred from their empirical studies that understanding proportionality is predicated on establishing "a relationship between two laws of progression" (p. 137), and Tal, to our judgment, has achieved just that. Moreover, differentiating between transformation situations of "fixed distance" (or "fixed difference") and "changing distance" (or "changing difference") has been repeatedly noted by mathematics-education researchers as pivotal for developing from additive to multiplicative dynamic concept images, such as in scaling (e.g., Karplus, Pulos, & Stage, 1983).

This example illustrates how educators' multimodal candidate understandings provide a powerful tool for being responsive to leaners' ideas. By the end of this episode,

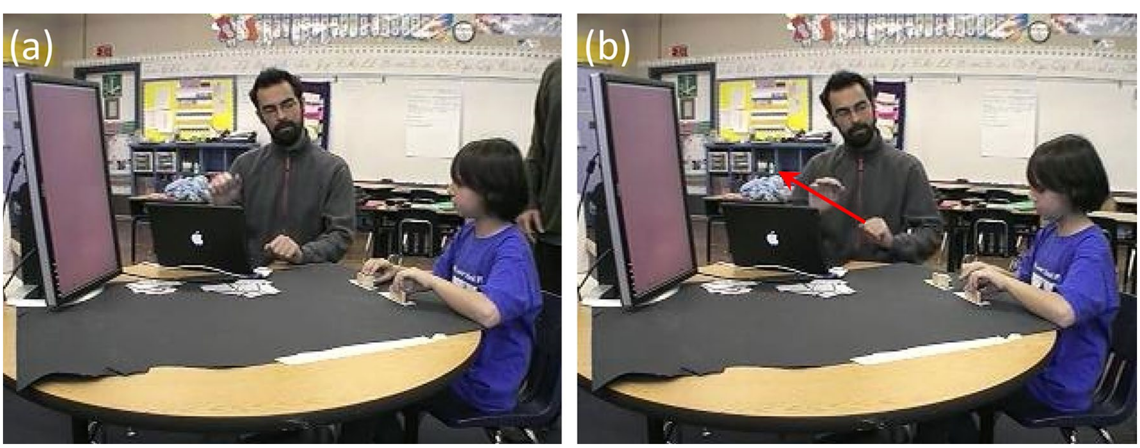

Fig. 13 **a** Dev holds his hands up as remotes and **b** traces a line between them

Dev had checked his *interpretation* of Tal's idea numerous times, allowing him to attune his guidance in ways that productively advanced Tal's understanding towards a new mathematical understanding. At first, Tal's same-distance idea was underspecified and ambiguous to the tutors. There are many potentially relevant distances that she might have been attending to. In order to follow-up on this idea in a productive way, the tutors needed to first understand what she was perceiving. By using gesture and indexical language that mirrored Tal's, Dev was able to show Tal that he understood her without supplying his own verbal description. This way, the task of articulating and specifying the pattern mathematically was left to Tal. *Summarizing* Tal's idea and reflecting it back to her provided the opportunity for Tal to revise her discovery.

6.3 Elaborating contributions: collaboratively building multimodal ideas together through co-constructed gesture

Another crucial aspect of responsive teaching and clinical interviewing, is taking up learners' ideas and helping them extend and connect these ideas to new disciplinary understandings. In the case of embodied learning technologies, building on learners' ideas often involves interacting *with* learners' gestures. This can include educators and learners simultaneously gesturing in different physical spaces or each contributing to the same gesture in the same physical gesture space. By co-constructing gestures, educators can help steer and formulate ideas in productive new directions while keeping them grounded in learners' initial observations and discoveries. In Excerpt 3 (Appendix 2) we present an example of a tutor and learner co-constructing an embodied, dynamic representation through gesture.

Fig. 15 Ela uses her hands to explain her idea about iteratively adding one half

When the episode begins, Ela, a student, has been working with the MIT-P with tutors Dor and Dev. She is exploring the secret ratio 2:3, with the grid and numbers turned on. While she has identified many green-producing number pairs, when asked, she is unable to predict where the right hand would be if the left hand was at 10. Dor encourages Ela to use her hands to explain her technique (similar to the case in Excerpt 1) and Ela illustrates that to turn the screen green, she raises her left hand one unit and then, looking to locate the right hand, adds a half unit *more*, thus iteratively raising her hands in the *a-per-Δ* sequential strategy (Abrahamson et al., 2014) of 1-per-1/2-more, which functionally amounts to raising the hands 1-per-1.5 (Fig. 15).

Dor sees an opportunity to build on Ela's embodied idea, towards a multiplicative understanding (Excerpt 3.1, Appendix 2). He asks Ela to raise her left hand above the desk and

