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Towards an Enactivist Mathematics Pedagogy

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Abstract

Enactivism theorizes thinking as situated doing. Mathematical thinking, specifically, is handling imaginary objects, and learning is coming to perceive objects and reflecting on this activity. Putting theory to practice, Abrahamson's embodied-design collaborative interdisciplinary research program has been designing and evaluating interactive tablet applications centered on motor-control tasks whose perceptual solutions then form the basis for understanding mathematical ideas (e.g., proportion). Analysis of multimodal data of students' hand- and eye-movement as well as their linguistic and gestural expressions has pointed to the key role of emergent perceptual structures that form the developmental interface between motor coordination and conceptual articulation. Through timely tutorial intervention or peer interaction, these perceptual structures rise to the students' discursive consciousness as "things" they can describe, measure, analyze, model, and symbolize with culturally accepted words, diagrams, and signs—they become mathematical entities with enactive meanings. We explain the theoretical background of enactivist mathematics pedagogy, demonstrate its technological implementation, list its principles, and then present a case study of a mathematics teacher who applied her graduate-school experiences in enactivist inquiry to create spontaneous classroom activities promoting student insight into challenging concepts. Students' enactment of coordinated movement forms gave rise to new perceptual structures modeled as mathematical content.

1. Introduction: What Does it Mean to Grasp a Mathematical Concept?

Experienced mathematics teachers know intuitively when a student understands a concept. "You got it," they might exclaim, "now hold on to that idea!" Students, in turn, know what the teacher means—they *feel* that they got it and that they can hold on to the idea. Yet, is an idea truly a thing one can *get* and *hold*? In what sense is a thought a tangible object one can grasp? Why might we want to refer to an idea, which, by definition, is immaterial, as bearing even vague likeness to a sensuous concrete thing? And, wait, is this question even important?

We believe this question is important, and here we invite you, our gentle reader, to dwell with us, for a while, on this question and see where this may take us. Spoiler alert: This question will take us toward advocating for what we will be calling an enactivist mathematics pedagogy. In short, for now, by *enactivist mathematics pedagogy* we mean to foreground embodied, dynamical, and imaginary facets of teaching and learning mathematics. Whereas these facets of conceptual practice are often ignored, marginalized, or trivialized, we believe they are constitutive of our cognitive phenomenology, specifically, what we experience as we think mathematically. We will soon be unpacking this notion and exploring its implications for the design and facilitation of instructional activities and classroom discourse. But, for now, let us return to how we all apparently think about thinking.

The would-be materiality of thoughts is entrenched in language. At least, in English, Dutch, or Hebrew we often use phrases such as “to grasp an idea,” “build an argument,” or “unpack a notion,” expressions which suggest our cultures’ prevalent, if implicit, belief that cognitive activity is somehow akin to manipulating things. In fact, the word “comprehend,” namely to understand, is rooted in the Latin verb *prehendere*, to grasp or to seize, from which we have other words, such as *prehensile*, *apprehend*, and, well, *prison*. “Even the English word *get* comes to us from the same ancient root that led to the Latin *prehendere*” (Merriam Webster Dictionary, 2020). In all these are implicated the hands—groping, grabbing, grasping—thus populating language with tacit lexical evocations of hands as vital participants in the sense-making process and, as such, conjuring an object-like quality of ideas. But are these turns of phrase all just curious fortuitous remnants of obsolete folk psychology, the type of speech forms people resort to in attempting to describe ineffable mental activity, as per the cognitive semantics theory of conceptual metaphor (Lakoff & Johnson, 1980)? Or, rather, could these heritage linguistic nuggets offer useful etymological clues into our shared pre-semantic cognitive phenomenology, what it *feels* like to think?

That is, the objective immateriality of ideas notwithstanding, what if the subjective psychological *experience* of thinking was, indeed, very much like that of handling material objects? In mathematics classrooms, we usually discuss ideas as though they transcend materiality, and we hardly talk about the subjective experience of mathematical thinking. By avoiding how it feels to think, by denuding concepts of their corporeality, by exorcising cognition of flesh, touch, and movement, we unwittingly bow to an epistemological tradition that separates the mind from the body (Damasio, 2000; Radford, 2009; Rotman, 1993). But is that the only way of thinking about thinking? Still, is all this erudite ontological theorization of ideas—as either immaterial and non-sensory or quasi-material and sensuous—ultimately important for what teachers and students do in classrooms?

We submit that this conversation about mathematical epistemology is worthwhile and even momentous for educational practitioners. We believe that attending together to the inherently prehensile and sensuous experience of mathematics will elevate its schooling by legitimizing classroom discourse on the sensorimotor phenomenology of thinking about the meanings of symbolic artifacts, such as diagrams, tables, and equations. We envision classroom *epistemic climates* (Feucht, 2010), where students and teachers share a candid, safe, and productive space of reckoning together with the core subjective experience of mathematics learning and thinking, an experience that, we will argue, is situated, sensorial, dynamical, and actionable; an experience of figuring stuff out and getting work done by handling things; an experience that may be personal, multimodal, pre-semiotic, and nuanced, yet is real, undeniable, and critical. Furthermore, we will argue that acknowledging the embodied nature of mathematical thinking bears direct impact on the quality of educational materials that could be available for instructional practice. That is, a conversation about the lived experience of mathematical concepts should result in how we design, build, evaluate, and facilitate opportunities for learning, for example by envisioning and engineering new genres of interactive resources compatible with principles of enactivist mathematics pedagogy. For, as enactivist phenomenologist Claire Petitmengin (2007) asks:

> are our teaching methods well adapted? For at present, teaching consists in most cases of transmitting conceptual and discursive contents of knowledge. The intention is to fix a meaning, not to initiate a movement. Which teaching methods, instead of *transmitting* contents, could elicit the gestures which allow access to the source experience that gives these contents coherence and meaning? Such a teaching approach, based more on initiation than transmission, by enabling children and students to come into contact with the depth of their experience, could re-enchant the classroom. (p. 79, original italics)

Yet, what movements should we initiate that would foster conceptually meaningful gestures? More broadly, this chapter we will address the following questions:

1. What philosophical and theoretical ideas underlie the notion of mathematical thinking as an embodied experience? What empirical research supports this thesis?
2. What might be a mathematics pedagogy informed by these ideas? In particular, what digital resources and instructional methodologies could enable teachers to take advantage of scientific research on embodied mathematics?
3. How can we prepare teachers to implement enactivist mathematics pedagogy?

