Digital Experiences in Mathematics Education
https://doi.org/10.1007/s40751-020-00068-2

Shaping Perception: Designing for Participatory Facilitation of Collaborative Geometry

Leah F. Rosenbaum[1] · Japleen Kaur[1] · Dor Abrahamson[1]

Published online: 15 May 2020

Abstract
Geometris is a technologically enabled, body-scale, educational game, in which collaborating players recreate target polygons, prompted on a digital screen, by activating a set of sensors–vertices on a large physical mat interface. We report on an evaluation study that draws on theoretical frameworks from ecological dynamics, genetic epistemology, and socio-cultural semiotics. Micro-analysis of three adult–child groups at play implicates two design features supporting mediated development of geometry skills: (1) spatial distribution across two displays – the screen and the mat – poses cross-display figural mapping as a tactical problem whose perceptual solution constitutes the game's learning objective; (2) a multi-sensor input interface – the mat's 'vertices' – enables flexible divisions of group labor for scaffolding solution enactment. We put forth the construct of *participatory facilitation* – an emergent interaction pattern in groups with inter-personal differences in content-domain knowledge and sensori-motor co-ordination. We tentatively generalize principles for designing informal educational activity architecture that create opportunities for relative experts to enculturate content learning via participatory facilitation.

Keywords Collaboration · Ecological dynamics · Educational technology · Geometry · Scaffolding

Introduction

Let us start with the design

Consider the following scenario. You and a young person – perhaps your child or student – are standing on a gray carpet with a ring of colored pads (Fig. 1a). On the

✉ Leah F. Rosenbaum
leahr@berkeley.edu

[1] Graduate School of Education, University of California, Berkeley, Berkeley Way West, Room 4110, 2121 Berkeley Way, Berkeley, CA 94720-1670, USA

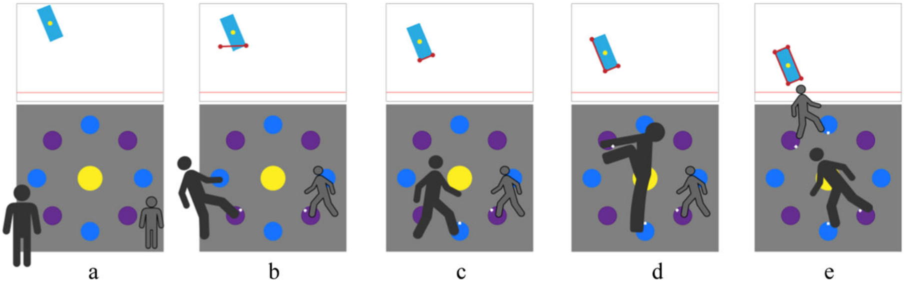

Fig. 1 *Geometris* gameplay: **a** the problem scenario; **b** an initial attempt; **c** an adjusted attempt; **d** testing a mapping; **e** completing the target shape (icons by Bradley Avison and James Keuning from Noun Project)

floor, in front of the mat, a screen displays a slowly descending rectangle. You are asked to make that rectangle by pressing the colored pads on the mat before the rectangle reaches the red line on-screen. You are confused: the rectangle is composed of four vertices and four edges. The carpet has an array of circles but no edges. Unsure, you step on the pad nearest you, and an LED on the pad turns on. Your young partner also steps on a pad, illuminating their LED. At the same time, two points appear on-screen, connected by a line. One point is on the rectangle's bottom-right vertex (bingo!), but the other point is to the left of the rectangle (Fig. 1b).

On a hunch, you step over one pad. Your on-screen point now appears on the rectangle's bottom-left corner (good!), and a line appears on its bottom edge (Fig. 1c). Hmm, the pads on the floor seem to map to the rectangle's vertices on the screen: left to the left, right to the right, and perhaps the two yellow circles correspond – the little one on the screen and the large one on the mat? There are still two vertices left. Lunging forward, you press a pad near the top of the mat with your hand. A third point appears on the rectangle along with one long edge (Fig. 1d). Ah hah! So up on the mat is up on the screen, too. A warning tone sounds, as the rectangle nears the bottom of the screen... The last corner of your rectangle must be to your right, but you are precariously balanced. Can your young partner stretch that far? You decide to reposition, ask your partner to take your spot, and return to the bottom of the mat. Having switched spots, the rectangle is complete (Fig. 1e). Go team! A high-pitched "Yay!" sounds from the game console, and the rectangle is replaced by another falling shape.[1]

What, if anything, have you learned from this scenario? What skills, assumptions, and understandings did you use to establish a figural mapping between the rectangle on the screen and the pads on the mat? What, if anything, has your young partner learned? As more shapes appear on-screen, how might you structure your play to help your young partner to develop those same figural mapping skills?

This article reports on an empirical evaluation of *Geometris*, the game described above. *Geometris* is rooted in the assumption that geometric knowledge rests in visualization and imagination of 2D shapes and operations upon them (CCSSI 2017; NCTM 2000). Research suggests that such spatial reasoning skills are highly correlated

[1] Each *Geometris* level contains eight target shapes, each of which must be completed by simultaneously activating its vertices, regardless of order. *Geometris* also includes an untimed Practice level. See Durán-López et al. (2017) for details on the hardware and the software.

with children's mathematical learning and achievement (Gilligan et al. 2019; Okamoto et al. 2014; Wolfgang et al. 2003) and can be improved with training (Uttal et al. 2013).

Geometris was designed to create playful opportunities for children to encounter and grapple with challenges of geometric and spatial reasoning. We argue that two qualities of the design support interactions conducive to pursuing this learning objective. First, we explain how *Geometris*' spatial distribution across two displays – a physical floor interface and an adjacent digital screen, also on the floor – poses the perceptual mapping of geometric figures across the displays as an emergent problem of enacting the game mechanics, a problem whose solution constitutes the learning objective. Second, through analysis of three focal groups, we illustrate how the game's large-scale, multi-sensor floor interface affords unusual pedagogical opportunities for adult participants to enculturate young participants into the target mathematical practice.

Our analysis has led us to propose the construct of *participatory facilitation*, a behavioral pattern observed within groups with inter-personal differences in content-domain knowledge and sensorimotor co-ordination, such as between parents and children. This study looks to characterize how *Geometris*' activity architecture creates opportunities for participants to scaffold collaborative achievement of the game objective through exercising participatory facilitation. We position this work within broader research efforts to understand how novel digital technologies enable new forms of participation in educational tasks (e.g. Hegedus and Penuel 2008). We now situate *Geometris* in the context of related mathematics learning tasks and explain its design rationale.

Game Style and Related Works

Geometris is a collaborative, body-scale, player-versus-environment game that challenges players to map shapes from a 6x3ft screen onto a 6x6ft sensor array (see Fig. 2). Similar to early work by Nemirovsky et al. (1998), *Geometris* occupies a middle ground between, on the one hand, sedentary activity at the desktop scale, such as using traditional pencil-and-paper or Dynamic Geometry software and its variants for individuals (Howison et al. 2011; Leung et al. 2013) or pairs (Nemirovsky et al. 2013), and, on the other hand, ambulatory activity at the city-block scale, such as *Walking Scale Geometry* (Ma 2017) or *GPS Graphing* (Hall et al. 2015).

Geometris is similar to these latter exemplars in that it requires players to tackle the posed challenges collaboratively. And, as in technologically enabled collaborative simulations that use a large screen to display collective activity of iconized participants (e.g. Enyedy et al. 2015; Wilensky and Stroup 2000), *Geometris* players must locate and monitor virtual traces of their individual actions in co-ordination of the enactment of a collective configuration – for example, distributing a rectangle among their bodies while recognizing oneself at a particular point. In addition, *Geometris* includes design features, such as a time limit, music, and the possibility for failure, that classify it as an educational game rather than a playful learning activity (e.g. Kelton and Ma 2020; Price and Duffy 2018).