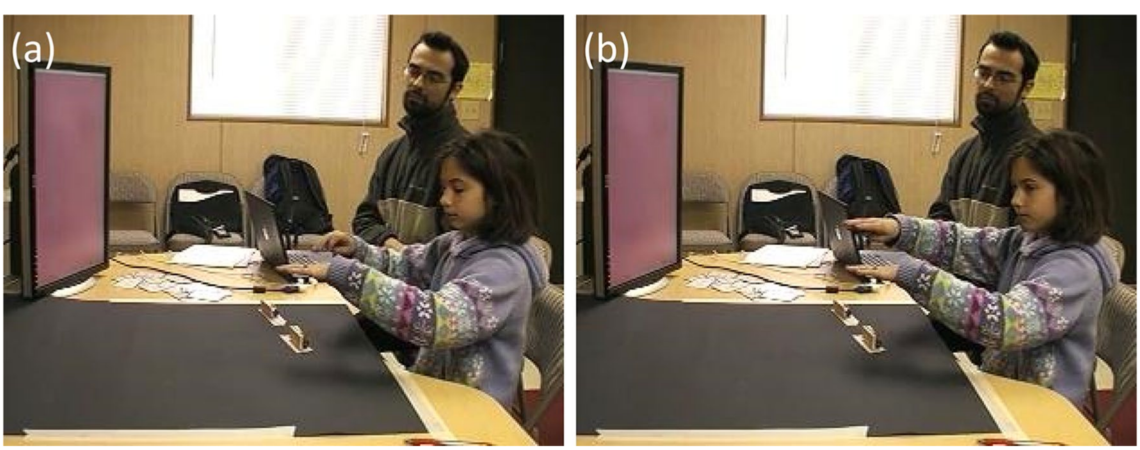

Fig. 16 **a** Ela hovers her right hand in the air, demonstrating uncertainty, **b** but then places it one and half times as high as the left hand as instructed by Dor

then he asks her to position her right hand so that it is half more than her left hand's height. Ela goes to move her right hand into place, but she visibly hesitates, hovering her hand in the air, displaying uncertainty (Fig. 16a). Dor does not intervene, and gives her a moment. After the moment, Ela successfully slides her right hand into place, in a position that is the height of her left hand, plus another half *of that entire height* high (Fig. 16b). Together Dor and Ela have co-constructed a dynamic way of representing the relationship between the left- and right-hand heights: the right hand is always the height of the left hand, plus another half of that height high. Dor uses the shared reference of this dynamic, embodied representation to connect it to a way of describing what it performs mathematically. Instead of saying *half more*, Dor explains, they can say one-and-a-half *times* as much.

Dor then challenges Ela to quantitatively predict the position of the right hand when given the position of the left hand in Excerpt 3.3. Dor reaches out and touches Ela's left hand and asks her to put it at 10 (3.3. 42–44; see studies of touch as part of giving directives in socialization processes, M. H. Goodwin & Cekaite, 2018). Then he asks her to figure out what number her right hand would be at (3.3 45). She looks at her hands and, after a pause, she correctly answers 15 (3.3. 46). The new, embodied idea that the right hand is half-the-height-of-the-left *more* appears to be helpful to Ela, who earlier could not predict 15 from 10 with her original strategy.

Next, Dor asks Ela to predict more pairs with the new approach, and he reaches in to interact with Ela's gesture to provide guidance when she gets stuck. When Dor asks about the left hand at two, after two long pauses Ela offers two and a half[1] instead of three (3.3 54–58). In response, Dor reaches under Ela's left hand and uses his thumb and index finger to bracket the vertical interval between Ela's hand and the table (Fig. 18), saying "take two" (3.3 59). Then, he slides his hand over towards Ela's right hand and decreases the bracketed vertical interval of his thumb and index finger by about half (Fig. 19) while saying "take half of two which is one" (3.3 60–65). Dor trails off his explanation, but with this embodied hint, Ela is able to answer correctly, and supplies the number three (3.3 66). Here, Dor uses his own gesture to interact with and enhance Ela's gesture, helping guide her attention to aspects of her own hand positions that can be used to predict the number that goes with two. Later on in the conversation, Ela repeats Dor's contracting-thumb-and-index-finger gesture to compare the two different heights as she talks about how the right-hand remote has to travel faster because it travels one-and-a-half times the distance of the left hand in the same amount of time. We take this as evidence that Dor's gestural intervention provided a helpful new way for Ela to make sense of the relationship between the two hand heights in the case of 2:3.

[1] Ela correctly predicted the pair 2 and 3 with her original additive method, but incorrectly predicted 2.5 when given 2, which further suggests she is applying Dor's multiplicative method: When children transition from additive to multiplicative reasoning with ratio, they sometimes struggle with pairs they before could predict (Van Dooren, De Bock, & Vershaffel, 2010).

Excerpt 3.3 Dor and Ela co-create a series of relations by hand

Dor: So [if you- if this hand were at uh ten where would that one be
[*((Dor reaches out and touches Ela's left hand and then points to Ela's*
right hand))
Where would that one be
Ela: At um, (1.1) at fifteen=
Dor: =Uhuh [If- if yo::ur- if this hand were do↓wn at one
[*((Dor points to Ela's left hand and Ela lowers her left hand so that it*
is almost on the table))
[where should that one be
[*((Ela brings her right hand down to a few inches above the left)))*
Ela: At:t: um (0.9) one and a half
Dor: Uhuh
Dor: [And if this one were at two
[*((Dor points at Ela's left hand))*
Ela: [At (0.9) tw- um (1.9) two and a half
[*((Ela raises her left hand, and then raises her right hand a few inches above*
her left hand))
Dor: If you take tw::o
[and take half of two
[*((**Fig. 18:** Dor moves his hand under Ela's left hand in a pinch shape))*

Fig. 18

which is [o:::ne so it's
[*((**Fig. 19:** Dor decreases the space between his index finger and thumb to*
about half to make a smaller pinch shape and then he slides his hand
towards Ela's right hand))

