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Research paper

# Modeling nonlinear dynamics of fluency development in an embodied-design mathematics learning environment with Recurrence Quantification Analysis ☆

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ABSTRACT

Although cognitive activity has been modeled through the lens of dynamical systems theory, the field lacks robust demonstrations in the learning of mathematical concepts. One empirical context demonstrating potential for closing this gap is embodied design, wherein students learn to enact new movement patterns that instantiate mathematical schemes. Changes in students' perceptuomotor behavior in such contexts have been described as bearing markers of systemic phase transitions, but no research to date has characterized these dynamics quantitatively. This study applied a nonlinear analysis method, continuous cross-Recurrence Quantification Analysis (RQA), to touchscreen data excerpts from 39 study participants working with the Mathematics Imagery Trainer on the Parallel Bars problem. We then conducted linear regression analysis of a panel of five RQA metrics on learning phase (Exploration, Discovery, and Fluency) to identify how nonlinear dynamics changed as fluency developed. Results showed an increase in determinism from the Exploration to the Discovery phase, and an increase in recurrence rate, trapping time, mean line length, and normalized entropy from Discovery to Fluency phases. To put these dynamics in context, we qualitatively contrasted the RQA metric trajectories of two case study participants who developed different degrees of fluency. Our results support the hypothesized existence of phase transitions in the human–technology dynamical system during a math learning task. More broadly, this study illustrates the purchase of nonlinear methods on multimodal mathematics learning data and reveals perceptuomotor learning dynamics informative for the design and use of embodied-interaction technologies.



Embodied approaches to epistemology that center dynamic body-environment in knowledge development (Newen, Bruin, & Gallagher, 2018) face the task of explicating human capacity for engaging in higher cognition such as mathematical reasoning. To scale these heights, scholars have been elaborating relevant theory (Bruineberg, Chemero, & Rietveld, 2019; Gallese & Lakoff, 2005; Maturana & Varela, 1992; Menary, 2015; Stephen, Dixon and Isenhower, 2009; Winter & Yoshimi, 2020), conjecturing implications for educational practice (Núñez, Edwards, & Matos, 1999; Shapiro & Stolz, 2019), and evaluating their positions empirically (Gallagher & Lindgren, 2015; Hutto, Kirchhoff, & Abrahamson, 2015). In particular, design-based educational researchers of cognition, teaching, and learning have been investigating relations between enacting movement forms and developing Science, Technology, Engineering, and Mathematics (STEM) concepts (DeLiema, Enyedy, & Danish, 2019; Duijzer, Shayan, Bakker, van der Schaaf, & Abrahamson, 2017; Lindgren, Morphew, Kang, & Junokas, 2019; Segal, Tversky, & Black, 2014; Smith, King, & Hoyte, 2014; Walkington, Chelule, Woods, & Nathan, 2019; Zohar & Levy, 2016). Considerable research on the relation between learning to move in new ways and learning new mathematical concepts has centered upon educational activities created according to principles of *embodied design* (Abrahamson et al., 2020; Abrahamson & Sánchez-García, 2016). Embodied design is a pedagogical approach to building learning environments that seeks to ground STEM content in students' inherent

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perceptuomotor capacity (Abrahamson, 2009, 2014, 2019). Qualitative analysis of learners solving embodied-design problems resonates with dynamical systems theory, which views development as self-organized and emergent from decentralized interactions. Embodied-design learners' interactions with their environment give rise to new movement coordination patterns unfolding at multiple timescales (e.g. Duijzer et al., 2017). Do learners indeed constitute a complex dynamical system? Is embodied mathematics learning a non-linear process? Do students' behaviors, like other dynamical systems, transition across a succession of distinct dynamically stable phase states, as a horse transitions from walk to trot, canter, and gallop?