Geometris was designed as a game through which learners could recognize and collaboratively enact geometric and spatial relationships. As the game's levels progress, the software challenges players to make polygons of increasingly higher order (Fig. 3),

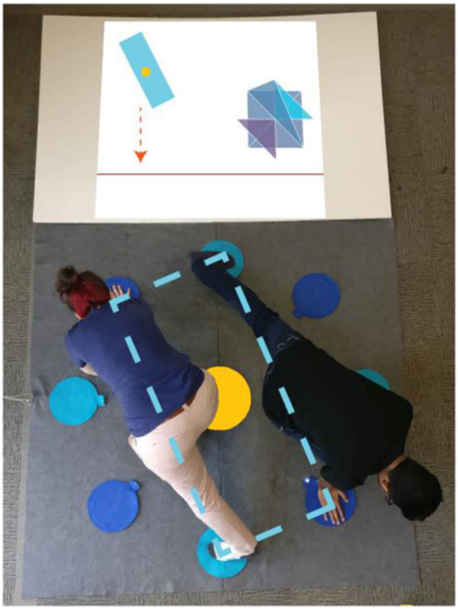

Fig. 2 The *Geometris* environment

even as the time limit remains constant. This design poses a trade-off. The mathematical degrees of freedom decrease across these levels, as there are fewer ways to define higher-order polygons (e.g. pentagons and hexagons) versus lower-order polygons (e.g. triangles) on the sensor array. However, players' co-ordination challenge increases. With more active vertices, players must develop a stronger sense of their location on the mat relative to the digital display – that is, a stronger figural mapping – in order to adjust their actions. As such, players must co-ordinate their actions more precisely to manage this increasing number of vertices within the time limit, making play more difficult.

Geometris was designed so that the target mathematics is intrinsic to gameplay, that is, the disciplinary content is instantiated in the game's tactics as well as its strategy (Habgood and Ainsworth 2011; Holbert and Wilensky 2014; Kafai 1996). As such, the moment-to-moment goals of mapping between the game's two displays both mobilize authentic geometrical reasoning in the game's granular tactics and exercise spatial reasoning in the game's broad strategy. In this sense, *Geometris* is more similar to games such as *The Logical Journey of the Zoombinis* (Broderbund 1996) or *Rolly's Adventure* (Williams-Pierce 2016), in that mathematics is intrinsic to play, and is less similar to edutainment games, such as *Math Blaster*, that present educational content between rounds of content-irrelevant play.

Design Rationale: Implementing Constructivist and Semiotic-Socio-Cultural Theory

Geometris was designed so that participants' attempts to perform the task exercise a set of disciplinarily favorable perceptual skills that most young children have not yet developed (Piaget et al. 1960). Given appropriate

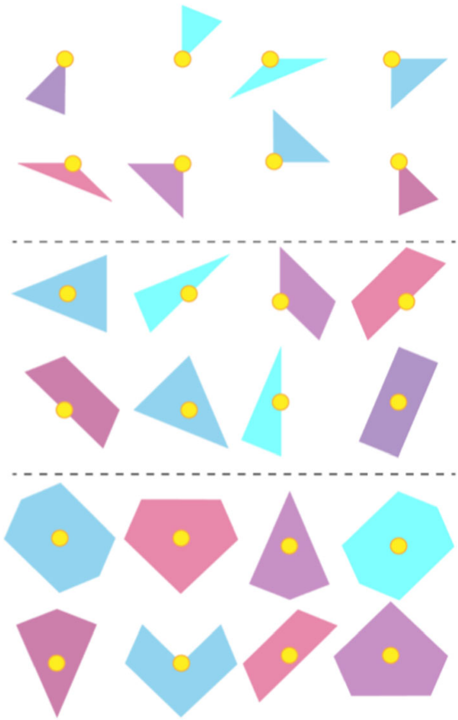

Fig. 3 *Geometris* levels: Level 1 (top) includes triangles, Level 2 (middle) includes quadrilaterals, Level 3 (bottom) includes higher-order polygons

mediation from accompanying adults, young *Geometris* players could thus become enculturated into forms of spatial reasoning believed to serve their mathematics learning.

From a cultural–semiotic perspective, its figural-mapping challenge demands perceiving two sensory displays as mutually referential (Duval 2006) or otherwise equivalent (Sfard and Lavie 2005). By what conventions might a child come to perceive a collection of four distributed points as a geometrically significant form, that is, as a rectangle? These displays are superficially different, yet they could become affiliated as 'the same' by endorsing mathematical perspectives (Abrahamson 2002; Bartolini Bussi and Mariotti 2008; Newman et al. 1989; Sfard 2002).[2] As such, teaching new mathematical concepts could be viewed as fostering learners' perceptual signification of the discipline's iconic displays in terms of selected features of sensory-rich concrete situations (Abrahamson 2009, 2012a, 2012b, 2014; Abrahamson and Wilensky 2007; Fyfe et al. 2014).

Rather than unidirectional fading from concrete to abstract, as Fyfe et al. (2014) propose, we interpret this challenge as encouraging repeated back-and-forth referencing and linking between concrete, enactive resources on the game mat and iconic shapes on the display screen. In a similar vein, *Geometris* seeks to foster geometrical semiosis through occasioning opportunities for the mathematical practice of figural

[2] Whereas perceptual affiliation of sensory stimuli is a Gestalt perception, highlighting a Gestalt in the context of mathematical activity marks it for learners as a culturally significant referent.

correspondences between polygons (vertices connected by edges) and their schematic rendition (vertices only). These pedagogically targeted norms are designed to emerge authentically through the collaborating players' efforts to communicate about figural elements relevant to the co-ordinated enactment of their joint actions (Abrahamson and Sánchez-García 2016; Barnes et al. 1996; Flood 2018; Shvarts and Abrahamson 2019; Wittgenstein 1953). Finally, some researchers argue that distributing tasks across physical and digital displays interferes with learning by increasing cognitive load beyond productive levels (Mayer 2005; Rau and Schmidt 2019; Sweller et al. 1998). In our game, however, the use of two displays intentionally introduces a figural mapping challenge whose perceptuomotor solution, in the form of new geometrical structures, is the pedagogical objective.

The game's environment is designed to support players in recognizing figural similarities across its two displays. As shapes appear on-screen, players move on the physical interface.[3] Once players realize that their actions on the mat are reflected on-screen, they learn to attend to the screen for feedback, even as they move on the physical interface. Thus, the game is designed to foster bi-lateral significations of its displays: the digital display mediates perception of the mat in terms of imagined lines and shapes, while the physical display mediates perception of the digital shapes by highlighting their vertices. Additional geometry notions – such as side length, angle measure, translation, dilation, rotation, and symmetry – are embedded into the task design as pre-symbolic, embodied experiences. Such context-bound skills can later be reconceived as instantiations of disciplinary mathematics content (see DeLiema et al. 2019, for examples in STEM-based play).

Having outlined *Geometris'* design and rationale, we next review a trio of theoretical frameworks that collectively offer both task-specific and socio-cultural perspectives on informal learning. We then describe the setting, methods, and analytic practices for a subsequent discussion of empirical results. Our analysis considers the varied, multi-modal resources that adults, in three different study groups, used in scaffolding children's *Geometris* play. Certain forms of that scaffolding activity, we explain, were productively constrained by particular design decisions, which we elaborate. To close, we reflect on those design decisions we interpret as most pedagogically influential and suggest their potentially broader value within mathematics learning environments.

Theoretical Frameworks

In evaluating *Geometris* gameplay, we draw on three theoretical frameworks. At the most granular level, we draw on ecological dynamics to consider players' actions as constrained by elements and interactions within the collaborative task environment. We then draw on cognitive anthropology to implicate the central challenge in performing those actions as entraining the perception–action routine of figural mapping. Finally, we take a socio-cultural perspective on means by which this perception–action routine is entrained through collaborative work.

[3] Very young children occasionally try to interact with the projection screen, perhaps based on experience with touchscreens.