Fig. 19

Ela: Uh three
Dor: Yeah, [and when this one is say at four
[*((Dor points to Ela's left hand))*
Ela: Six::
Dor: Uhuh an:d uh six
((Dor points to Ela's hands))
(1.9)
Ela: °Um ah six° nine
Dor: °Yeah, yeah°

Overall, this episode demonstrates the pedagogical utility of building from learners' gestured ideas and co-constructing new gestures that elaborate and extend these ideas. Ela started out using her hands to demonstrate an additive scheme. This gesture became a *substrate* (C. Goodwin, 2018) that Dor built on when he instructed her on how to experience her gestured demonstration as a functional multiplicative relation between the heights of the left and right hand. When Ela became stuck, Dor used his own hands to literally *reach into* her gesture and co-gesture a representation with her. Co-constructing this gesture impacted Ela, who later applied it to the new situation of thinking about relations between speeds. Past work has shown that students collaboratively co-construct gestures to build on each other's ideas (Singer et al., 2008; Walkington et al., 2019; Yoon et al., 2011). Here, we see that co-constructing a gesture is a useful responsive-teaching strategy to *build from* and *elaborate* learners' initial embodied ideas (e.g., Ela's additive scheme) and connect them with new disciplinary understandings (e.g., the functional multiplicative scheme Dor and Ela co-construct). Ela and Dor's collaborative achievement is significant. Non-integer functional factors between two numbers composing a ratio are challenging landmarks *en route* to mastering rational numbers (e.g., Vanhille & Baroody, 2002).

7 Discussion

Our investigation contributes to filling current gaps in our understanding of how learning can be facilitated with digital technologies designed to deliberately incorporate the body into learning mathematics. Embodied learning technologies pose unique problems for teaching, since they engage learners' bodies in explorations of mathematical content through discovery. Educators must find ways to guide learners towards disciplinary mathematics understandings working from the substrate of learners' embodied experiences of perceiving and moving. In our analysis, we demonstrated how responsive-teaching strategies of: (1) eliciting and probing ideas, (2) summarizing and offering interpretations of ideas; and (3) building from and elaborating ideas are adapted for engaging with learners' multimodally-expressed embodied contributions. Our examples show how these practices can put educators in a better, more informed position to make decisions for customizing instruction in the moment—namely, what kinds of guidance and scaffolding to offer next—in order to help students connect their embodied experiences with the cultural forms of mathematics. In this sense, these strategies allow for *embodied* proximal formative assessment, the just-in-time adjustment of instruction to students' needs.

Following Wertsch's recommendation (1984), we were able to examine how zones of proximal development emerge with embodied learning technologies by exploring the fine details of tutor–student interactions. We treated learners' multimodally-expressed embodied ideas as substrates (C. Goodwin, 2018), and investigated how tutors built on and transformed these contributions (in Tal and Ela's case) or encouraged learners themselves to build on and transform these contributions (in Ben's case). Our approach, drawing on EMCA and Goodwin's Co-Operative Action framework, presented both advantages and limitations for understanding this process. By focusing on the fine details of these processes, we were able to uncover the precise interactional work that tutors and students do to successfully negotiate mathematical discoveries with the MIT-P. By characterizing these techniques in detail, we have made it possible for other researchers to identify and compare whether similar practices are deployed by educators working in different settings with different students and different embodied mathematics learning technologies. The tangible, concrete particulars from our cases can also serve as a resource and inspiration for practitioners to experiment with as part of their own responsive praxis. Following Erickson (1992), we believe microanalytic investigations of teaching practices can be especially valuable to practitioners because they illustrate the details of practices-in-use rather than presenting more general, abstracted categories of practice.

At the same time, our approach has several limitations. First, it does not allow us to determine if these embodied responsive teaching practices have an impact on children's longer-term ontogenetic mathematical development over time. We can only understand how meanings are generated in the moment. A second limitation is that we have identified and characterized just a small subset of practices. Our work points to the need for more research to investigate embodied responsive teaching both in and outside of technology-enabled embodied mathematics learning settings. Although previous studies have identified some ways instructors are responsive to students' gestures (e.g., Arzarello et al. 2009), in general, very little attention has been paid to how educators and students' gestures and other multimodal resources come into contact and interanimate each other dialogically (Bahktin, 1981). Future work could examine: (1) how embodied responsive teaching practices impact longitudinal mathematics learning outcomes; and (2) additional forms and functions of embodied responsive teaching practices for embodied learning technologies.

Finally, we also demonstrated that "entering the child's [embodied] mind" creates important opportunities: (1) to gauge learners' epistemic stance; and (2) for learners to reflect on their own embodied ideas. Eliciting, engaging, and building on learners embodied ideas provide opportunities for learners to practice *embodied* metacognition: they are given a chance to deliberately reflect on how they are using their bodies to think. This appears to support mathematical discovery. In addition, we saw that attending to gesture was not only helpful for understanding the *representational content* of ideas, but that it can also help educators appreciate learners' *epistemic stances,* that is, how certain they feel about the knowledge they are sharing. Monitoring learners' gestures for embodied displays of epistemic stance presents a powerful resource for responding appropriately and productively to learners in the moment. In the cases we presented, tutors responded to learners' uncertainty by encouraging them to continue and elaborate their explanations, signaling to learners that their ideas were valuable and worth exploring, even when incomplete or uncertain. In all three cases, rather than give up, learners continued to work through their hesitation, sharing their unfolding thoughts, and making productive discoveries. We suspect that educators' responsiveness to learners' multimodally-expressed embodied thinking sends an important epistemic message to students that the body is a valuable and legitimate resource for doing mathematics.