To date, research on the embodied-design learning process has analyzed data from students' multimodal perceptuomotor behavior using qualitative methodologies (e.g., Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011), statistical modeling (e.g., Duijzer et al., 2017; Ou, Andrade, Alberto, Bakker and Bechger, 2020; Ou, Andrade, Alberto, van Helden and Bakker, 2020) and machine learning (e.g., Abdullah et al., 2017). Our current study is motivated by the belief that in light of the seemingly dynamical processes observed, educational researchers could enrich the learning sciences' toolkit by taking up theoretical and methodological tools from the movement sciences (cf. Beilock, 2008), specifically dynamical systems theory and coordination dynamics. We propose and test an addition to the toolkit for multimodal learning analytics of embodied design from the dynamical systems field that we argue offers traction on dynamical features in embodied learning data: Recurrence Quantification Analysis (RQA). This paper aims to bring RQA to bear upon data from an embodied-design context: the Mathematics Imagery Trainer for Proportion—Parallel Bars problem (henceforth, MIT-P). We use RQA to quantitatively characterize the nonlinear dynamics of bimanual coordination associated with different phases of embodied learning: Exploration, Discovery, and Fluency. We propose that RQA stands to enable education researchers to get an analytic handle on the microprocesses of discovery, in turn informing teaching techniques and the design of educational technology.

To frame this study, we first review embodied cognition in STEM education research. We then introduce our theoretical framework, drawn from coordination dynamics and dynamical systems, and present how this can be applied to our study context, the MIT-P. Finally, we provide an introduction to the central method in this paper, cross-RQA.

### *0.1. Embodied cognition and STEM education research*

In recent decades, theories of embodied cognition in the cognitive sciences have brought the dynamic interactions within and between body and environment to the forefront in research on learning and cognition. Philosophers of cognitive science are far from any consensus on the validity, viability, coherence, or reach of embodied cognition models (Barsalou, 2010; Kiverstein and Clark, 2009; Newen et al., 2018; Shapiro, 2014). However, embodied cognition has stirred significant resonance within the learning sciences and STEM education (Hall and Nemirovsky, 2012). Recent technological innovations for multimodal measurement also provide unprecedented opportunities for measuring perceptuomotor learning processes. For example, eye tracking has enabled STEM researchers to measure how study participants attend to visual displays (Abrahamson, Shayan, Bakker, & Van der Schaaf, 2016; Alemdag & Cagiltay, 2018; Richardson & Spivey, 2004; Shvarts, 2018). The collective evolution of theory, technology, and methodology has led researchers to investigate the role of physical movement in STEM learning (Abrahamson, 2018; Abrahamson & Bakker, 2016; Brooks, Barner, Frank, & Goldin-Meadow, 2018; Fadjo, Hallman, Harris, & Black, 2009; Hall and Nemirovsky, 2012; Kim, Roth, & Thom, 2011; Lindgren & Johnson-Glenberg, 2013; Nathan & Walkington, 2017; Sinclair & Gol Tabaghi, 2010), in particular as they solve problems designed to foster early grips on STEM concepts (Hutto et al., 2015). We focus here on mathematics education, in which embodied perspectives have led researchers to explore how students' development of new perceptual orientations towards sensory displays carry mathematical meanings (Abrahamson, Lee, Negrete, & Gutiérrez, 2014; Alibali & Nathan, 2012; Ferrara, 2014; Núñez et al., 1999; Radford, 2015; Roth, 2014; Sinclair & de Freitas, 2014). This line of research rejuvenates a long tradition in cognitive developmental psychology of tracing the ontogenesis of conceptual learning in the perceptual organization of motor exploration (Allen & Bickhard, 2013; Arsalidou & Pascual-Leone, 2016; Piaget, 1971).

Situated enactment theory (Chemero, 2009; Clancey, 2008; Greeno, 1998; Hutto & Myin, 2013; Noë, 2006) suggests that when students develop new motor control capacity, they are also developing new perceptual forms that guide these actions, forming new cognitive structures (Varela, Thompson, & Rosch, 1991). Enactivist epistemology rejects modeling the mind as an isolated input–output module that processes amodal symbolic information; instead, cognition is conceptualized as inherently modal activity that extends through the body into the natural and cultural ecologies (Anderson, 2003; Kiverstein, 2012; Wilson, 2002). A stronger position supported by cognitive psychology studies asserts that the enactment of complex motor actions is contingent not on developing new motor coordinations per se, but rather on identifying perceptual Gestalt structures in the environment that facilitate coordinated movement (Mechsner, 2003, 2004; Mechsner, Kerzel, Knoblich, & Prinz, 2001). These views lend researchers of mathematics cognition a renewed epistemological foundation to theorize conceptual learning as hinging on developing perceptual patterns that emerge as students attempt to accomplish tasks within natural and cultural situations (e.g., Steffe & Kieren, 1994; von Glasersfeld, 1987). In turn, as we explain in the next section, foregrounding the role of situated perceptions in cognitive processes has suggested to some researchers that conceptual learning is a nonlinear process that may resemble change processes in nonlinear material systems, namely, phase transitions between dynamically stable states.