The Dynamics of Joint Action

To make sense of players' collaborative, goal-oriented work in *Geometris*, we follow Abrahamson and Sánchez-García (2016) in applying constructs from ecological dynamics to mathematics education research. Researchers of *ecological dynamics* model skill acquisition in terms of the evolving dynamics between actors and the task environment (Vilar et al. 2012). Three categories of constraints in the actor–environment system fundamentally shape players' activity: *task*, *environmental*, and *organismic* constraints (Newell 1986).

The *Geometris* task is to reconstruct shapes. Task constraints are: (1) players must simultaneously activate precisely those pads corresponding to a shape's vertices; and they must do so (2) within a limited time duration; and (3) in collaboration with a partner. Players must also develop shared, mutually intelligible vocabulary and frames of reference to convey their confusion, planning, instruction, and feedback.

Environmental constraints include: (a) the game's deliberately disjoint physical layout, which requires that players develop fluent perceptual routines for figural mapping between two spatially disparate displays; and (b) the range of deliberate features of the activity resources, such as the mat's large size, as well as incidental features, such as color selections for the sensor pads.

Finally, players are constrained by their own organismic qualities, such as their capacity to access information vital for task completion, physical size, number of limbs, and dexterity. The game's set of features – simultaneous sensor activation (task constraint) and a mat interface (environmental constraint) that is larger than the players' body size (organismic constraint) – limit an individual's ability to enact a solution alone.[4] In turn, the multi-element quality of solution inputs enables a flexible distribution of labor among collaborating players. For example, a dyad could share a target set of 5 vertices at ratios of 1–4, 2–3, 3–2 or 4–1.

Players can also alter task and environmental constraints for one another through feedback and instruction that contain more information than the recipient might otherwise access (Newell and Ranganathan 2010). For example, one player might confirm their partner's hesitant movements toward a particular vertex, offering otherwise unavailable intermediate task feedback. Note that players who are new to the game, but slightly more expert than their partner(s), likely continue to learn even as they teach their partner(s). As such, it could be expected that their learning and teaching goals may, at times, compete, such as in offering incorrect instructions.

Entrained Perception as *the* Problem

By design, the *Geometris* environment requires players to develop a cross-display figural mapping. Such entrained or skill-mediated perceptual routines are well-documented within the research literature, whether as professional vision (Goodwin 1994), disciplined perception (Stevens and Hall 1998), or educated perception (Goldstone et al. 2009). We agree with Goodwin that "all vision is perspectival" (p. 606) and we believe that, within *Geometris*, entrained perception is not just *a* problem, but rather *the* problem. We draw on the tripartite role that Goodwin outlines for relative

[4] We observed one teenaged player make a hexagon by himself using his head, knees, feet, and elbows. Such contortion is atypical within *Geometris* play.

experts in entraining novices' perception: (1) *highlighting* elements of the environment as task-relevant; (2) *coding* those elements into disciplinary categories; (3) *creating and interpreting graphical representations* that collapse information across space and time.

Furthermore, we draw on Abrahamson et al.'s (2012) expansion of Goodwin's framework from perception per se to perception-for-action. That is, to develop a successful strategy, a novice must entrain their "perceptuo*motor* – not just perceptual – orientation toward the activity" (p. 77; emphasis in original). Novices must learn how to orient and adjust their physical position, for example, maintaining their gaze on the screen, in ways that support sense-making of environmental information that changes in response to their movements. We see *Geometris* players guiding novices to achieve these entrainments to varying extents, as we aim to illustrate.

Scaffolding Perception within Co-Operative Work

Finally, we draw on socio–cultural theories of learning to describe how co-operative activity entrains new action–perception routines. In studying childhood development, Vygotsky (1934/2001) differentiated between *real forms* – the intuitive ways that children perceive and act upon the world – and *ideal forms* – culturally specific and sanctioned ways of perceiving and acting. Importantly, the gradual transition between the two occurs through co-enacting ideal forms. Vygotsky's (1930/1978) famed 'zone of proximal development' captures the difference in operational outcomes when children enact real forms versus when they co-enact ideal forms with adults or more capable peers. Taking a systemic reading of Vygotsky (Shvarts and Abrahamson 2019), we apply these concepts to *Geometris* gameplay, interpreting figural mapping as an ideal perceptual form that players can learn through co-enactment. We also attend to the means by which relative experts scaffold novice partners toward this ideal.

Inspired by several Soviet researchers, such as Nikolai Bernstein (for a review, see Shvarts and Bakker 2019), the construct of scaffolding has come to be understood as "'controlling' those elements of the task that are initially beyond the learner's capacity, thus permitting him to concentrate upon […] those elements that are within his range of competence" (Wood et al. 1976, p. 90). These authors delineate categories of these 'controlling' actions: reducing degrees of freedom, highlighting salient environmental features, modeling desired actions, and offering feedback and encouragement.

As we aim to illustrate below, adults perform many of these functions in their *Geometris* play with children. In so doing, we invoke Cazden (1981) to differentiate between scaffolds oriented toward *performance*, that is completing the task at hand, and those oriented toward *competence*, that is gaining "understanding from which answers to similar questions can be generated alone" (p. 7). Importantly, Cazden does not valorize one form of assistance over the other. We take similar care to consider both as pedagogically useful within *Geometris* play.

Research Questions

1. What are common participatory facilitation techniques in informal, body-scale, collaborative mathematical play?
2. How is effective participatory facilitation of informal, body-scale, collaborative mathematical play enabled and constrained by design features?

Answering these questions, we maintain, could contribute both towards developing theories of learning and optimizing environments for engagement and learning.

Methods

From mid-June to mid-July, 2018, *Geometris* was installed at a family-oriented science museum in the western US. During installation, it was attended by volunteer facilitators from the museum who introduced it to visitors and offered varying forms of facilitation. Audio-video recordings were made over two days during the installation, capturing roughly seventy visitor groups. The camera was positioned near the ceiling at roughly a 45^{o} angle to the floor, capturing visitors as well as the mat and on-floor screen display.

In first reviewing the recordings, we noted the following features of each visitor group: how many players were present and their approximate ages; how many levels they played and of what difficulty; characteristic play behavior; whether an adult was present and, if so, whether they observed, played, or coached. This initial review revealed that visitor groups exhibited a wide range of play behavior that seemed to vary with group dynamics and game level.

Of particular interest for this study were cases in which one player developed a figural-mapping strategy before their partner(s) and then began to facilitate the group's play toward apparently shifting goals. At times, these relative experts[5] leveraged their figural mapping skill to create scenarios to help their partners develop *competence* in figural mapping. At other times, especially when time was running low, these relative experts seemed to prioritize *performance*, focusing the group's energies on completing each shape. Relatively expert players seemed to transition between these two goals, apparently driven by ad hoc design features. We term such play, in which one player supports another's work within collaborative play, *participatory facilitation*. As we conceive it, *participatory facilitation* incorporates informal facilitation and scaffolding toward goals of both competence (learning an ideal perceptual form) and performance (winning the game[6]). We are interested in the conditions that seem to support pursuit of one goal or the other.

In the course of the data analysis, the research team came to consider participatory facilitation as a potentially valuable pedagogical technique in informal mathematics learning environments. We therefore decided to narrow our study focus to groups who displayed this interaction style, and we investigated the pedagogical means employed by the relatively expert players to scaffold novices' play. Three focal groups were selected for analysis. In these groups, the expert players seemed to shift their facilitation goals. By examining why they did so, we observed consistent relations between design factors and facilitation characteristics. Incidentally, these same groups exhibited a moderate to high frequency of utterances, which was conducive for our qualitative analysis of observed behavior. We analyzed these relative experts' multi-modal utterances, including speech, gestures, body position, and

[5] We describe these players as *relative* experts, because they, along with their partners, were equally new to the game. Nevertheless, their perceived expertise *relative to their partners* seemed to sanction their informal teaching behavior through facilitation of play, which is our phenomenon of interest.

[6] Because *Geometris*' design includes figural mapping as part of the game strategy, performance-oriented facilitation could still expose relative motives to game states through which they could learn this ideal form, if incidentally.