8 Conclusion

Our goal in this paper was to identify and characterize ways educators can be responsive to learners' multimodally-expressed embodied ideas when they use embodied learning technologies for mathematics. We presented three ways that educators can be responsive that included: (1) explicitly encouraging learners to use gesture and being aware of gesture–speech mismatches; (2) using multimodal candidate understandings; and (3) co-constructing multimodally-expressed embodied ideas using gesture. Our fine-grained examples, analyzed multimodally moment-by-moment, show how these practices can help learners make new discoveries and connect their embodied experiences with mathematical ways of perceiving and organizing the world.

Overall, we have contributed to enriching the field's understanding of responsive-teaching and clinical-interviewing practices with digital mathematics-learning technologies by extending the discursive focus on these practices to also include multimodal communication like gesture. Our study helps bring to light "seen-but-unnoticed" (Garfinkel, 2002) aspects of embodied instructional work with digital technologies that are frequently overlooked. We hope future inquiries will continue to investigate the embodied nature of responsive teaching with embodied learning technologies.

Appendix 1 Adapted conversation analysis transcript conventions from Jefferson (2004)

Sample	Meaning
[so he was [you're it	Brackets denote overlapping talk or bodily activity
Does=	Equals indicates latching: one utterance follows the other unusually quickly
(2.5)	A number in parentheses indicates a timed pause
(.)	A period in parentheses: pause less than a tenth of a second
#only#	Pound signs indicate exaggerated voice quality
. or ↓	Falling pitch, note punctuation not used grammatically
? or ↑	Rising pitch, note punctuation not used grammatically
,	Comma indicates slight falling pitch
if this-	Dash indicates cut off, abrupt stop, or unfinished words
<not yet>	A less than and more than symbol indicate slower or drawn out speech
>hey you<	A more than and less than symbol indicate rushed or quickly spoken speech
°maybe°	Degree signs indicate quiet speech or whisper
TAKE IT BACK	Capitals denote loud speech, shouting
Take it now	Underline denotes emphasis (voiced stress)
So::o	Colons indicate elongated word or syllable
hhh	h's indicate audible exhalation
(and)	Single parentheses indicate analyst uncertainty/approximation
((RH flick))	Double parentheses indicate annotation of bodily activity

Appendix 2 Full transcripts of three examples

Excerpt 1. Dor asks Ben to use his hands to show how to make the screen green

Excerpt 1.1

```
Dor: Could you show us with your hands- without them-
     just lift- start with your [two ha:nds on-
                                [((Ben holds his two hands out in front of
                                  him as if he is holding remotes))
     on- on the- on the [surface
                        [((Fig. 2: Dor puts his hands flat on the table and
                          Ben then puts his hands flat on the table))
```

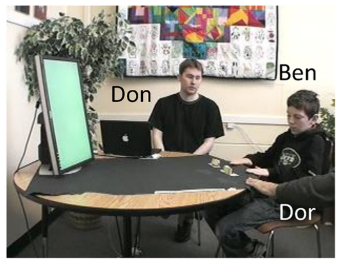

Fig. 2

```
    and now show how would you do it if you were just doing-
    if you- if you were holding it- how would your hands go up
```

Excerpt 1.2 Ben elaborates his original idea

```
Ben:  Mmmm [I would start the right hand=
           [((Fig. 3: Ben slowly raises his hands so that his right hand
             continuously remains about twice as high as his left hand))

```

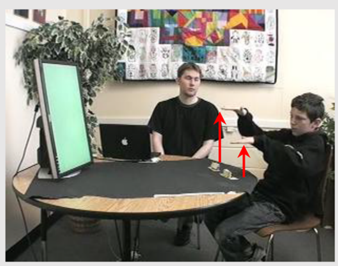

Fig. 3

Fig. 4

```
Dor: [Mhmm
Ben: [=going a little faster than the- um left hand
Dor: Uhuh
Ben: Then you [keep them (1.8)
              [((Fig. 4: Ben lowers his hands so that his right hand continuously
                remains about twice as high as his left hand))
      [going even a-paced
      [((Ben raises his right and left hand so that his right hand stays
        higher than left))
      you have to keep them in- [(3.0)    tch       um- same pa::ce basically
                                [((Fig. 5: Ben spins his right hand in the air))
```

Fig. 5

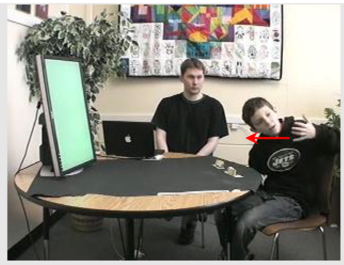

Fig. 6

25 [sort of
26 [*((**Fig. 6:** Ben moves his hand, torso, and head side to side))*
27 Dor: Uhuh
28 Ben: Yeah you have to keep them [going the same pace
29 [*((Ben raises his hands so that his right*
30 *hand rises above his left hand))*
31 [so it's really complicated
32 [*((Ben lowers his hands so that his right hand continuously remains higher*
33 *than his left hand))*
34 [when you come back down so::o (1.6)
35 [*((Ben picks up the remotes and raises them so that the screen stays*
36 *green, then lowers them))*