As a framework, embodied design draws from enactivism's theoretical argumentation for the successful enactment of new movement patterns as constitutive of mathematical learning. Within mathematics education research, embodied design radicalizes earlier positions respecting the putative imagistic–kinaesthetic core of mathematical reasoning, as expressed by neo-Piagetians (Arnon et al., 2013; Kalchman, Moss, & Case, 2000; Steffe & Kieren, 1994; von Glasersfeld, 1987), enactivists (Kamii & DeClark, 1985), and cognitive linguists (Núñez et al., 1999). As such, embodied design aligns better with lines of research (Nemirovsky, Kelton, & Rhodehamel, 2013; Nemirovsky, Tierney, & Wright, 1998; Sinclair & de Freitas, 2014; Sinclair & Gol Tabaghi, 2010; Vogelstein, Brady, & Hall, 2019) that seek to theorize students' increasing capacity to perform coordinated motor actions as intrinsic, rather than merely contextual, to grounding mathematical concepts. Embodied design reimagines instructional design from an enactivist lens, crystalizing and studying the implications of this theoretical view for pedagogical practice.

### *0.2. Coordination dynamics and dynamical systems theory*

Coordination dynamics has been inspired greatly by the dynamical systems approach (Kelso, 1995; Richardson & Chemero, 2014; Stephen, Boncoddo, Magnuson and Dixon, 2009; Stephen, Dixon et al., 2009; Thelen & Smith, 1994, 2006), especially in the effort to model the emergence of movement forms. These models define and operationalize theoretical constructs that illuminate what is common to the structure and dynamics of diverse systems, including biological organisms, as they agitate and transition between functionally adaptive states. In this article, we draw on these interrelated fields to understand embodied learning of curricular content. We argue that constructs and techniques from these fields can be productively applied to analyzing learning as goal-oriented adaptation in situated interaction (Allen & Bickhard, 2013; Anderson, Richardson, & Chemero, 2012; Hilpert & Marchand, 2018; Jacobson, Kapur, & Reimann, 2016; Stephen, Dixon et al., 2009).

A core construct in coordination dynamics is the *phase*: a stable dynamic or time-evolutionary state of a system. A *phase transition* is a change from one phase to another, such as from the liquid to the gaseous state of water. Stable, functionally adaptive higher-order patterned phase states, here liquid form and gaseous form, are also referred to as *attractors*. Phase transition results from a change in *control parameters* (Kostrubiec, Zanone, Fuchs, & Kelso, 2012) in the task or environment. In the case of water, the control parameters of pressure and temperature drive water's transition from a liquid to a gaseous phase state. Phase transition can manifest as *shift* or *bifurcation*. Shift is a slight change of pattern maintaining the overall configuration, whereas bifurcation is an abrupt reconfiguration of the pattern (Kostrubiec et al., 2012). The result of a phase transition is a change in an emergent relational quality created through the interaction between the system's components, known as an *order parameter* (Kelso, 2000). For liquid and gas, the order parameter is density. In living systems, an example of a phase transition is a horse's transitions from walking to trotting, cantering, and galloping, each an attractor state (Schöner, Jiang, & Kelso, 1990). The order parameter is leg movement configuration; for example in walking, the horse will move each leg in sequence, while in trotting it will move legs in diagonal pairs. The phase transitions among these attractors are driven by changes in speed (control parameter).