Table 1 Categories of scaffolding in *Geometris* play

Code	Description	Example
Reduce complexity	Reduce degrees of freedom in the task	Expert activates two of three vertices of a target triangle
Direction maintenance	Direct novice's attention to the task	"Where do you have to go?"
Highlight	Point out features of the environment as task-relevant	
Mat		"That one (*points to pad*)."
Screen		"We're trying to make that triangle (*points to screen*)."
Across displays		"The yellow dot here (*points to mat*) is the yellow circle there (*points to screen*)."
Feedback	Evaluate completed action	"That's not right, is it?"
Instruction	Coach the novice on future action using…	
Direct	… specific commands	"Get the blue one (*points to pad*)."
Indirect	… general guidance	"Keep going..."

gaze, and categorized them using constructs from the earlier reviewed theoretical frameworks of constructivism, ecological dynamics, and socio-cultural theory (see Table 1).

The video data were divided into segments by utterance and coded independently by two researchers. The researchers first trained their coding on three practice groups and then coded the three focal groups. Inter-rater reliability was above 80%.

Results and Analysis

We first describe the play of each focal group and then highlight patterns in participatory facilitation across all three groups.

Jared & Audrey

Jared[7] (5 years old) and his mother, Audrey, began in the Practice level, which is not time-constrained. Audrey activated pads with her feet and hands, while Jared remained in one spot, looking between the mat, his mother, and the screen. When Audrey indicated they were ready, they began Level 1.

In Level 1, Audrey quickly and consistently activated the central yellow pad and, after the first shape, one other vertex (see Fig. 4). Jared worked to complete each shape. Audrey typically described this task, pointing to the screen and saying, "OK, now we have to make that one" (4 times) and asking her son, "Where do you need to go?" (6 times). Over a period of 2 minutes, Audrey often highlighted features of the mat (9 times) and screen (8 times), though she did not overtly link them. She also gave frequent feedback (11 times), both redirecting her son's efforts ("That's not right, is it?") and affirming his work ("You got

[7] All names are pseudonyms.

it!"). Audrey's instructions were predominantly indirect (9 indirect versus 4 direct), encouraging Jared to "try it" and "keep going." On the last shape, Audrey posed an extra challenge for her son. Previously, she had activated vertices such that Jared was relatively close to the last necessary vertex (1 or 2 pads away, Fig. 4a–g). On the last shape (Fig. 4h), Audrey moved her right foot forward from one necessary vertex to a different one, forcing Jared to move all the way around the circle. After quite a few steps (and mis-steps), Jared completed the shape in time.

As the pair played at Level 2, they communicated much less (~15 utterances versus ~40 in Level 1). Audrey again occupied two target vertices, leaving Jared to find either one or two vertices of each shape. She took longer to establish her position and repositioned herself on half the shapes, moving simultaneously with her son. As a result, Audrey had less time to focus on instructing her son, and Jared had less independent movement time, that is, time when he was the only player moving. Audrey reiterated the shape-making task objective only once, and she tended to give direct instruction (5 direct versus 4 indirect), often highlighting a place on the mat by pointing ("Go there", "Get these ones"). She offered about half as much feedback as in Level 1. For their last shape, Audrey physically positioned Jared, nudging his hips forward so he stepped off an unnecessary pad.

Joy & Mike

Joy (8 years old) and her father, Mike, approached the *Geometris* exhibit while a previous pair was playing. When this earlier dyad left half-way through Level 1, Joy and Mike stepped in. As they played the last half of Level 1, Mike consistently positioned himself on two of the three target vertices. The pair traded off completing shapes, with Mike twice working around Joy's position and twice setting her up to finish the shape. Mike highlighted three environmental features and gave feedback 3 times, though once was inaccurate.

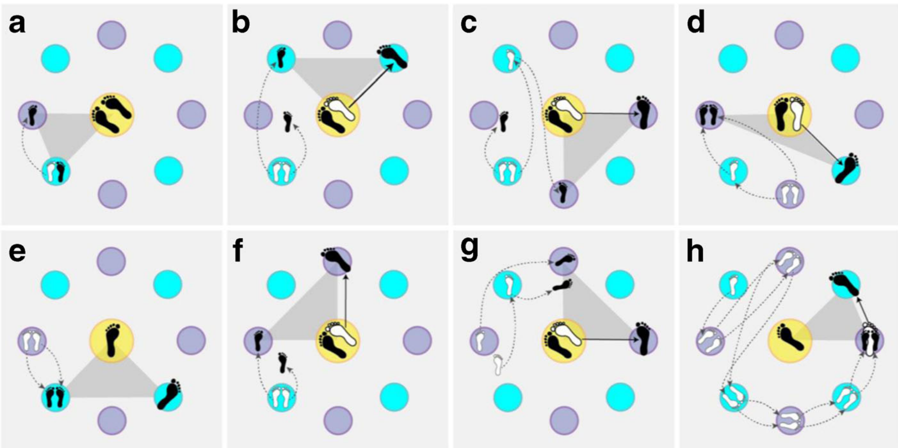

Fig. 4 Audrey's (big feet) and Jared's (little feet) positions during Level 1. Each square represents work on one target shape (in gray). White footprints represent temporary positions. (icons by James Keuning of Noun Project)

Next, they played Level 2. Mike typically positioned himself on two vertices of each shape before posing the problem to his daughter, “Where are you going to go?” (6 times). Joy worked to find the remaining one or two vertices to complete each shape. Mike scaffolded his daughter’s work with frequent feedback (22 times), evenly split between affirming her work (“Yes!”, “Perfect!”) and redirecting her efforts (“Nope”, “Not there”). Mike often paired feedback with indirect instruction in the phrase “Yes! And?” (4 times). He highlighted environmental features occasionally (7 times), usually to accompany feedback (“Yup, blue”) or instruction (“You do purple”).

When the pair played Level 3, their co-ordination patterns changed. Whereas in Level 2 Mike set his position and then prompted his daughter, in Level 3 he continued to change his position as they worked on five of the eight shapes,[8] moving simultaneously to his daughter and leaving less time for dedicated instruction. Mike no longer asked Joy where she should go. Instead of scaffolding his daughter’s exploration with feedback, Mike tended to give direct instruction such as “Go there” or “Get that purple one” (17 times). He often paired instructions with gestures that highlighted environmental features, highlighting parts of the mat (21 times) and once linking the mat and screen. He gave feedback half as often as in Level 2, skewed more toward redirecting (6 times) than affirming (4 times). As time ran down on one shape, Mike lifted his daughter’s feet off the mat to release extra pads.

Evan, Max & Leslie

Evan (8 years old) and Max (10 years old) visited the *Geometris* exhibit with Leslie, their mother. While a volunteer facilitator guided Evan and Max through the Practice level, Leslie observed and asked questions from the sidelines.

As the boys played Level 1, Leslie commented from the back of the mat (Fig. 5a), while the volunteer facilitator explained the mapping and gave feedback. On the second shape, Leslie stepped forward to give Evan feedback on his position (Fig. 5b), linking it to the on-screen display (“Do you see how this foot is not where you want it to be? You want it to be... straight shot, right?”). She then explained the figural mapping by linking the mat and screen (“So the yellow dot is the first yellow dot. You want to make your yellow lines go around the shape of the whole shape.”). With time running out, Leslie moved to complete the third shape (Fig. 5c), stopping herself as Evan got there. She remained adjacent to the screen just off the mat (Fig. 5d), posing the problem to her children (“How are you going to make this one?”) and giving direct instructions (“You’re going to get that one”). With time again running low on the fifth shape, Leslie offered a stream of feedback (“Nope, nope, nope, nope, nope”) and stopped herself just short of stepping onto the mat. She then brought her hands to her face, grinned sheepishly, and stepped back to the edge of the area (Fig. 5e), where she remained for the rest of the level.

In contrast to the other two groups, Leslie’s spontaneous facilitation did not change as dramatically when her boys played Level 2. As in Level 1, the volunteer facilitator continued to give direction and feedback. Leslie remained adjacent to the mat, at times removing her shoe as if to step on the mat, but never moving onto it. She directed her

[8] In repositioning, Mike occasionally activated and held unnecessary vertices.