2. Thinking as Doing: From Enactivist Philosophy to Educational Practice

If we, education practitioners and researchers, are to build together an enactivist mathematics pedagogy where concepts emerge from new ways of moving, then we need ways of talking about relations between doing and thinking. This section appeals to the scholarship of metaphysical philosophy and cognitive science concerning the emergence of meaning from experience in the natural and cultural environments. We introduce an apparently unfordable epistemic schism between unreflective doing and linguistic thinking, then we explain how, nevertheless, new ways of thinking could rise through reflecting on doing. In particular, we draw on phenomenological, constructivist, and enactivist work to motivate learning environments that create conditions for students to develop perceptual strategies for moving in new ways that, reflected upon, become dynamical instantiations of mathematical concepts.

2.1 The Problem of Language

We are prisoners of our language. Grammar, with its immutable Subject–Verb–Object syntactic structure, surreptitiously subscribes us to a particular world order we ourselves complicity corroborate each time we speak, write, and read—the myth of us each being an autonomous agent operating on an external reality (cf. Barton, 2008, on non-dualist languages). In some sense, yes, sure, that is true. I am here, coffee mug is there, and now I LIFT MUG, Subject Verb Object. But then again, in another sense, this linguistic segregation of agent (me), operation (lifting), and patient (mug) does not capture my experience *before I stopped to think about it*. That is, until I began reflecting on my action, I was not splitting the world from myself. Rather, I was immersed in unreflective doing, in which I and the ready-to-hand mug were an irreducible unreflective action unit, what the German phenomenologist philosopher Martin Heidegger calls the *Dasein* (Heidegger, 1962). That is, I wasn't thinking, I was just doing, drinking.

We are trapped, because if I try to capture a glimpse of the pre-reflective drinking *Dasein*, then, by virtue of so doing, it materializes and fractures into its constituent grammatical elements; immediately and irredeemably my phenomenologically irreducible I–MUG action-unit is severed through what French phenomenology philosopher Maurice Merleau–Ponty (1945/2005) dubbed *representational intentionality*. By virtue of inspecting the pre-reflective, we necessarily bring forth a new world order that is no longer who we were a split second prior. This is slippery epistemological territory with a syntactical twist in the road: Whenever we stop to think about what we are doing, we think *about* how we are doing it, *what* we are working *on*, and *how* we are doing this. And so, quite logically, our thinking-self will rationalize that our lived life consists of an "I" that is separate from, yet somehow entangled with, the stuff out there. Again, a quick visual scan will make evident that the mug is not a material part of my body, and so, it is enticing to posit the mug and I as fundamentally distinct entities that become related through my action upon it. Yes, that makes sense. And yet making sense is not the issue at hand, at least, not yet. Rather, it is about a moment prior to making sense. It's about our mind's innate capacity to be one with the world. Formal learning, along with inscribing and representing, begins when we stop to think about it, when we begin to think about doing.

A caveat here is due, though, that our broad philosophical brushstrokes are brazenly obscuring epistemological and ontological nuances dear to phenomenologists. Closer inspection of these distinctions would only contribute to a pedagogical framework seeking to ground its practice in philosophical *terra firma* even as it looks to evaluate the firmness of these very hallowed grounds against empirical data of learning micro-processes. Indeed, the early 20th century, whence phenomenology, constructivism, cultural–historical psychology, ecological psychology, and dynamical systems theory burgeoned, continues to offer educational researchers a stimulating palette of ontic attributions to sensory perception. Curiously, though, from across their rival theses respecting the nature and provenance of meaning—be it epigenetic realization, cultural appropriation, of some propitious synergy thereof—many educational theorists locate learning in interaction, as we now elaborate.

2.2 Educational Philosophy

Pedagogical philosophers have tended to agree that learning should begin from immersed purposeful experiences. During the Enlightenment, Genevan philosopher Jean-Jacques Rousseau (1755/1979) considered how best to teach Emile, the model student of futurist pedagogy:

> What is the use of all these symbols; why not begin by showing him the real thing so that he may at least know what you are talking about? .... As a general rule—never substitute the symbol for the thing signified, unless it is impossible to show the thing itself; for the child's attention is so taken up with the symbol that he will forget what it signifies. (Book III, p. 170)

Operating in like spirit, the German inventor of kindergarten, Friedrich Fröbel (1885/2005), pioneered pedagogical regimens consisting of material resources, such as a yarn ball—"gifts," he called these—for children to play with informally, many years before they are to formally inspect the properties of a sphere. Similarly, in Italy, Maria Montessori (1949/1967) created materials and articulated precise instructions for students to engage with these materials. This learning-by-doing reform-oriented zeitgeist resonated with United States philosopher John Dewey's educational program to establish schools where formal study in vested in purposeful experiences:

> [C]areful inspection of methods which are permanently successful in formal education, whether in arithmetic or learning to read, or studying geography, or learning physics or a foreign language, will reveal that they depend for their efficiency upon the fact that they go back to the type of the situation which causes reflection out of school in ordinary life. They give the pupils something to do, not something to learn; and the doing is of such a nature as to demand thinking, or the intentional noting of connections; learning naturally results. (Dewey, 1916/1944, p. 154)

Thus, both phenomenological philosophy and pedagogical practice converged historically on a recommendation to launch learning from engaged doing.

2.3 Constructivism

Learning-by-doing, as a philosophy of knowledge and a pedagogical program, would find a powerful ally in the experimental scientific research of Swiss cognitive-developmental psychologist Jean Piaget, whose theory of genetic epistemology implicates knowledge as the result of explorative interactions with the environment (Piaget, 1968). For Piaget, the world comes forth to us only inasmuch as we can absorb it into our purposeful engagement:

> Knowing does not really imply making a copy of reality but, rather, reacting to it and transforming it (either apparently or effectively) in such a way as to include it functionally in the transformation systems with which these acts are linked. (Piaget, 1971, p. 6)

As we assimilate the world onto our action plans, posits Piaget, we widen our scope of actionable forms (e.g., a mug with no handle, a larger mug, a square mug) even as we accommodate our motor schemes to enable this actionable expansion (a variety of new grasps). Thus, *action schemata* are formed, to be implemented and differentiated through future encounters. For example, if the world has occasioned us situations in which we move our hands with a *fixed* spatial interval between them, such as when we carry a large plate from the cupboard to the dinner table, then we would need to adapt this action schema upon an encounter with a new situation that, it turns out, requires us to handle an object by constantly *changing* the spatial interval between our hands. As Piaget's ideas permeated into schools, genetic epistemology became known as the constructivist pedagogical principle of building on what students know (Kamii & DeClark, 1985; cf. Phillips, 1995). By that token, we shall later see, one might design a learning activity in which a proportional schema (moving the hands at different rates, i.e., with a changing interval) is born as a differentiation of an additive schema (moving the hands at the same rate, i.e., with a fixed interval).