Excerpt 1.3 Ben elaborates his idea further with an analogy

37 Don: What're you saying=
38 Ben: =You see if you keep them in the same like pace like
39 [same like for a car
40 [*((Ben spins his right hand in the air))*
41 like [if you wanted to do this with a car
42 [*((**Fig. 7:** Ben moves his flattened right hand forward out in front of him*
43 *horizontally with his left hand trailing it, and repeats the*
44 *motion))*

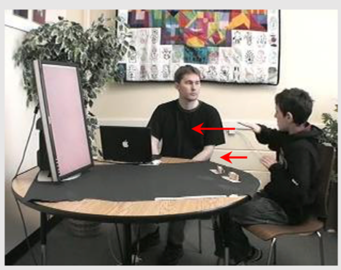

Fig. 7

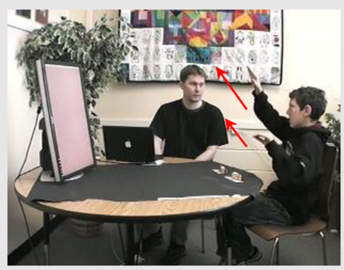

Fig. 8

45 Ben: [It would sort of be
46 [*((Ben moves his right hand forward horizontally in front of him in a*
47 *pinch shape))*
48 [the same (.) speed limit
49 Don: [Mhhm
50 Ben: [Like this one is going twenty
51 [*((Ben points to the left remote and then the right remote))*
52 Don: Okay
53 Ben: [And this one is going fifty and they have to keep on going s::o
54 [*((**Fig. 8:** Ben moves his flattened right hand forwards and up 45 degrees from*
55 *his body with his left hand following, and then moves them both down))*
56 Don: Okay

Excerpt 2. Dev uses gesture to check his understanding of Tal's idea

Excerpt 2.1

Tal: [I– it's about the same distance
     [((Tal points between the remotes she and Dev hold))
Dev: What's the– What's about the same distance
Tal: [Fr–
     [((Tal points between the remote Dev holds and the remote she is
       holding twice, tracing a diagonal line))

Fig. 9


Fig. 10

     um, [this to this:
         [((Fig. 9: Tal again points between the remote Dev holds and the remote she
           is holding, tracing a diagonal line))
((Tal points again between the remote Dev holds and the remote she is
  holding, tracing a diagonal line))
     it's a– diagonal [li–
                      [((Fig. 10: Tal opens her hand and waves it back and
                        forth))
Dev: Oh so sort of [from here to here↑
                   [((Fig. 11: Dev points on the screen between the two
                     cursors))


Fig. 11

Tal: Yeah and notice like if you move one of them=
Dev: =Mhmm
Tal: The– the color changes↑
Dev: Mhmm so if you keep yours still [and I move mine
                                     [((Dev moves his cursor up and down and
                                       the screen turns red))
     [the °color changes°
Tal: [Yeah
(1.0)
Tal: Oh! And that's why [it was flashing when I was moving it up and down
                        [((Tal moves her cursor up and down the screen making
                          it flash red))
     because like I had touched (.) the sp– [like the space=

```
31                                            ⌊((Tal waves her hand in the air
32                                              between the remotes))
33 Dev: =Mhmm right
```

Excerpt 2.2 Dev uses gesture to check his understanding of Tal's revised idea

```
34 Dev: Uhm, so let's see, if I'm really like ↓lo:w do:wn °here↓°
35 ((Dev moves his cursor so it is a few inches from the bottom of the screen and
36   Tal simultaneously moves her cursor, they pause when the screen turns green))
37 Tal: Oh (.) Oh! Here's °another° space
38 Dev: °Mhhmm°
39 (4.2)
40 Dev: So it's about green↑=
41 Tal: =Yeah
42 ((Tal holds her hand palm up))
43 Dev: ⌈And let's take a look at if I'm ↑up here
44      ⌊((Dev moves his cursor to the very top of the screen and Tal follows
45         and stops her cursor when the screen turns green))
46 ((Dor reaches in to adjust Tal's remote in her hand and speaks inaudibly))
47 Tal: ⌈So that's not °the same distance° right there
48      ⌊((Tal points quickly back and forth between the remote Dev holds
49         and the remote she holds))
50 Dev: Okay say more about that
51      What's the distance that you are talking about (.) a-gain?
52 Tal: Um, the distance ⌈between this and- that=
53                      ⌊((Tal points more slowly between the remote she is
54                         holding and the remote Dev is holding, tracing a
55                         diagonal line))
56      =Like it seems-
57 ((Tal waves her open hand between the remotes))
58 Dev: ⌈°This and this↑°=
59      ⌊((Fig. 12: Dev points between the remote he is holding and the remote
60         Tal is holding, tracing a diagonal line))
```