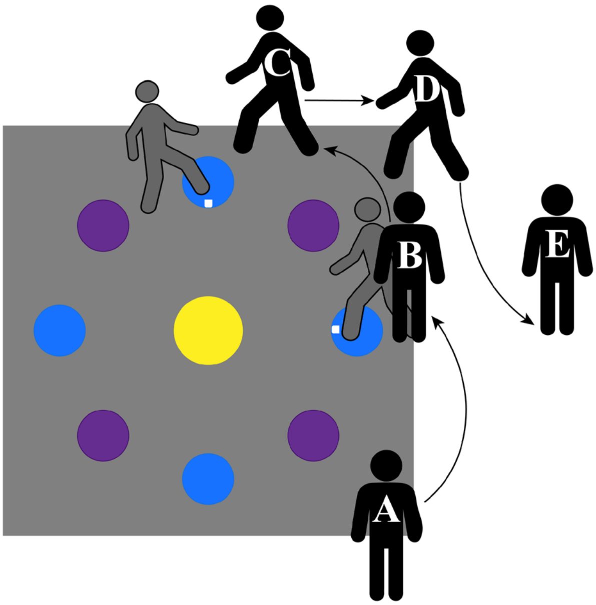

Fig. 5 Leslie's positions (black) in Level 1. Starting at A, she moved to B to give feedback ("Do you see how this foot is not where you want it to be?"). She advanced to C ("Top! That one, that one, that one, that one."), stopping short of completing the shape. She continued instructions and feedback from D ("Nope, nope, nope, nope, nope"), before stepping back to E for the rest of the level. (icon by Bradley Avison of Noun Project)

sons' attention twice (half as often as in Level 1) but gave feedback over twice as often (17 times in Level 2 versus 7 times in Level 1). The most notable change occurred in her highlighting patterns. Whereas in Level 1, she highlighted features of the mat and screen and linked them 2 or 3 times each, in Level 2, Leslie predominantly highlighted features of the mat (5 times), highlighting the screen twice and linking them only once. Consistent with the other groups, she instructed more often in this harder level (9 times in Level 2 versus 6 times in Level 1), and her instructions were predominantly direct (7 direct versus 2 indirect).

Patterns of Facilitation – When Time Runs Low, Parents Stop Teaching

Despite their varying group compositions and play styles, we see certain similarities across these adults' participatory facilitation of their children's *Geometris* play. During easier rounds, all three adults repeatedly directed their children's attention to the mapping challenge (Fig. 6). They overtly described the task (Leslie: "You want to make your yellow lines go around the shape" and Audrey: "See? We're making that triangle") or asked their children, "Where do you have to go?" (Audrey and Mike). They made space for their children to explore by setting up a simplified task scenario (Audrey and Mike) or stepping back (Leslie) (Fig. 6). They offered frequent feedback on their children's work, and their instruction tended to be indirect, encouraging their children's continued exploration (Audrey's "Keep going" and Mike's "Yes! And?"; Fig. 6). We interpret this cluster of facilitation behavior to suggest an orientation toward the children's developing competence at learning the game's rules and strategy, and –

as a by-product – figural mapping skill, in these relatively easy, less time-pressured scenarios.

These facilitation patterns changed as the challenge increased or when time ran low (Fig. 6). The adults stopped describing the task or asking their children, "Where do you have to go?"[9] Their instructions increased in number and became predominantly direct, typically paired by mat-only highlighting ("Go there (*pointing*)!", "The blue one"; Fig. 6). Such direct instruction left little room for the children to explore or get things wrong, thus reducing the frequency of feedback (Fig. 6). These changes in facilitation style fundamentally altered the task for the relatively novice players from one of figuring out where to go to one of going where they were told. We interpret this cluster of facilitation behavior to suggest an orientation toward performance, that is, winning regardless of the child's understanding during challenging, time-pressured scenarios. We acknowledge that successful performance of the game task seemed to be the parents' consistent goal and, through this analysis, highlight adults' different strategies toward that goal based on task difficulty, with differential impacts on children's opportunities to learn the target figural mapping skills.

We also note a relative infrequency of screen-based and cross-display highlighting (Fig. 6). Audrey connected features of the mat and screen only once and Mike only twice. Leslie highlighted cross-display features the most, though this highlighting decreased as the challenge increased (3 times in Level 1 versus once in Level 2). Considering the literature on entrained perception (Goodwin 1994; Stevens and Hall 1998) and entrained perception-for-action (Abrahamson et al. 2012), highlighting the cross-display correspondences more frequently could have better supported the children in connecting their on-mat movements with on-screen environmental information, both increasing their efficacy at the game (the parents' goal) and, as a by-product, their figural mapping skill (the designers' goal).

Divisions of Labor on the Large-Scale, Multi-Sensor Interface

We also emphasize the role of the mat interface in adults' participatory facilitation, in particular their use (or dis-use) of the mat to scaffold their children's play. In easier rounds of play, Audrey and Mike would set their position, activating one or two pads, before prompting their children's work ("Where do you have to go?"). This routine established an implicit norm of "I go, you go" – a sequential rather than simultaneous movement co-ordination that simplified the child's task by: (1) reducing the remaining work; (2) clarifying the task of finding oneself in the display (the moving point is the moving person); and (3) removing the need to negotiate a distribution of labor. That routine broke down during more challenging play.

As the increasingly complex shapes increased the mapping challenge, parents took longer to position themselves (see vertical lines in Fig. 6, Reduce Complexity), which both decreased time for their children to explore and meant that players moved simultaneously, eliminating the three simplifying benefits described above. Though

[9] While this decrease could result from fatigue, there is no demonstrable decrease across each group's first round of play.