2.4 Enactivism

In historical hindsight, Piaget's intellectual contributions would be pooled with a variety of research efforts across diverse disciplines. Cognitive scientists acknowledge that multiple strains of philosophy and research—in fields as divergent as robotics, linguistics, movement studies, neuroscience, and anthropology—are increasingly amounting to a *pan-disciplinary turn* toward founding a theory of knowledge upon humans' lived phenomenology rather than on the questionable assumption that the brain processes information like a computer (Freeman & Núñez, 1999). As writes Chilean cognitive psychologist Francisco Varela (1999):

> [T]here are strong indications that within the loose federation of sciences dealing with knowledge and cognition—the cognitive sciences—the conviction is slowly growing that ….a radical paradigm shift is imminent. At the very center of this emerging view is the conviction that the proper units of knowledge are primarily concrete, embodied, incorporated, lived; that knowledge is about situatedness; and that the uniqueness of knowledge, its historicity and context, is not a "noise" concealing an abstract configuration in its true essence. The concrete is not a step toward something else: it is both where we are and how we get to where we will be. (p. 7)

Varela is speaking to the enactivist principle that, similar to Piaget's view, conceptualizes knowledge, or, rather, *knowing*, as coming forth from purposeful sensorimotor interaction with the world. According to enactivist philosophers, we should therefore theorize human phenomenology of thinking, even imaginary thinking about fanciful entities, as concrete, situated, and modal (vested in sensorial and actionable modalities). As Varela, Thompson, and Rosch (1991) submit:

> In a nutshell, the enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided. (pp. 172–173)

Thus perception, or, rather, perceiving, is an action-oriented adaptive cognitive capacity. Biological organisms develop their capacity to interact with the environment by selectively and iteratively consolidating effective sensorimotor patterns into durable cognitive structures for responding to increasingly complex contingencies (Piaget, 1971). As such, a child's cognitive development depends on its embodied interactions in natural ecologies (Adolph et al., 2018; Allen & Bickhard, 2013; Hubert & Wiesel, 1962; Turvey, 2019). In turn, cultural ecologies nurture the child's nature through participation in the social enactment of heritage practices utilizing historical artifacts of both material and immaterial forms (Malafouris, 2013; Reed & Bril, 1996; Saxe, 2012; Stetsenko, 2002; Tomasello, 2019; Vygotsky, 1930/1978). In particular, expert mathematical practices include routines for manipulating interactive computational devices, such as an abacus (Stigler, 1984), whereas pedagogical mathematical practices include simplified devices for nurturing early adaptation (Fröbel, 1885/2005; Montessori, 1967/1949). Mathematics educators writ large—teachers, guardians, researchers, designers, policy-makers, and the broader community—are the cultural collective charged with nurturing students' natural perceptual capacity to build cognitive structures enabling their effective participation in the social enactment of conceptual practice. As such, designers of mathematics learning environments need to know in great detail how cognitive structures emerge from attempting to move purposefully, which cognitive structures lead to understanding targeted curricular concepts, which recurrent sensorimotor patterns evolve into these structures, and which activity architectures facilitate these processes.

As we will demonstrate from our own research, cognitive structures emerge from recurrent sensorimotor patterns by way of the organism adapting its perceptual orientation to the environment, and *this adaptation may include attending to new perceptual structures that appear to come forth from the environment*. For example, to simultaneously manipulate two cursors on a screen, it might be advantageous for us to "see" an imaginary line connecting between the cursors. This line would be a new perceptual structure. As we will further explain, a perceptual structure may then rise to our consciousness.

2.5 Cognitive Science on Perceiving and Knowing

The world of objects, per a broad reading of enactivism, consists not only of material entities but, rather, of anything that is given to perception, such as inscriptions penned on paper—for example, mathematical propositions—or digital recreations of material entities, namely virtual objects. Cognitive scientists have demonstrated that we implicitly treat mathematical notations, including algebraic symbols, as things we can literally grab, as when moving x to the other side of the equation (Harrison et al., in press; Landy & Goldstone, 2007). Likewise, mathematics-education researchers have called to consider interactive digital objects as virtual manipulatives (Sarama & Clements, 2009).

Enactivist philosophy is in close dialogue with the ecological psychology of James and Eleanor Gibson (e.g., Gibson, 1966; see also Heft, 1989; Turvey, 2019)—a dialogue so close that the similarities and differences of these two perspectives are heatedly debated (Di Paolo et al., 2020). The similarity, roughly, is in a commitment to conceptualizing the human–environment relation as a pre-reflective synthetic *a priori*. Conscious analytic knowledge begins when we stop to think about what we are doing. Heidegger and others called this experience of stopping to think a moment of breakdown, when ready-to-hand ways of operating on the world apparently fail us (Koschmann et al., 1998). As we will elaborate, enactive mathematics pedagogy designs for breakdowns, where students become conscious of their perception by realizing its contextual inadequacy to organize their movement solution to a motor-control problem, and then, once they have solved the problem, they notice how their perception has adapted. Through teachers' guidance, talking about perception becomes mathematical discourse.

2.6 Mathematics Education Research

Our own research, as we shall soon discuss, led us to adopt the enactivist theorization of sensory perception (Hutto et al., 2015). Perception, we maintain, is the epistemic interface between unconscious doing associated with everyday informal experiences and conscious knowing associated with formal disciplinary practice. As one engages in new sensorimotor activities, like juggling, one is naturally inclined to formulate new ways of perceiving the world so as to enable the effective coordination of one's motor actions (Mechsner, 2004). As such, we submit, at the heart of the paradigm shift in how cognitive scientists are considering knowledge is a rethinking of sensory perception (Abrahamson, 2019). For example, knowing what a cube *is* sprouts from and carries forth the haptic–kinesthetic–visual perceptual experience of handling a cube (Roth, 2009). This integrated multimodal knowing is henceforth conjured in the absence of material objects. Calling these present–absent objects "phantasms," Nemirovsky et al. (2012) submit that mathematical discourse and thinking are one and the same cognitive activity:

> [R]ather than conceiving of a mental realm different in kind from talk and gesture, and in which multiple types of computational, representational, and cognitive processing take place, we propose that what we often refer to as "mental" consists of inhibited and condensed bodily activities engaging the same kind of perceptual and motor events that others could echo and refract, if allowed to expand and reach the periphery of the nervous system. (pp. 160–161)

2.7 Summary of Section 2

In this section, we drew on phenomenological, constructivist, and enactivist theory to argue that subjective phenomenology of mathematical concepts is perceptuomotor—it is about purposely acting on stuff to get something done, whether the situation is actual or imaginary, and that this requires perceiving the situation in a particular way that enables our actions to be effective. We now further submit that mathematical phenomenology is *necessarily* perceptuomotor, because it draws on cognitive architecture that evolved to promote our species' adaptive motor interactions with perceptual features of the environment (Abrahamson, in press). By this token, all knowing is doing, even when this doing is imaginary actions on imaginary objects. Thought is action, albeit unconsummated action (Vygotsky, 1926/1997, pp. 161–163). The core of mathematical knowing is not what you know *about* a concept but your capacity to *enact* the concept as perceptuomotor activity. As Pirie and Kieren (1989) write, "if one wishes to see if a person knows how to play the piano, the person might actually be asked to play. Then the observer determines if this action is effective" (p. 8). In like spirit, Freudenthal (1971) believed geometry should be taught "just like swimming" (p. 159; see Trninic, 2018), and von Glasersfeld (1983) analogized teaching mathematics to teaching athletics using technologies that enable athletes to see what they are doing.