A classic example from coordination dynamics (Haken, Kelso, & Bunz, 1985) examines finger movement coordination. Study participants are asked to wag both index fingers simultaneously in parallel (like windshield wipers) at an accelerating rate. Invariably, at some point, the fingers begin moving not in parallel but in symmetry, moving towards the midline at the same time. Researchers model this change in movement patterns as a phase transition between two dynamically stable systemic attractors: parallel movement and symmetrical movement. The angle of offset between the fingers is the order parameter; speed of movement is the control parameter. These results have been analyzed in terms of neuro-muscular (Kelso, 1984) as well as perceptual (Mechsner et al., 2001) constraints. Coordination dynamics constructs have been extended beyond movement patterns to describe phenomena such as the development of human agency. Infants' discovery of their causation of events in their world has been modeled as a phase transition in the infant–environment system (Kelso, 2016). We propose that fluency acquisition in embodied design functions the same way, with different patterns of movement fluency corresponding to different phases in the system.

### *0.3. Study context: The Mathematics Imagery Trainer for Proportion*

Building upon prior qualitative analyses (e.g., Abrahamson et al., 2014), we apply coordination dynamics constructs to analyze data collected in an embodied design environment, the MIT-P. In the MIT-P variant used in this dataset, users manipulated two parallel bars (Fig. 1) originating at the bottom of the screen. The bars appear once the user touches the screen and are initially red. Participants change the bars' height by sliding the tops of the bars upwards or downwards with their index fingers. As their heights are manipulated, the bars turn green whenever the ratio of the left bar's height to that of the right bar's height fulfills a secret ratio, here 1:2. That is, the left bar needs to be half the height of the right bar for the bars to turn green. The goal of the task is to discover how to move both fingers over the screen while maintaining the bars green, such that the right finger moves at twice the speed of the left. In later stages of the interview, a grid and then numbers are overlaid onto the screen, and other secret ratios are explored. We limited our analysis in this paper to the stage before the introduction of these additional elements.

Participants exhibit a rich range of strategies and trajectories in working with the MIT-P, but most pass through two key transitions. The first is finding a green position for the first time. Upon finding green, participants begin to move in more directed ways based on their concepts of what invariant feature elicited the green feedback. The second key transition is central to the design goal of the MIT-P. The MIT-P activity was designed to challenge students to move in a new way that grounds the concept of proportionality. In particular, it challenges students' common "additive" assumption that the difference between two quantities should remain constant as the quantities increase, i.e. that 1:2 is equal to 2:3 is equal to 3:4. The movement pattern that corresponds to this assumption is to move the hands while maintaining an equal distance between them. The MIT-P presents a challenge to this approach because moving in this way will repeatedly bring the learner out of the green since in fact the distance between the hands must grow and shrink as the hands move over the screen: a "multiplicative" way of moving. Discovering how to move multiplicatively is the critical breakthrough that grounds students' ability to move fluently in green.

We propose that MIT-P activity milestones can be conceptualized as phase transitions. Note several similarities between Kelso's classical finger experiment and the MIT-P activity: both contexts present participants with a bimanual motor-control task, and both procedures engender a transition from one way of moving to another. From a dynamic-systems theoretical perspective, both contexts introduce a task demand that perturbs a functioning enactment to the point that it becomes untenable and is then reconfigured. In the MIT-P task, we submit, the control parameter is conceptual: the technology's embedded mathematical function and its particular numerical values, such as a 1:2 ratio, perturb students who have not yet learned proportional reasoning. Learners are required to transition into a movement form they cannot as yet define formally. Displacing the fingers at equivalent speeds fails to yield the task criterion of success (green feedback), so a new movement form must be sought. The MIT-P's order parameter is the distance between the two hands relative to the height: a fixed hand-to-hand distance results in red feedback, whereas a distance that grows at a particular rate correlative to the hands' overall height results in the favorable green feedback.