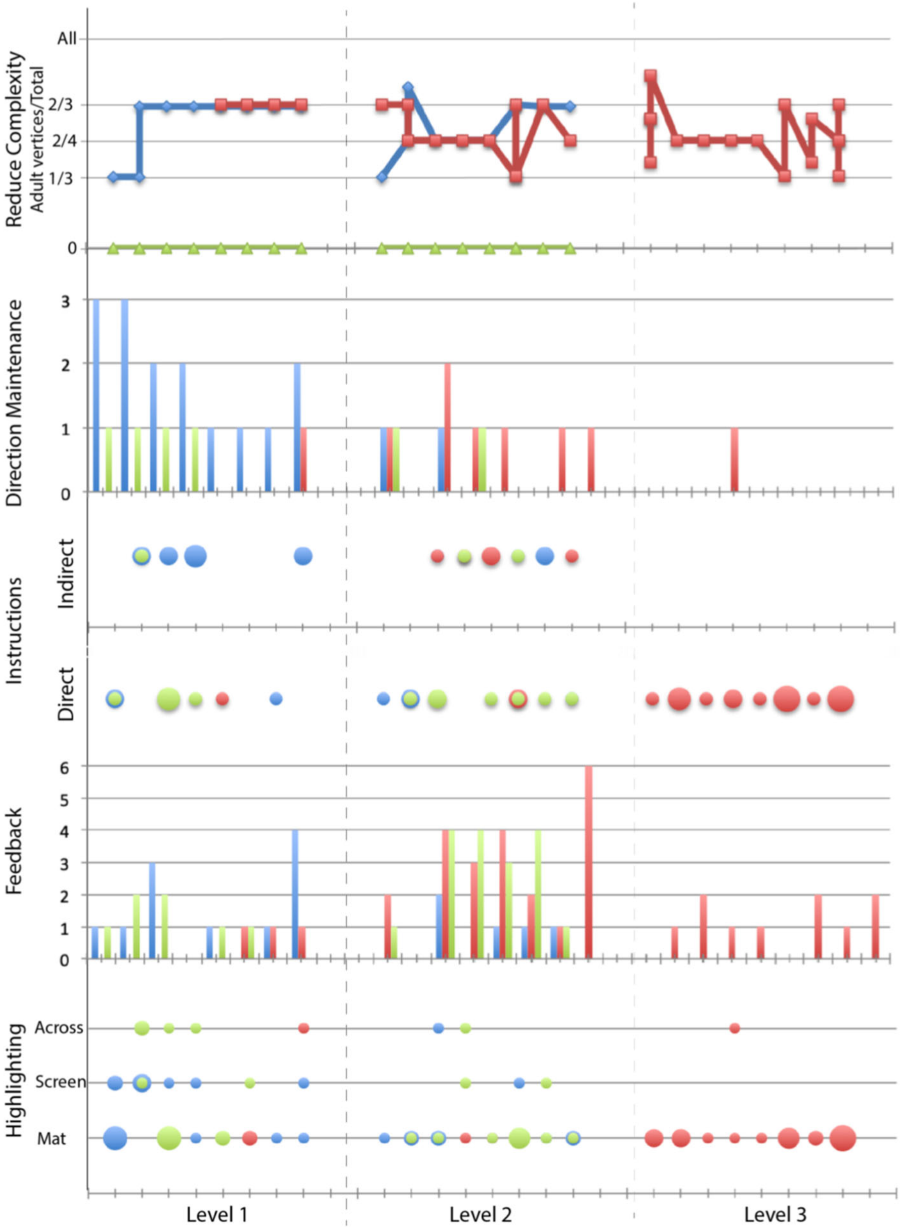

Fig. 6 Audrey (blue), Mike (red), and Leslie's (green) participatory facilitation behavior by level of *Geometris* play, indicated per shape. From the top: the degree to which each adult reduced task complexity by activating vertices; frequency of directing children's attention to the task; the type and number (bubble size) of instructions; frequency of feedback; location and number (bubble size) of highlighted environmental features. Only Mike and Joy played Level 3

she did not activate pads, Leslie exhibited a similar pattern of behavior. When her sons were exploring or performing well, she stood back from the mat. When they struggled, she stepped forward, stopping herself just short of activating vertices. Ironically, by doing more during challenging moments, parents likely complicated their children's figural-mapping task by introducing more variables into the physical and digital displays, thereby obfuscating cause-and-effect relationships between them.

Discussion and Implications

By its design, *Geometris'* distribution across two displays (here, a physical interface and a digital screen) poses cross-display figural mapping as the key challenge for players. By its design, it also affords resources for scaffolding this learning objective, namely the large-scale, multi-sensor mat interface. As the above cases illustrate, parents' participatory facilitation of their children's play involved flexible use of the game mat among other scaffolding techniques.

The above cases also illustrate that parents' participatory facilitation changed in similar and pedagogically relevant ways. In easier rounds, facilitation supported children in exploring the game's rules and developing successful strategies based on figural mapping. Parents took up consistent positions, posed consistent problems and guided their children using feedback. When play became more challenging, those facilitation patterns changed, and children's exploration was replaced by instruction-following. Parents moved more, often simultaneously to their children; they stopped overtly framing the task; and they directly instructed their children precisely where to go. From a design perspective, these changes compromised opportunities to notice cross-display similarities in several ways. Simultaneous movement obfuscates cause–effect relationships across displays. At the same time, the decrease in adults' problem-posing and cross-display highlighting meant that those relationships were not highlighted either verbally or gesturally. In these challenging and time-pressured moments, parents made it harder for their children to develop figural mapping skills that would make them more effective players. In noting these pedagogical differences between spontaneous facilitation styles, we aim to avoid the role of critic. Rather, we are encouraged to see exemplar scenarios where each style emerges and to consider design choices that appear to influence facilitation. We also note that these changes may be due to parents' relative, rather than absolute, expertise at the game. With more experience and a stronger sense for the figural mapping across all levels, it could be that these adults would maintain competence-oriented scaffolding techniques throughout play.

We use the remaining space here to reflect on *Geometris'* dual-display design and flexible user interface and to propose directions for future work.

In Dual-Display Designs, as Elsewhere, Mechanics Matter

We claim that *Geometris'* dual-display design poses for players the cognitive and perceptual challenge of figural mapping. Distributed over physical and digital media, *Geometris* prompts players to establish a mapping between their inputs on the mat and corresponding outputs on the screen. Importantly, this functional mapping alone does not pose the figural mapping challenge that we, as designers, deem relevant to mathematics learning. Rather, figural mapping additionally requires that the means through which players engage with these distributed displays to perform the tasks – that is, the sensorimotor actions players enact to solve emergent problems – are designed to constitute the target mathematics skills.

For contrast, consider Brain Dive (Kiili and Perttula 2012), another body-scale educational game for mathematics. In Brain Dive, a basic arithmetic problem is projected on a large screen. A host of fish also appear on-screen, each labeled with a number. The player, monitored by a motion tracker, jumps to direct their shark up or down in the water column to eat the fish labeled with the answer to the arithmetic problem. Whereas Brain Dive thus requires a particular functional mapping between physical movement and changes to the digital display, the required spatial–dynamical physical act (jumping) does not enact the logico–mathematical process of the content in question (an arithmetic operation). As a result, the fostered sensorimotor perceptions are unrelated to the target concept.

In *Geometris*, the tactic of moving in physical space is precisely reflected in the appearance of points and lines in digital space. Thus, the central figural mapping challenge arises not from the use of two displays per se but from design choices that engender interactions with those displays that are congruent with the mathematical skills for engaging with the target concepts. In *Geometris*, human bodies can collectively enact, subtend, inscribe, and configure mathematical objects. We propose that sensorimotor congruence between game mechanics and disciplinary practice is a pedagogically useful characteristic of body-scale mathematics learning environments.

Flexible Divisions of Labor Accommodate Flexible Facilitation Goals

We also claim that *Geometris'* large-scale, multi-sensor mat interface creates opportunities for spontaneous facilitation to scaffold novices' play. Recalling our selection criteria for the data analysis, we chose focal groups in which a relatively expert player both facilitated play and seemed to exhibit a change in their facilitation goals, switching between performance and competence. As we aimed to illustrate above, the physical interface serves both these goals. The game mat is a shared physical interface with sensors agnostic to individual agents. It requires simultaneous physical contact with a discrete set of sensors and, as such, that set is given to multiple valid partitions. Relatively expert players sometimes foster consistent partitions, as in the "I go, you go" routines of Audrey and Mike's first levels, conducive to novices' developing competence. At other times, relative experts leverage this flexible partitioning toward performance, as when each of the three adults, above, moved to complete shapes. Importantly, the transition between these two modes was rapid, reversible, and required little overt co-ordination. We argue that this ability to switch seamlessly between facilitation goals while maintaining authentic play bears pedagogical value.

For contrast, consider the game of catch: two people must stand at a distance in order to pass a ball back and forth. If a novice demonstrates a need for significant instruction, for example, if they keep throwing the ball into the ground or hold their baseball mitt at their hip, the more expert player must leave their position, approach the novice, and guide them, perhaps by positioning their hands and limbs, perhaps by doing a throw together, etc. By requiring close physical proximity, these moments of competence building are incompatible with normative play.[10] Once the expert deems the novice ready, they return to their separate positions and resume play.

In *Geometris*, the physical interface requires no such delineation between teaching–learning and authentic play. Teaching and learning can be seamlessly integrated into play, as when a relative expert completes more or less of the target shape, gives more direct or indirect instruction, or gives more specific or general feedback. Such scaffolds can be taken up, cast off, and taken up again within the norms of gameplay. We do not argue that *Geometris* always results in such teaching behavior – the examples, above, show it does not. Rather, we argue that an interface which supports the ability to switch seamlessly between performance and competence goals respects novice learners' agency by maintaining their role as contributors to authentic play through varying degrees of scaffolding. Counter to familiar accounts of scaffolds monotonically fading, we document iterative cycles of deploying and fading participation scaffolds. Given that the target figural mapping skill is one of entrained perception, specifically perception of environmental information as mediated by an ideal form of instrument use (Vygotsky 1930/1978), we argue that learning environments that enable flexibly distributed labor create especially auspicious conditions under which that perception can be encouraged, tested, and developed.