Up to this point, we have highlighted enactivist philosophy and cited research from movement sciences as supporting its tenets. Yet how would all this philosophical, theoretical, and empirical work actually play out in practice, in the case of teaching and learning a mathematical concept? What, for example, would be a perceptuomotor experience corresponding to the concept of a proportion, a sine, or a parabola? How would we offer students this perceptuomotor experience? How would we know that they are indeed partaking in this perceptuomotor experience? Would all students have the same perceptuomotor experience? What if they do not? And how, in practice, does having a perceptuomotor experience support forms of classroom practice more traditionally associated with the mathematical discipline, such as writing symbolic notations on paper and then applying procedures to these symbols? That is, how could students transition from moving to writing, all the while maintaining the perceptuomotor experience as the symbols' conventional meanings? As they transition from moving to writing, do these meanings remain the same or do they change?

3. From Emergent Perceptual Structures to Articulated Mathematical Meanings

And so... IF: (1) you are convinced by the enactivist narrative, to wit, that cognitive structures emerge from repeated patterns of perceptually guided action; AND (2) you are impressed by cognitive-developmental and movement-sciences research on the constitutive role of perception in purposeful interaction; THEN, where does this leave you as a person who is passionate about education? In particular, what might all this research mean for creating interactive mathematics learning activities, where concepts are to come forth from movement via perception? This section describes a research program that brings enactivist philosophy and movement sciences to practice by designing and evaluating technological environments where students learn mathematical concepts by moving in new ways and then reflecting on how they are enacting these movements.

3.1 An Example of Enactivist Mathematics Pedagogy

Consider this paradigmatic example from our research, which we call the Mathematics Imagery Trainer for Proportion (for a project review, see Abrahamson, 2019). You're opposite a screen. Nearby remote-sensors detect where your hands are. You're asked to place your hands flat down on a desk in front of you. Then you're asked to lift both hands and move them up or down, as if through invisible vertical shafts (see Figure 1). You're not told how, specifically, to move the hands. Instead, the screen gives you streaming feedback on the quality of your performance: the screen shows cursors corresponding to the heights of your hands above the desk and signals to you whether your hands are at locations that, per some undisclosed manual choreography, are correct (green), incorrect (red) or in between (color gradations from green to red; in Figure 1, red is rendered as dark grey, green as light grey). You are working on an interactive bimanual motor-control problem of manipulating two virtual objects to change a property of the environment, and you are searching for a movement form—how the cursors should move—as well as a perceptual solution for enacting this movement motorically.

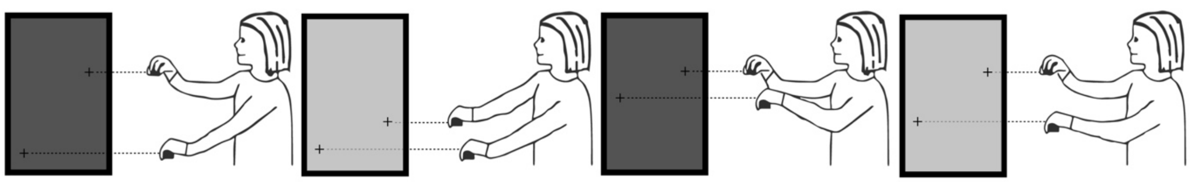


Figure 1. The Mathematics Imagery Trainer for Proportion—the case of a 1:2 ratio.

What you don't yet know is that the screen turns green whenever your hands happen to be in locations that correspond to a certain mathematical relation. For example, if the system is programmed according to proportionality, then, for the screen to be green, the height of your left hand over the desk as compared to the height of your right hand over the desk needs to accord with the hidden ratio, say 1:2. Of course, many combinations of left-and-right hand heights would yield a 1:2 relation, such as 5 cm and 10 cm, 10 cm and 20 cm, or 15 cm and 30 cm above the desk, respectively, and each combination would make the screen green. You are asked not only to find green places for the hands but also to move the hands all the while *keeping* the screen green. That means you should move your hands at different rates, here relating as 1:2, so that the right hand should move twice as fast as the left hand to keep the screen in constant green. Can you do that? Most people find this coordination challenging!

If you're anything like our study participants, here's how you might go about solving this bimanual motor-control problem. As you attempt to move your hands "in green," you might initially try moving just one hand up and down (red) or both hands side-by-side up and down (still red). At some point, you'll flail your hands in search of green *anywhere*. When you find it, you will freeze your hands. Prompted to move your hands in green, you will hesitate. You expect that to keep the green feedback constant, *something* should stay the same about the situation, perhaps some*thing* about your actions, and yet you don't know what. The only 'thing' you see is the distance between your hands. And so, you decide to move both hands *keeping the same distance between them*. Lo, doing so only violates the ratio (red), leading to breakdown and reassessment. Undaunted, you believe still that your fixed-distance theory is correct only that you messed up in implementing this theory, and so you adjust the distance until you are back in green. Once again, you're asked to move your hands in green, and once again you keep a constant distance between the hands, you get red, and you correct back to green. After several such cycles of move-and-correct, it eventually dawns on you that you can preempt the red by introducing the correction into the action itself. You think to yourself, "Given that it will need a bigger distance, I should raise my hands *even as I make this distance bigger.*" Soon you exclaim, "Oh, so the higher the hands, the bigger the distance between them!" With some practice, you soon become proficient at moving in green.