Fig. 12

```
61 Tal: =Yeah it seems that that's like the re- like how it turns green↑
62 Dev: Uhuh
63 (1.5)
64 Tal: And like it can get greener and less green
65 ((Dor adjusts the remote Tal is holding, then Dev))
66 Dor: °One of them is giving us trouble°
67 Tal: Uh, thanks. Like it can get darker green and red: and then lighter green and
68      brown (1.9)
69      ⌈all depending on which- how you move each one=
70      ⌊((Tal points to the remote she is holding))
71 Dev: =Okay
72 Dev: So, I'll put this one down and you can take it yourself
73 ((Dev places his remote on the desk and Tal places hers on the desk))
74      and now ⌈are you talking about the distance-
75              ⌊((Fig. 13a: Dev holds his hands up as if holding the remotes and
76                 raises one of his hands))
```

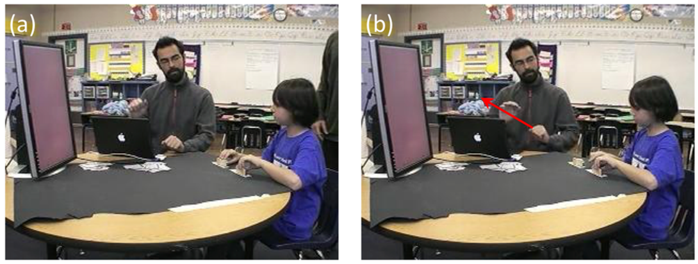

Fig. 13

```
77        [sort of the diagonal  [distance
78        [((Fig. 13b: Dev traces|the diagonal between his hands))
79  Tal:                         |Yeah yeah between
80                               [((Tal points back and forth to each remote on the
81                                 desk))
```

Excerpt 2.3 Pursuing Tal's idea leads to a new idea

```
82  Dor: O::h kay (.) and what else↑ do we know about the distance,
83       Like [if I came (.) he::re, °what do y- what do you want to do°
84            [((Dor moves his cursor to a point high on the screen))
85  ((Tal moves her cursor up until the screen turns green))
86  Tal: I'd- I'd move up↑ [°and then- it would become green again°
87  Dor:                   [Uhhuh                                      Uhuh
88  Dor: °Okay° (.) and if I'm down- [>let's look what happens<
89                                   [((Dor moves his cursor to a lower point on the
90                                      screen to "one"))
91  Dor: Say I'm [all the way down here
92               [((Tal begins moving her cursor down the screen and stops when the
93                  screen turns green))
94  Tal: °So that's at two, and that's at o:ne°
95  Dor: °Yeah, one and two°
96  Tal: So basically [like um (.) if you put like either one at one point,
97                    [((Tal points to the screen))
98       you'd be able to find a green
99  Dor: Huh↑
100 Dor: [What if this one is at two,
101      [((Dor moves his cursor to two))
102                    [where do you think that one should be
103                    [((Tal moves her cursor until the screen turns green))
104      (2.2)
105 Tal: Oh and [they're getting farther apart as it goes up
106             [((Fig. 14: Tal makes a bracketing gesture))
```

Fig. 14

```
107 Dor: O:h
```

Excerpt 3. Dor guides Ela through a new way to use hands to think about 2:3

Fig. 15

Excerpt 3.1

Dor: I want offer a slightly different way of looking at exactly the same thing and
tell me if it makes sense. (2.3) What if- if you put your- ah left hand
anywhere just at [some height
[((Ela picks up the left remote))
[You know what even without that for a moment
[((Dor points to Ela's hands))
((Ela puts down the remote and raises her left hand approximately 10 inches (~25 cm) from
the desk))
Dor: I just- to see if you can make sense,
Now look how high your hand is above [the desk
[((Dor sweeps his hand across the desk))
[((Ela looks at her hands then back to Dor and nods))
[Ela: Mhmm
Dor: And put your ri::ght hand half more than that
((**Fig. 16a:** Ela raises her right hand and hesitates))

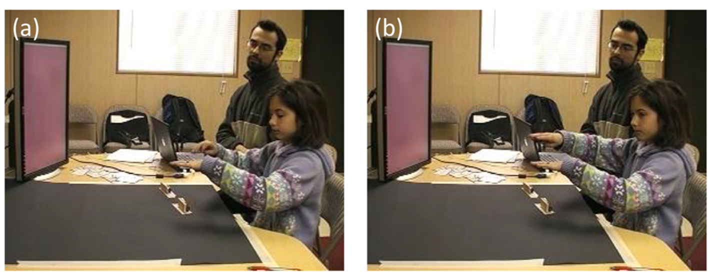

Fig. 16

((**Fig. 16b:** Ela then places her right hand approximately 6 inches above her left
hand, then she adjusts and lowers it another 2 inches))
Dor: Did that make sense the way I put it?
Ela: Yeah
Dor: It's different from what you said before
Dya- right it's a different [way of looking at it
Ela: [Yeah

Excerpt 3.2 Dor and Ela reflect on the new dynamic representation

Dor: And I wanta- so if you lift your [l::eft hand,
[((Dor point towards Ela's left hand))
lift it up a bi::↑t

Fig. 17

```
((Fig. 17: Dor raises his left hand and then Ela raises her left hand))
((Ela raises her left hand up two inches))
Dor: Now where would- should the right hand be
((Ela raises her right hand, then she drops it down to the table, then she brings
  it to half the height of the left hand and pauses, then brings it up above the
  left hand))
Ela: At half as much as that
Dor: Right half mo::re [of that
Ela:                   [Yeah
Dor: And when we speak in- kind of mathematically instead of saying half more
     it's another way of saying it, which is it's one and a half ti:mes=
Ela: =Yeah [one and a half times as much
Dor:       [Do y- do y-
Dor: Do you see what I mean, yeah one and a half times as much
     kind of a funny way of putting it one and a half times as much but that's what
     I mean
```