Limitations and Future Work

We identify some limitations to the above analysis and identify areas for continued work. While on display, *Geometris* was monitored by at least one volunteer facilitator at all times. These facilitators adopted a variety of approaches, from encouraging visitors to 'figure it out', to overtly explaining the figural mapping and giving frequent feedback, to playing alongside visitors. Such facilitation undoubtedly impacted guests' experiences of the designed environment and, likely, the play behavior and participatory facilitation we observed from adults. As it was outside the scope of our role as researchers to standardize these volunteers' facilitation behavior, we simply noted for each visitor group the frequency and type of volunteer facilitation.

It could also be that some visitors facilitated their partners' play in ways not captured by our analysis. Visitors may choose to facilitate in these more subtle ways, without directly observable behavior, or by choosing not to get involved. While we tracked changes in participation behaviors – for example, Leslie's stepping up and pulling back – our analysis does not include adults who chose non-participation from the start as their means of facilitating their children's playful exploration.

To guide future work, we also identify factors, both from within the designed environment and from the exhibit setting, that seemed to influence adults' participatory facilitation. We associate the major change in facilitation behavior documented above with the game element of time, specifically a lack thereof. Moments when time was running low tended to be the same moments that adults shifted from posing problems and giving feedback to instructing their children directly. More work is needed to evaluate this apparent relationship. Additionally, the groups varied along multiple dimensions external to the design: child age, number of children, and the volunteer

[10] We can imagine other types of competence-oriented facilitation – such as increasing the distance between players, modeling form, or throwing certain types of passes – as compatible with normative play.

facilitator's engagement level, to name a few. Presumably, all of these factors influence how adults engaged with the game and how their participatory facilitation unfolded. For example, Leslie seemed to attend more to Evan, her eight-year-old son, than to Max, her ten-year-old. Work remains to evaluate the impact of each dimension on facilitation behavior, so that players of all ages and group compositions may access and grapple with the game's figural-mapping challenge in ways conducive to their mathematics learning.

Conclusions

This article reported on the empirical evaluation of *Geometris*, a collaborative, body-scale, geometry game. We argued that two qualities of the design support interactions conducive to pursuing the game's learning objective of developing geometrical and spatial reasoning. First, *Geometris'* spatial distribution across two displays poses figural mapping as the central challenge for players. Second, the flexible divisions of labor enabled by the game's body-scale, multi-sensor interface open possibilities for addressing that challenge. We propose that this dialectical design architecture – engineering activity features that pose domain-relevant problems for task performance even as they create conditions for teaching and learning how to overcome those problems – is pedagogically desirable in educational designs.

We additionally identified, characterized, and exemplified the phenomenon of participatory facilitation, a spontaneous pedagogical practice, in which relative experts alter their own play in consistent ways to facilitate novices' contribution toward shared task outcomes. By delineating the flexible adoption, adaptation, casting-off, and redeployment of scaffolding behavior that occurs within participatory facilitation, we offer this construct as a contribution to theoretically oriented literature on (in-)formal instruction. By implicating design decisions that enable and shape particular participatory facilitation behavior, namely imposing a time constraint on task completion as well as furnishing a large-scale, multi-sensor user interface, we also suggest the value of this construct for pragmatically oriented literature on educational activity design.

More broadly, we see the above cases of *Geometris* play as examples of immersive adult–child co-play in an informal educational setting. Other scholars have identified trade-offs of such immersive educational designs. For example, children may be so immersed in play that they seldom reflect on the domain-relevant relationships that designers intended to highlight (Malinverni et al. 2016), with verbal description implicated as a critical component of reflection on action (Nathan and Walkington 2017). We have extended these ideas to participatory facilitation of play. Adults who facilitate immersive educational designs for their children may themselves become sufficiently immersed in play so as to change their facilitation behavior, shifting from a competence orientation to a performance orientation. These findings echo research on informal science education, which suggests that, in collaborative tasks, adults may bear the brunt of cognitive work rather than cede control to their children (Gleason and Schauble 1999). Especially given the informal educational setting of the cases presented in this article, we aim to avoid elevating certain facilitation goals and behavior over others. Rather, we suggest that identifying design elements that tip this balance of facilitation goals could be beneficial for the design of immersive activity in contexts that do seek to promote competence-oriented mathematics teaching and learning.

Acknowledgements The authors wish to thank the DEME Guest Editors and the 3 anonymous reviewers for their formative comments on this manuscript. We also wish to thank the exhibits staff, volunteers, and participating visitors at the Lawrence Hall of Science, University of California Berkeley, for making this study possible. The ideas presented in this article were workshopped and refined with members of the Embodied Design Research Laboratory at the University of California, Berkeley. Geometris was collaboratively designed and created by Elena Durán-López, Ganesh V. Iyer, and Leah F. Rosenbaum, with significant guidance from Professor Kimiko Ryokai and Dr. Noura Howell.

Compliance with Ethical Standards

Conflict of Interest On behalf of all authors, the corresponding author claims no conflicts of interest.

References

Abrahamson, D. (2002). When “the same” is the same as different differences: Aliya reconciles her perceptual judgment of proportional equivalence with her additive computation skills. In D. Mewborn, P. Sztajn, E. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.), *Proceedings of the twenty-fourth annual meeting of the north American chapter of the International Group for the Psychology of mathematics education* (Vol. 4, pp. 1658–1661). Columbus: PME-NA.

Abrahamson, D. (2009). Orchestrating semiotic leaps from tacit to cultural quantitative reasoning: The case of anticipating experimental outcomes of a quasi-binomial random generator. *Cognition and Instruction, 27*(3), 175–224.

Abrahamson, D. (2012a). Discovery reconceived: Product before process. *For the Learning of Mathematics, 32*(1), 8–15.

Abrahamson, D. (2012b). Rethinking intensive quantities via guided mediated abduction. *Journal of the Learning Sciences, 21*(4), 626–649.

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction, 2*(1), 1–16.

Abrahamson, D., & Sánchez-García, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *Journal of the Learning Sciences, 25*(2), 203–239.

Abrahamson, D., & Wilensky, U. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning, 12*(1), 23–55.

Abrahamson, D., Gutiérrez, J., Charoenying, T., Negrete, A., & Bumbacher, E. (2012). Fostering hooks and shifts: Tutorial tactics for guided mathematical discovery. *Technology, Knowledge, and Learning, 17*(1–2), 61–86.

Barnes, B., Henry, J., & Bloor, D. (1996). *Scientific knowledge: A sociological analysis*. Chicago: University of Chicago Press.

Bartolini Bussi, M., & Mariotti, M. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, B. Sriraman, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). New York, NY: Routledge.

Broderbund. (1996). *Logical journey of the Zoombinis*. Novato: Broderbund Software (video game).

Cazden, C. (1981). Performance before competence: Assistance to child discourse in the zone of proximal development. *Quarterly Newsletter of the Laboratory of Comparative Human Cognition, 3*(1), 5–8.

CCSSI (2017). *Standards for Mathematical Practice*. (http://www.corestandards.org/Math/Practice/).

DeLiema, D., Enyedy, N., & Danish, J. (2019). Roles, rules, and keys: How different play configurations shape collaborative science inquiry. *Journal of the Learning Sciences, 28*(4–5), 513–555.

Durán-López, E., Iyer, G., & Rosenbaum, L. (2017). *Geometris*: A collaborative embodied geometry game. In G. Mark, S. Fussel, F. Mueller, & J. Tanenbaum (Eds.), *Proceedings of the 35th conference on human factors in computing systems* (pp. 214–217). Denver: CHI.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131.

Enyedy, N., Danish, J., & DeLiema, D. (2015). Liminal blends: How students blend symbols, experiences, and their own bodies together in order to co-construct meaning in a collaborative augmented reality learning environment. *International Journal of Computer-Supported Collaborative Learning, 10*(1), 7–34.

Flood, V. (2018). Multimodal revoicing as an interactional mechanism for connecting scientific and everyday concepts. *Human Development, 61*(3), 145–173.

Fyfe, E., McNeil, N., Son, J., & Goldstone, R. (2014). Concreteness fading in mathematics and science instruction: A systematic review. *Educational Psychology Review, 26*(1), 9–25.