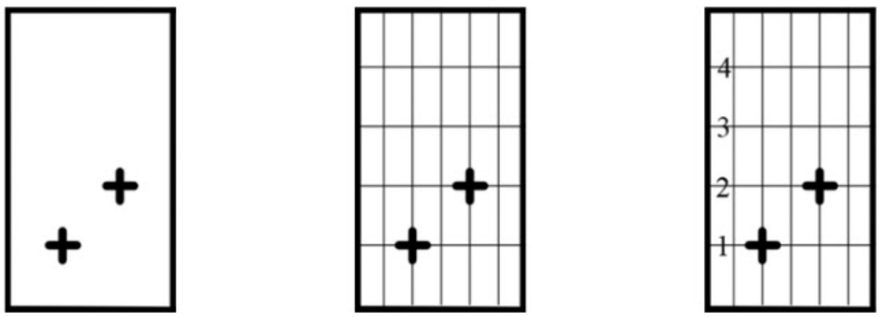


Figure 2. The Mathematics Imagery Trainer for Proportion: Interface schematics with, from left: cursors only, supplemented grid overlay, supplemented numbers

What we find remarkable in this process is that nobody told you to look at the spatial interval between your hands, where, in fact, there is nothing to see! We also note that, once you began attending to that empty space, your speech and gestures suggested that you were treating it as a thing that you were handling. This behavioral phenomenon is curious for philosophers, cognitive scientists, and mathematics-education researchers. Philosophers might ask how emptiness—the blatant negation of substance, a spatial region in the sensory manifold that is patently devoid of stimuli—is objectified as a bonafide object and, moreover, a utensil of sorts. Cognitive scientists, in particular those interested in the intersection of perception, action, and cognition (PAC), would hypothesize that locating a "handle" on the environment reduces a mentally daunting bimanual coordination to a cognitively cheaper, more manageable, and coherent action, where perception, moreover, the imaginary perception of a new Gestalt "laid over" reality, takes over much of the cognitive burden of motor complexity (Mechsner, 2004; Piaget, 1968). Mathematics-education researchers would note that once you began attending to that interval, you were able to articulate a verbal proposition connoting a rudimentary understanding of covariation—"The higher my hands go, the bigger the distance between them" (cf. Pirie & Kieren, 1989). They would also note that you thus became prepared to use mathematical instruments for re-describing your actions quantitatively. Indeed, the activity then proceeds to introduce mathematical resources for quantitative re-description, including a grid and numerals, by which proportional distances can be determined (see Figure 2). In the course of adopting these resources, students transition spontaneously into quantitative proportional reasoning—"For every 1 I go up on the left, I go up 2 on the right" (Abrahamson, 2019).

We call this interactive digital technology the Mathematics Imagery Trainer, because it occasions opportunities for students to develop informal action-oriented perceptual structures (e.g., the interval covarying with height) that ground formal mathematical concepts (e.g., proportionality), even before students know what they are learning—even before they realize they are learning mathematics. We have referred to these perceptual structures as *attentional anchors* (Abrahamson & Sánchez–García, 2016), we have used eye-tracking instruments to document the emergence and types of attentional anchors (Duijzer et al., 2017), and we have reported how reflecting on competing attentional anchors grounds further mathematical insight (Abrahamson et al., 2014). For example, students notice that their right hand is moving twice as fast as their left hand, and they explain this pattern as commensurate with the hands' 1-per-2 increments. (A tablet version of the Mathematics Imagery Trainer is freely available at www.tinyURL.com/FreeMITP .)

In closing this section, we wish to highlight several design principles that have emerged from our research program as bearing on classroom practice. Readers are referred to Abrahamson, Nathan et al. (2020, Appendix B) for additional citations of specific studies from our own research program supporting each of the following principles.

3.2 Principles for Enactivist Pedagogy Design and Facilitation

- *Problem analysis.* Realize that there are multiple legitimate perceptuomotor strategies for solving a motor-control problem and prepare for teaching by familiarizing yourself with the tasks' various perceptual patterns and strategies. This work requires you to access the implicit dynamical images of your own conceptual reasoning about a specific mathematical idea.
- *Concepts as tools*. Build a movement problem whose solution requires attending to the situation in a new way. The problem may be individual, such as a bimanual motor-control problem, or it may be collaborative, such as two or more students needing to coordinate their physical actions at a desk or in the classroom space (Abrahamson, 2019; Kelton & Ma, 2018; Zohar et al., 2018). The target concept emerges as a mathematical conversation about the perceptual solution. Students can learn also through engaged peripheral participation (King & Smith, 2018; Rosenbaum et al., 2020)
- *Generic objects*. Engage students with generic (non-iconic) objects that bear minimal resemblance to familiar objects, so that students will draw on a wide range of perceptuomotor capacities, minimally constrained by preconceptions of what an object *is* and, therefore, what one might *do* with it (Rosen et al., 2018).
- *Scaffolding*. Support student engagement by monitoring what they are perceiving and doing, reorienting their attention to the interaction space, prompting them to reenact their own strategies, and structuring their reflection on their own actions (Flood et al., 2020).
- *Inclusive design*. Build and facilitate learning resources that respect and accommodate students' multi-aspectual diversity, which may be sensorimotor, epistemological (Turkle & Papert, 1991), cultural–linguistic (Barton, 2008; Verran, 2001), or other, tapping students' particular ways of being as learning resources (Tancredi et al., in press) and supporting collaborative interactions.
- *Enactive empathizing*. As you monitor students' work, anticipate and suggest advantageous instructional moves through imagining that you yourself are engaged in the activity (Confrey, 1991; Davis, 1994; Shvarts & Abrahamson, 2019).
- *Revoicing*. In your conversations with students, use multi-modal revoicing—including verbal and gestural utterances—to selectively filter and shape their ideas into normative mathematical ideas (Flood, 2018).

- *Coordinating*. When students devise more than a single perceptuomotor interaction strategy, encourage them to compare between these different strategies and determine how the strategies are complementarity, as a means of arriving at new mathematical coordinations (Abrahamson et al., 2014). This obtains both for the case of a single student comparing their own perceptual orientations and for the case of two or more students bearing different orientations (Bamberger, 2016).
- *Mathematical appropriation.* To transition students from movement to mathematics, introduce symbolic artifacts into the activity space, such as a grid of lines, numerals, or rhythmic structures. Students will recognize in these supplementary resources immediate utilities for enhancing the enactment, explanation, evaluation, or coordination of their strategy for the task at hand. Yet, in so doing, students will begin to engage these resources as frames of reference that may reconfigure their enactment strategy to incorporate reference to the symbolic artifacts, thus signifying their actions in normative mathematical nomenclature (Abrahamson & Trninic, 2015).
- *Elaboration and analysis.* To extend the activities, have students recreate the activity space on paper or board, including the objects they manipulated, both actual and imaginary. Students might then use further mathematical instruments for construction and measurement, such as a ruler, compass, or protractor (Bongers, 2020).
- *An enactivist epistemic climate.* Throughout the lesson, cultivate in the classroom an epistemic climate that respects, valorizes, and normalizes orientations to knowledge as embodied and enactive. In particular, encourage students to describe their perceptuomotor sense-making of mathematical concepts, such as through:
    - explaining the idiosyncratic metaphors, similes, and imaginary animations by which they mobilize mathematical concepts as enacted processes (Abrahamson et al., 2012)
    - gesturing as they explain their reasoning, such as in:
        - Being: first-person dynamical experiences "in" mathematical representations (Gerofsky, 2011; Ochs et al., 1996);
        - Linking: re-perceiving features of the activity materials, or action sequences on the materials, as congruent with familiar schemes or symbolic forms (Abrahamson, 2009); and
        - Miming: reenacting in front of students action sequences "lifted" from the worksheet, board, or digital medium (Fuson & Abrahamson, 2005; Rasmussen et al., 2004).