Excerpt 3.3 Dor and Ela co-create a series of relations by hand

```
Dor: So [if you- if this hand were at uh ten where would that one be
        [((Dor reaches out and touches Ela's left hand and then points to Ela's
          right hand))
     Where would that one be
Ela: At um, (1.1) at fifteen=
Dor: =Uhuh [If- if yo::ur- if this hand were do↓wn at one
           [((Dor points to Ela's left hand and Ela lowers her left hand so that it
              is almost on the table))
     [where should that one be
     [((Ela brings her right hand down to a few inches above the left))
Ela: At:t: um (0.9) one and a half
Dor: Uhuh
Dor: [And if this one were at two
     [((Dor points at Ela's left hand))
Ela:      [At (0.9) tw- um (1.9) two and a half
          [((Ela raises her left hand, and then raises her right hand a few inches above
             her left hand))
Dor: If you take tw::o
     [and take half of two
     [((Fig. 18: Dor moves his hand under Ela's left hand in a pinch shape))
```

Fig. 18

which is [o:::ne so it's
[*((**Fig. 19:** Dor decreases the space between his index finger and thumb to about half to make a smaller pinch shape and then he slides his hand towards Ela's right hand))*

Fig. 19

Ela: Uh three
Dor: Yeah, [and when this one is say at four
[*((Dor points to Ela's left hand))*
Ela: Six::
Dor: Uhuh an:d uh six
((Dor points to Ela's hands))
(1.9)
Ela: °Um ah six° nine
Dor: °Yeah, yeah°

References

Abrahamson, D., Lee, R. G., Negrete, A. G., & Gutiérrez, J. F. (2014). Coordinating visualizations of polysemous action: Values added for grounding proportion. *ZDM Mathematics Education, 46*(1), 79–93.

Abrahamson, D., & Trninic, D. (2014). Bringing forth mathematical concepts: signifying sensorimotor enactment in fields of promoted action. *ZDM Mathematics Education*, *47*(2), 295–306.

Alibali, M. W., & Goldin-Meadow, S. (1993). Gesture-speech mismatch and mechanisms of learning: What the hands reveal about a child's state of mind. *Cognitive Psychology, 25*, 468–523.

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences, 21*(2), 247–286.

Alibali, M. W., Nathan, M. J., Boncoddo, R., & Pier, E. (2019). Managing common ground in the classroom: Teachers use gestures to support students' contributions to classroom discourse. *ZDM Mathematics Education, 51*(2), 347–360.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Bahktin, M. (1981). *The dialogic imaginiation* (M. Holquist (ed.)). University of Texas Press.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93*(4), 373–397.

Ball, Deborah Loewenberg, Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematicak knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (pp. 433–456). Macmillan.

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31–37.

Clark, M. R., Berenson, S. B., & Cavey, L. O. (2003). A comparison of ration and fractions and their roles as tools in proportions reasoning. *Journal of Mathematical Behavior, 22*, 297–317.

Coffey, J. E., Hammer, D., Levin, D. M., & Grant, T. (2011). The missing disciplinary substance of formative assessment. *Journal of Research in Science Teaching, 48*(10), 1109–1136.

Crowder, E. M. (1996). Gestures at work in sense-making science talk. *The Journal of the Learning Sciences, 5*(3), 173–208.

de Freitas, E., & Sinclair, N. (2013). New materialist ontologies in mathematics education: the body in/of mathematics. *Educational Studies in Mathematics, 83*(3), 453–470.

diSessa, A. A. (2007). An interactional analysis of clinical interviewing. *Cognition and Instruction, 25*(4), 523–565.

Erickson, F. (1992). Ethnographic microanalysis of interaction. In M. D. LeCompte, W. L. Millroy, & J. Preissle (Eds.), *The handbook*