Gilligan, K., Hodgkiss, A., Thomas, M., & Farran, E. (2019). The developmental relations between spatial cognition and mathematics in primary school children. *Developmental Science, 22*(4), #e12786.

Gleason, M., & Schauble, L. (1999). Parents'assistance of their children's scientific reasoning. *Cognition and Instruction, 17*(4), 343–378.

Goldstone, R., Landy, D., & Son, J. (2009). The education of perception. *Topics in Cognitive Science, 2*(2), 265–284.

Goodwin, C. (1994). Professional vision. *American Anthropologist, 96*(3), 603–633.

Habgood, M., & Ainsworth, S. (2011). Motivating children to learn effectively: Exploring the value of intrinsic integration in educational games. *The Journal of the Learning Sciences, 20*(2), 169–206.

Hall, R., Ma, J., & Nemirovsky, R. (2015). Rescaling bodies in/as representational instruments in GPS drawing. In V. Lee (Ed.), *Learning technologies and the body* (pp. 112–131). New York: NY: Routledge.

Hegedus, S., & Penuel, W. (2008). Studying new forms of participation and identity in mathematics classrooms with integrated communication and representational infrastructures. *Educational Studies in Mathematics, 68*(2), 171–183.

Holbert, N., & Wilensky, U. (2014). Constructible authentic representations: Designing video games that enable players to utilize knowledge developed in-game to reason about science. *Technology, Knowledge and Learning, 19*(1–2), 53–79.

Howison, M., Trninic, D., Reinholz, D., & Abrahamson, D. (2011). The mathematical imagery trainer: From embodied interaction to conceptual learning. In G. Fitzpatrick, C. Gutwin, B. Begole, W. Kellogg, & D. Tan (Eds.), *Proceedings of the annual meeting of the Association for Computer Machinery Special Interest Group on computer–human interaction* (pp. 1989–1998). New York: ACM Press.

Kafai, Y. (1996). Learning design by making games: Children's development of design strategies in the creation of a complex computational artifact. In Y. Kafai & M. Resnick (Eds.), *Constructionism in practice: Designing, thinking and learning in a digital world* (pp. 71–96). Mahwah: Lawrence Erlbaum Associates.

Kelton, M. & Ma, J. (2020, On-line). Assembling a torus: Family mobilities in an immersive mathematics exhibition. *Cognition and Instruction*, (30).

Kiili, K., & Perttula, P. (2012). Exerbraining for schools: Combining body and brain training. *Procedia Computer Science, 15*, 163–173.

Leung, A., Baccaglini-Frank, A., & Mariotti, M. (2013). Discernment of invariants in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460.

Ma, J. (2017). Multi-party, whole-body interactions in mathematical activity. *Cognition and Instruction, 35*(2), 141–164.

Malinverni, L., Ackermann, E. & Pares, N. (2016). Experience as an object to think with: From sensing-in-action to making-sense of action in full-body interaction learning environments. In Proceedings of the tenth international conference on tangible, embedded, and embodied interaction (pp. 332–339). Eindhoven, The Netherlands: ACM.

Mayer, R. (2005). Cognitive theory of multimedia learning. In R. Mayer (Ed.), *The Cambridge handbook of multimedia learning* (pp. 31–48). New York: Cambridge University Press.

Nathan, M., & Walkington, C. (2017). Grounded and embodied mathematical cognition: Promoting mathematical insight and proof using action and language. *Cognitive Research: Principles and Implications, 2*(1), 20.

NCTM. (2000). *Principles and standards for school mathematics*. Reston: National Council of Teachers of Mathematics.

Nemirovsky, R., Kelton, M., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education, 44*(2), 372–415.

Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16*(2), 119–172.

Newell, K. (1986). Constraints on the development of coordination. In M. Wade & H. Whiting (Eds.), *Motor development in children: Aspects of co-ordination and control* (pp. 341–361). Amsterdam: Martinus Nijhoff Publishers.
Newell, K., & Ranganathan, R. (2010). Instructions as constraints in motor skill acquisition. In I. Renshaw, K. Davids, & G. Savelsbergh (Eds.), *Motor learning in practice: A constraints-led approach* (pp. 17–32). Florence: Routledge.
Newman, D., Griffin, P., & Cole, M. (1989). *The construction zone: Working for cognitive change in school.* New York, NY: Cambridge University Press.
Okamoto, Y., Weckbacher, L., & Hallowell, D. (2014). How is spatial reasoning related to mathematical thinking and how important is early exposure to spatial activities? In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group of the Psychology of Mathematics Education* (Vol. 1, pp. 177–179). Vancouver: PME.
Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry (E. Lunzer, trans.)*. New York: Basic Books.
Price, S., & Duffy, S. (2018). Opportunities and challenges of bodily interaction for geometry learning to inform technology design. *Multimodal Technologies and Interaction, 2*(3), #41.
Rau, M., & Schmidt, T. (2019). Disentangling conceptual and embodied mechanisms for learning with virtual and physical representations. In S. Isotani, A. Ogan, P. Hastings, B. McLaren, & R. Luckin (Eds.), *Artificial intelligence in education* (pp. 419–431). Cham: Springer.
Sfard, A. (2002). The interplay of intimations and implementations: Generating new discourse with new symbolic tools. *Journal of the Learning Sciences, 11*(2–3), 319–357.
Sfard, A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? Early numerical thinking revisited. *Cognition and Instruction, 23*(2), 237–309.
Shvarts, A. & Abrahamson, D. (2019). Dual-eye-tracking Vygotsky: A microgenetic account of a teaching/learning collaboration in an embodied-interaction technological tutorial for mathematics. *Learning, Culture and Social Interaction*, *22*, (#100316).
Shvarts, A., & Bakker, A. (2019). The early history of the scaffolding metaphor: Bernstein, Luria, Vygotsky, and before. *Mind, Culture, and Activity, 26*(1), 4–23.
Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In M. Lampert & M. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 107–149). Cambridge: Cambridge University Press.
Sweller, J., van Merrienboer, J., & Paas, F. (1998). Cognitive architecture and instructional design. *Educational Psychology Review, 10*(3), 251–296.
Uttal, D., Meadow, N., Tipton, E., Hand, L., Alden, A., Warren, C., & Newcombe, N. (2013). The malleability of spatial skills: A meta-analysis of training studies. *Psychological Bulletin, 139*(2), 352–402.
Vilar, L., Araújo, D., Davids, K., & Renshaw, I. (2012). The need for 'representative task design' in evaluating efficacy of skills tests in sport: A comment on Russell, Benton and Kingsley (2010). *Journal of Sports Sciences, 30*(16), 1727–1730.
Vygotsky, L. (1934/2001). *Lektsii po pedologii* [Lectures on paedology]. Izhevsk, Russia: Izdatel'kii dom Udmurtskii universitet.
Vygotsky, L. (1930/1978). Mind in society: The development of higher psychological processes. Cambridge: Harvard University Press.
Wilensky, U., & Stroup, W. (2000). Networked gridlock: Students enacting complex dynamic phenomena with the HubNet architecture. In B. Fishman & S. O'Connor-Divelbiss (Eds.), *The 4th International Conference of the Learning Sciences* (pp. 282–289). Mahwah: Lawrence Erbaum Associates.
Williams-Pierce, C. (2016). Provoking mathematical play through hidden deep structures. In C. Looi, J. Polman, U. Cress & P. Reimann (Eds), Proceedings of the 12th international conference of the learning sciences (vol. 2, pp. 1241–1242). Singapore: International Society of the Learning Sciences.
Wittgenstein, L. (1953). *Philosophical investigations (G. Anscombe, trans.)*. Upper Saddle River: Prentice Hall.
Wolfgang, C., Stannard, L., & Jones, I. (2003). Advanced constructional play with LEGOs among preschoolers as a predictor of later school achievement in mathematics. *Early Child Development and Care, 173*(5), 467–475.
Wood, D., Bruner, J., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry, 17*(2), 89–100.