What would classroom teaching look, if teachers implemented all these principles? As yet, we cannot say. But the next section will recount an episode of one teacher who used some of these principles.

3.3 Summary of Section 3

Whereas the first section introduced embodiment theory, this section described research that implemented the theory in the form of experimental technological designs for mathematics learning, and we listed a set of instructional guidelines resulting from our research studies evaluating these designs. We now turn from theory, design, and research to practice. We challenge ourselves by asking: *What enactivist design principles might mathematics teachers implement in their classroom professional practice, given the curricular, technological, and temporal constraints of institutional education?* That is, we will take new strides to explore what enactivist mathematics pedagogy might look like more broadly, if our recommendations for practice are adopted. For example, what non-digital activities might teachers design themselves, even on the fly, for whole-classroom participation? *An enactivist mathematics teacher, we will argue, is a teacher whose commitment to an enactivist mathematics pedagogy warrants an insistence that their students formulate perceptuomotor understandings of the concepts through engaging in guided physical activities, whether actual or imaginary, so that the meanings of symbolic forms, propositions, and procedures become grounded in the spatial dynamics of embodied action*.

4. From Graduate School to High School: A Case Study

If you have studied mathematics in mainstream regimens, it may be difficult to become that teacher who patiently listens to students making sense of new ideas (Ma & Singer–Gabella, 2011). In fact, some teachers unwittingly fall back on familiar patterns of traditional instruction (Cohen, 1990). What can be done about this? What forms of pre-service preparation or professional development might get teachers to buy into reform-oriented epistemological theories, such as enactivist perspectives on mathematical knowing? That is, how might teachers open up to consider a different way of thinking about thinking?

As teacher educators, we have found that if teachers are to empathize with their students, they should go through the very same experiences as will their students (Abrahamson, Nathan, et al., 2020, pp. 16–18). Therefore, in our pre-service courses and in-service workshops, we engage teachers in solving problems that give rise to reflection on the enactive quality of mathematical reasoning. We then have the teachers prepare thoroughly to teach the same activity, by figuring out all the different ways of thinking about the problem. Next, they take the activity to their classrooms, videotape their lessons, and then analyze these movies to make sense of the teaching–learning process. The cycle ends with the teachers writing up a report on their study (Abrahamson, Zolkower, & Stone, 2020; Barth–Cohen et al., 2018). In a similar approach, we guide pre-service teachers through a design-based research process of building and evaluating educational products (Abrahamson, 2018). This section relates the theory-to-practice story of Ms. Dutton, a teacher and co-author of this chapter. We argue that Ms. Dutton's development as a mathematics teacher was affected by her participation in a graduate course led by Professor Dor Abrahamson, also a co-author of this chapter.

How might an enactivist perspective on cognitive development empower mathematics teachers to offer their students experiences leading to conceptual understanding? Particularly, what types of activities in teacher-education programs would possibly prepare teachers to provide learning experiences that support students in grounding target content as meaningful enactment? Through the following first-person accounting of a classroom episode, Ms. Dutton puts forth that her theoretical and experiential grounding in enactivist pedagogy, during a teacher-education academic program, prepared her to improvise an enactivist activity that supported her students' conceptual understanding of slope.

Find the slope using the table of values.

x	0	1	2	3	4
y	0	2	4	6	8

Figure 3. Table of values on the board for students

Find the slope using the table of values.

x	0 ↓ +0	1 ↓ +1	2 ↓ +2	3 ↓ +3	4 ↓ +4
y	0	2	4	6	8

Figure 4. Marco's observation of the changing differences between *x* and *y* values

On this particular day in a 9th grade Algebra 1 classroom, my students were learning to find the slope of a line using a table of values (see Figure 3). It was October, so still early in the school year, when students are beginning to get to know each other. Looking at the two-way table, Jocelin, a confident student, claimed the slope is 2, "[because we] learned in 8th grade that the change in *y* over the change in *x* is the slope, and, in this case, that is 2." Marco, a student who, at this point in the year, had made it clear to me that he "did not like math," had a puzzled look on his face. Gesturing a finger up and down in each column from the *x* value to the *y* value—0 to 0 (difference of 0), 1 to 2 (difference of 1), 2 to 4 (difference of 2), etc.—Marco asked, "How can the slope be the same when the numbers are increasing?" (see Figure 4). If you were the teacher in this situation, how might you respond?

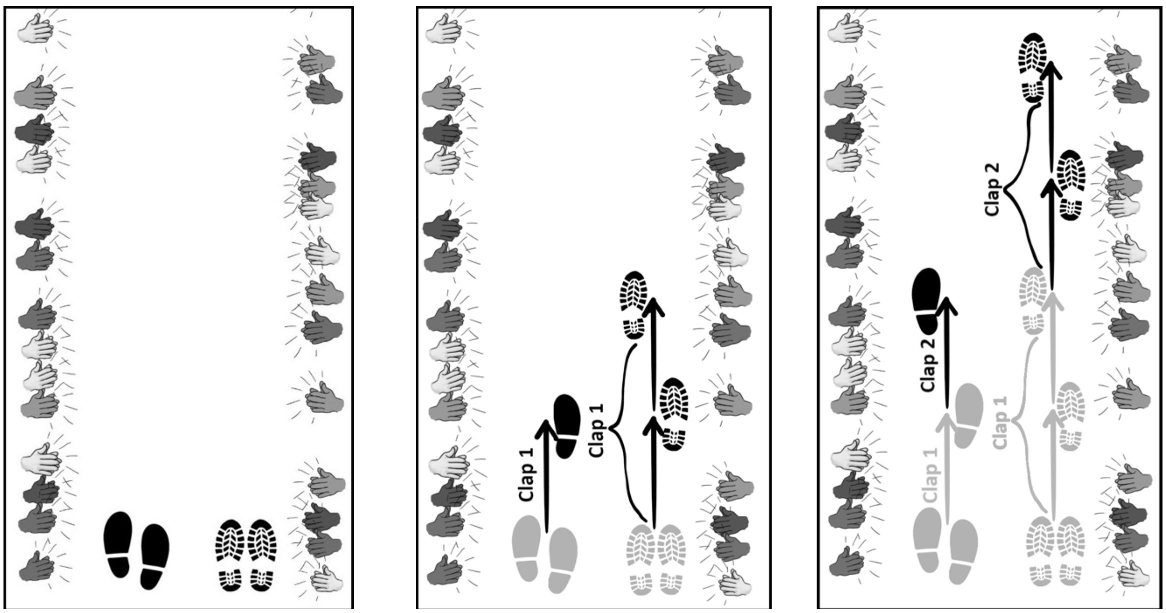
Find the slope using the table of values.