of qualitative research in education (pp. 201–225). Academic Press.
Erickson, F. (2007). Some Thoughts on Proximal Formative Assessment of Student Learning. In P. A. Moss (Ed.), *Evidence and Decision Making* (pp. 187–216). Blackwell Publishing.
Flood, V. J. (2018). Multimodal revoicing as an interactional mechanism for connecting scientific and everyday concepts. *Human Development, 61*, 145–173.
Flood, V. J., Amar, F. G., Nemirovsky, R., Harrer, B. W., Bruce, M. R. M., & Wittmann, M. C. (2015). Paying attention to gesture when students talk chemistry: Interactional resources for responsive teaching. *Journal of Chemical Education*, *92*(1), 11–22.
Garfinkel, H. (2002). *Ethnomethodology's program: Working out Durkheim's aphorism*. Rowman & Littlefield.
Ginsburg, H. P. (1997). *Entering the child's mind*. Cambridge University Press.
Goodwin, C. (2000). Action and embodiment within situated human interaction. *Journal of Pragmatics, 32*(10), 1489–1522.
Goodwin, C. (2018). *Co-operative action*. Cambridge University Press.
Goodwin, C., & Heritage, J. (1990). Conversation analysis. *Annual Review of Anthropology, 19*, 283–307.
Goodwin, M. H., & Cekaite, A. (2018). *Embodied family choreography: Practices of control, care, and mundane creativity*. Routledge.
Goodwin, M. H., & Goodwin, C. (1986). Gesture and coparticipation in the activity of searching for a word. *Semiotica, 62*(1–2), 51–76.
Hall, R., Ma, J. Y., & Nemirovsky, R. (2014). Rescaling bodies in/as reprsentational instruments in GPS drawing. In V. R. Lee (Ed.), *Learning technologies and the body: Integration and implementation in formal and informal learning environments* (pp. 112–131). Routledge.
Heritage, J. (1984). A change-of-state token and aspects of its sequential placement. In J. M. Atkinson & J. Heritage (Eds.), *Structures of social action* (pp. 299–345). Cambridge University Press, Cambridge
Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: an emerging framework of teaching moves. *ZDM Mathematics Education, 48*(1–2), 185–197.
Jefferson, G. (2004). Glossary of transcript symbols with an introduction. In G. H. Lerner (Ed.), *Conversation analysis: Studies from the first generation* (pp. 13–31). John Benjamins.
Karplus, R., Pulos, S., & Stage, E. K. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 45–89). New York: Academic Press.
Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied ming brings mathematics into being*. New York: Basic Books.
Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Information Age Publishing.
Lausberg, H., & Sloetjes, H. (2009). Coding gestural behavior with the NEUROGES-ELAN system. *Behavior Research Methods, Instruments, & Computers, 41*(3), 841–849.
Lineback, J. E. (2015). The redirection: An indicator of how teachers respond to student thinking. *Journal of the Learning Sciences, 24*(3), 419–460.
Mondada, L. (2019). Contemporary issues in conversation analysis: Embodiment and materiality, multimodality and multisensoriality in social interaction. *Journal of Pragmatics, 145*, 47–62.
Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2014). *Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit., 44*(2), 372–415.
Ochs, E. (1996). Linguistic resources for socializing humanity. In J. J. Gumperz & S. C. Levinson (Eds.), *Rethinking linguistic relativity* (pp. 407–437). Cambridge University Press.
Piaget, J., Grize, J. B., Szeminska, A., & Bang, V. (1968). *Epistemology and psychology of functions*. Presse Universitaire de France.
Pierson, J. L. (2008). *The relationship Between Patterns of Classroom Discourse and Mathematics Learning*.
Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM Mathematics Education, 46*(3), 349–361.
Robertson, A. D., Scherr, R. E., & Hammer, D. (2016). *Responsive teaching in science and mathematics*. Routledge.
Saxe, G. B., Dawson, V., Fall, R., & Howard, S. (1996). Culture and children's mathematical thinking. In R. Sternberg & T. Ben-Zeev (Eds.), *The nature of mathematical thinking* (pp. 119–144). Erlbaum.
Saxe, G. B., Gearhart, M., & Seltzer, M. (1999). Relations between classroom practices and student learning in the domain of fractions. *Cognition and Instruction, 17*(1), 1–24.
Schegloff, E. (1982). Discourse as an interactional achievement. In D. Tannen (Ed.), *Analyzing discourse: Text and talk* (pp. 71–93). Georgetown University Press.
Schegloff, E. (1991). Conversation analysis and socially shared cognition. In L. B. Resnick, J. M. Levine, & S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 150–170). APA.
Shein, P. (2012). Seeing with two eyes: A teacher's use of gestures in questioning and revoicing to engage english language learners in the repair of mathematical errors. *Journal for Research in Mathematics Education, 43*(2), 182–222.
Shvarts, A. (2018). A dual eye-tracking study of objectification as student–tutor joint activity appropriation. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education, Vol. 4,* (pp. 171–178). PME.
Shvarts, Anna, & Abrahamson, D. (2019). Dual-eye-tracking Vygotsky: A microgenetic account of a teaching/learning collaboration in an embodied-interaction technological tutorial for mathematics. *Learning, Culture, and Social Interaction*.
Singer, M., Radinsky, J., & Goldman, S. R. (2008). The role of gesture in meaning construction. *Discourse Processes, 45*(4–5), 365–386.
Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–234). SUNY.
Van Dooren, W., De Bock, D., & Vershaffel, L. (2010). From addition to multiplication … and back. The development of students' additive and multiplicative reasoning skills. *Cognition and Instruction*, *28*(3), 360–381.
Vanhille, L. S., & Baroody, A. J. (2002). Fraction instruction that fosters multiplicative reasoning. In B. H. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions* (pp. 224–236). National Council of Teachers of Mathematics.
Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41–59). SUNY Press.
Vygotsky, L. S. (1986). *Thought and language* (A. Kozulin (ed.)). MIT Press.

Walkington, C., Chelule, G., Woods, D., & Nathan, M. J. (2019). Collaborative gesture as a case of extended mathematical cognition. *Journal of Mathematical Behavior, 55*(March), 100683.

Wertsch, J. V. (1984). The zone of proximal development: Some conceptual issues. In B. Rogoff & J. V. Wertsch (Eds.), *Children's learning in the zone of proximal development* (pp. 7–18). Jossey-Bass.

Yoon, C., Thomas, M. O. J., & Dreyfus, T. (2011). Grounded blends and mathematical gesture spaces: Developing mathematical understandings via gestures. *Educational Studies in Mathematics, 78*(3), 371–393.