x	0 ↓ +0	1 ↓ +1	2 ↓ +2	3 ↓ +3	4 ↓ +4
y	0	2	4	6	8

$$\frac{\Delta y}{\Delta x} = \frac{\boxed{2}-\boxed{0}}{\boxed{1}-\boxed{0}} = \frac{\boxed{6}-\boxed{4}}{\boxed{3}-\boxed{2}} = \frac{2}{1} = 2$$

Figure 5. Slope as a challenging concept: Differences in *x*–*y* pairs increase (0, 1, 2, etc.), while ratios of *y* and *x* respective increments remain constant (2).

Whereas Jocelin sought teacher and classroom approval by reciting a familiar formula for slope, Marco, gazing at the numbers on the board, was observing an apparent ambiguity in ratio; that is, the differences in number pairs running along a sequence (vertical arrows) change, while the ratios of increments along these sequences ($\mathbf{\Delta}y$ over $\mathbf{\Delta}\, x$) remain constant (see Figure 5). I was concerned that the conversation between Jocelin and Marco might only lead to Jocelin "correcting" Marco's observation as insignificant to the definition of ratio. I sensed the need to ground both students and, with them, the whole classroom, in a shared enacted experience that could then be readily evoked for thinking together about the situated meanings of slope. How might I meet this sudden new instructional goal?



Set Up: Students line up in the hallway with Jocelin (closed shoe print) and Marco (open shoe print) shoulder to shoulder.

Clap 1: Jocelin walks 1 step, Marco walks 2 steps.

Clap 2: Jocelin walks 1 more step, Marco walks 2 more steps.

Figure 6. Progression in the hallway with students. Jocelin (full shoes) walks one step for every clap, and Marco (dashed shoes) walks two steps for every clap.

I then recalled Dor's work with the Mathematics Imagery Trainer for Proportion, which I had encountered during my graduate studies. I realized I could facilitate my goal of a shared enacted experience by having Jocelin and Marco walk in the hallway, analogous to the two hands in the Trainer. I would ask the two students to walk down the corridor, alongside each other, but at two distinct paces, with one moving twice as fast as the other. I hoped that all students would be able to see how the distance between the two walkers keeps increasing (Marco's observation), even as the ratio between their paces remains the same (Jocelin's observation). Yet how would I get Jocelin and Marco to coordinate their stepping? By keeping time! Placing Jocelin and Marco shoulder to shoulder in the middle of the corridor, I positioned all the other students along the corridor walls, and got them to clap their hands at a steady beat. I then instructed Jocelin to walk one step for every clap, and Marco to walk two steps for every clap (see Figure 6). Before they began walking, I asked all students to predict what would become of the distance between Marco and Jocelin and received varying answers (increase, decrease, and stay the same). Only once the two students walked out at their respective paces, keeping time with the clapping, did all students see that the distance between the two changed even as their respective paces remained constant.

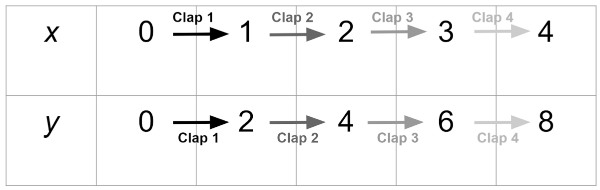


Figure 7: Clapping to keep time is *represented as* the temporal–spatial experience of "walking along" the rows of a two-way table.

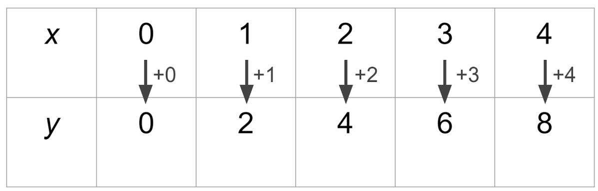


Figure 8: Increasing distance between two students is *represented as* changing differences between *x* and *y* ordered pairs.

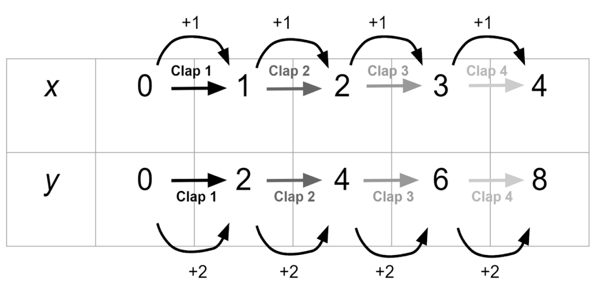
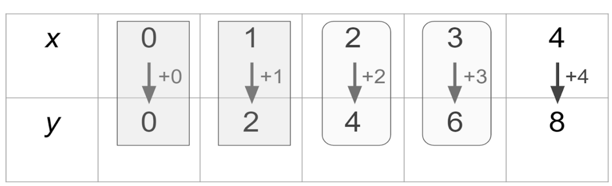


Figure 9: Walking consistent speeds is *represented as* consistent differences in consecutive *x* values and consecutive *y* values, respectively.



$$\frac{\Delta y}{\Delta x} = \frac{\boxed{2}-\boxed{0}}{\boxed{1}-\boxed{0}} = \frac{\boxed{6}-\boxed{4}}{\boxed{3}-\boxed{2}} = \frac{2}{1} = 2$$

Figure 10: Walking twice as far along the corridor, per each clap, is *represented as* consistent ratios of increments, viz. **Δ***y* over **Δ** *x* remains constant.

Upon returning to the classroom, I noticed a change: the conceptual understanding of slope was "clicking" for students. In the class debrief about the walking experiment, students avidly discussed not only the patterns they observed in the numbers but also how these patterns showed up in their lived experience and how these patterns related to each other. For example, students discussed: (a) where the clapping showed up in the problem (see Figure 7); (b) where the increasing distance between Jocelin and Marco showed up (see Figure 8); (c) where the consistent walking speeds showed up (see Figure 9); and, finally, (c) where the "this student is walking *twice as fast* as that student" showed up, which is the pivotal conceptual understanding of slope (see Figure 10). Before the hallway activity, students had been attending to the table's numerical features and executing calculational procedures. After the hallway experience, this same mathematical form became a model of a situation, because now it bore experiential meanings. Thus, students appeared to be coordinating between their collective movement in the hallway and the mathematical form on the board (i.e. the table of values). Importantly, this coordination was bidirectional (Dutton, 2018).

In reflecting on this experience, I have realized that my work, as a teacher, is to support students in formulating perceptuomotor understandings of the concepts through engaging in guided physical activities, so that the meanings of symbolic forms, propositions, and procedures become grounded in the spatial dynamics of embodied action. Doing so, I find, requires close empathic listening to students, who may not be using academic vocabulary yet are drawing deep conceptual connections between their lived enactments and formal mathematical signs. How could teachers learn to empathetically listen to students? How could teachers learn to implement an enactivist pedagogy that results in meaningful sense-making of mathematical signs? These questions warrant further exploration in teacher–researcher partnerships to develop robust teaching programs that prioritize enactment as a central component of teaching mathematics. In reflecting on my own experiences in graduate school and in many fellowship programs, I hope to provide a window onto the types of unique learning experiences that empowered me to provide high-quality instruction to my students both in the classroom and out in the hallway.

In graduate school, I recall numerous experiences in which my cohort was led to experience the enactive essence of mathematical situations *before* discussing them formally. Whether solving an unfamiliar mathematical puzzle or tackling a motor-control problem, my cohort of pre-service teachers would be grounded in a pre-conceptual perspective that forced us to feel the mathematical concepts that we had previously "mastered." We would then reflect on these enactments in light of the learning theories in the course syllabus. This enactivist orientation in our coursework, which provided teachers the appropriate scaffolds to first "play student" by feeling the mathematical concepts as we would hope our students would, and only *then* "play teacher," ultimately enabled me to confidently improvise the hallway activity with my students.

Figure 11. From left: (a) Illustration of a circle made on an etch-a-sketch. The left rotating knob controls an invisible stylus's vertical location, and the right rotating knob controls the same stylus's horizontal location. If two people are operating the device simultaneously, one knob each, how would they make a circle together?; (b) Ms. Elizabeth Dutton (seated on the on left) working the etch-a-sketch with Ms. Emily Gilden (seated on the right), with Ms. Annemarie Newell (between them) keeping time, "One Mississippi, two Mississippi, …"; resulting in (c) a quarter of a circle perimeter

For example, in one lesson during graduate school, I explored what a circle *is* with two fellow student-teachers by starting with the *making* of a circle (see Abrahamson & Bakker, 2016). We used an etch-a-sketch, whose mechanism challenges intuitive motor control (see Figure 11). In attempting to make the circle, we recognized that the left/right knob must slow down, while the up/down knob speeds up, and vice versa. This realization prompted the need for temporal coordination, as we turned the etch-a-sketch knobs. With one student clapping her hands and the other two students each rotating their respective knob, our cohort was able to coordinate so as to make a circle. In developing this coordination, we unwittingly enacted a naive perspective on trigonometric functions, *feeling* the trigonometric waves as we navigated the constraints of the etch-a-sketch (cf. Petitmengin, 2017, p. 144). Only *after* our embodied experience did we discuss what type of mathematical notions could ensue from the activity. Like my students, who, a few years later, would pace along the hallway, we pre-service teachers relied on our enactive experiences with the etch-a-sketch knobs to discuss how sine and cosine waves model the construction of a circle. To do so, we, coordinated back and forth between our experience (e.g., "How did we know when to slow down/speed up/change direction?") and the mathematical model ("Where is the slowing down-ness and speeding up-ness in the relationship between these sine-vs.-cosine waves?").

Through experiencing multiple pre-symbolic enactive experimentations, like the etch-a-sketch activity, I formed a pedagogical perspective on mathematical sense-making that values enactment as a vital part of learning and knowing mathematics. I was also able to cultivate a skill all teachers know is necessary for teaching yet is difficult to articulate—a form of empathetic listening to student sense-making, with which teachers can gauge whether and how their students' mathematical conceptions are productively grounded in enactment. We propose that *only through experiencing and understanding how we ourselves move between and coordinate intuitive pre-symbolic sense-making and formal mathematical models* can teachers empathize with their students' need to experience the enactive quality of mathematical concepts—to *experience* the math, before describing it in words or symbols.

5. Moving Forward: Practicing Classroom Enactivist Mathematics Pedagogy

Enactivist mathematics pedagogy is based on philosophical approaches to cognitive science that center on relations between physical movement and sensory perception in conceptual development. This pedagogical approach promotes students' active bodily participation in instructional activities, through which they develop dynamical perceptual patterns for making sense of mathematical concepts. The chapter introduced theoretical positions supporting enactivist mathematics pedagogy and then exemplified how this pedagogy is operationalized both in the forms of digital learning environments and classroom non-technological participatory simulations. A case study of Ms. Dutton suggested that mathematics teachers can learn to apply enactivist mathematics pedagogy in their own classroom through graduate-school preparation. This preparation includes, more broadly, active participation in a research community of practice engaged in the design-research of enactivist mathematics pedagogy, taking a practicum course that draws on enactivist theory to illuminate pedagogical problems and proposed solutions, and conducting a milestone empirical study of small-group mathematics learning.

Reflecting on the case study of Ms. Dutton's lesson, we discern examples of several enactivist pedagogy principles listed earlier. She created a movement activity that students had to enact collectively. In the course of enacting the proportional progression, the students were surprised by the unfolding figural dynamics of their classmates' mutual progression along the hallway. In particular, they attended to a perceptual structure, the distance between the two marching students that increased by one step per clap. This recurrent sensorimotor pattern became a new cognitive structure. Back in the classroom, the students were able to identify these movement patterns as numerical relations in the mathematical model on the whiteboard. In particular, the increasing-*distance* perceptual structure grounded the increasing *difference* between corresponding numbers on the chart, thus enabling the students to read the chart as a recounting of their shared hallway experience. In so doing, they evoked, re-enacted, and coordinated a plurality of idiosyncratic perceptions from the physical enactment.

Much remains to be achieved so as to explicate and promote our perspectives from research to practice, and graduate schools of education stand to play an important role in these efforts. Moreover, enactivist pedagogy should be a STEM-wide endeavor, if not beyond, because many of the principles we have delineated may well obtain in other disciplinary domains, such as science (Abrahamson & Lindgren, 2014; Enyedy et al., 2012; Lindgren & Johnson–Glenberg, 2013). We look forward to further engaging with a diverse interdisciplinary community of educational practitioners, technology engineers, and researchers interested in learning, teaching, and embodiment, as we envisage and pilot the futures of enactivist mathematics pedagogy.

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