### *0.4. Recurrence quantification analysis*

RQA is a nonlinear analysis method used for quantifying recurrence or coupling in a dynamic system. It offers a means to visualize and quantify dynamic characteristics of time series such as patterns in their repetition, periodicity, stability, order, and predictability (Balasubramaniam, Riley, & Turvey, 2000; Riley, Balasubramaniam, & Turvey, 1999). Originating in the field of physics, RQA has since been taken up for the study of complex systems in such diverse applications as physiology, economics, joint action, cognition, and communication. A full review of this literature is beyond the scope of this article. However, it is worth highlighting those few studies that have begun to apply RQA to research in academic contexts as well as research on problem-solving. Several studies use RQA to study the degree of synchrony

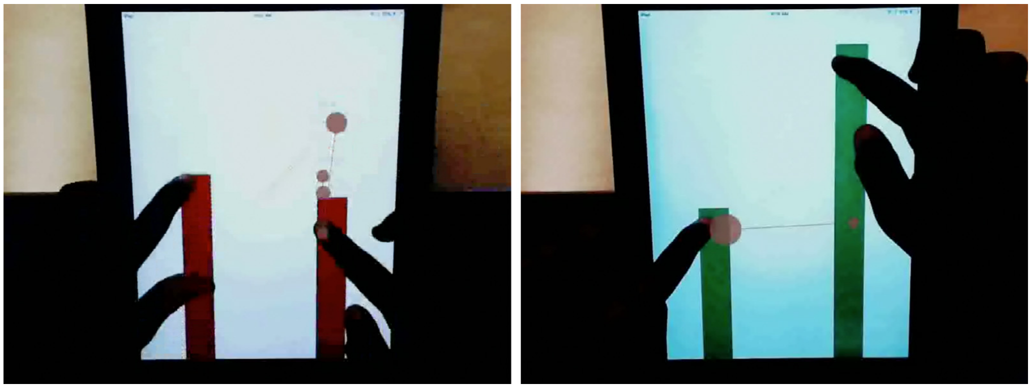

**Fig. 1.** User interacting with the MIT-P. *Note.* The bars turned green in the right image because the height of the left bar is in a 1:2 ratio with the height of the right bar. The pink circles show overlays of eye tracking data.

across team members on collaborative tasks, either electrodermal activity (Dindar, Alikhani, Malmberg, Järvelä, & Seppänen, 2019) or speech rate, body movement, and team interaction (Amon, Vrzakova, & D'Mello, 2019). Others use RQA to improve assessment, either in predicting self-explanation reading comprehension quality from linguistic sequences (Likens, McCarthy, Allen, & McNamara, 2018), comparing response time complexity across dyslexic and nondyslexic beginning readers' word-naming (Wijnants, Hasselman, Cox, Bosman, & Van Orden, 2012), or evaluating children's false beliefs through the dynamic stability of their hand movements in a balancing task (Fleuchaus, Kloos, Kiefer, & Silva, 2020). Of most relevance to the present study, Stephen, Boncoddo et al. (2009) use the RQA metric of entropy to predict the moment of insight in problem-solving about rotating gears. They model discovery of a consistent relation amongst gears as self-organization of a new cognitive structure. The present paper builds upon this work by examining the dynamics of both discovery and fluency-development in the context of intentionally cultivated, conceptual mathematical cognitive structures in an embodied design environment, guided by recent advances in applying embodiment theory to math pedagogy.

In the following paragraphs, we briefly introduce RQA and its foundation, the *recurrence plot*. Recurrence plots offer a visualization of the evolution of dynamical time series data. Here, we conduct cross-recurrence of two continuous time series, but recurrence plots can be constructed for single or multiple time series, and for continuous or categorical data.

Cross-recurrence plots map when two time-series have entered the same state. The unit of analysis is the coordination of the emergent system. For example, let us consider a hypothetical dataset of left and right-hand heights where the heights range from 1–10 inches. Each value in the time series is the mean height of that hand in a 1-second interval. To build a recurrence plot, we place the right-hand heights time series on the $x$-axis and the left-hand heights time series on the $y$-axis so that we can systematically compare every point in the right-hand series to every point in the right-hand series (Fig. 2). When there is alignment in states, a point is added to the plot. Blank spaces indicate a lack of alignment. For example, the first left hand height measurement, 3, appears in the first position on the $y$-axis. It is compared to every state in the right-hand time series along the x axis, and any time in the right-hand series that the height is 3, a blue point appears on the plot (here, the first and sixth columns). In the same way, the first left hand height measurement appearing in the first position on the $y$-axis, also 3, is compared to every height in the right-hand series. Applying this process to every number in each time series, we end up with a visualization that shows us all of the times the two time-series entered matching states, synchronously and asynchronously.

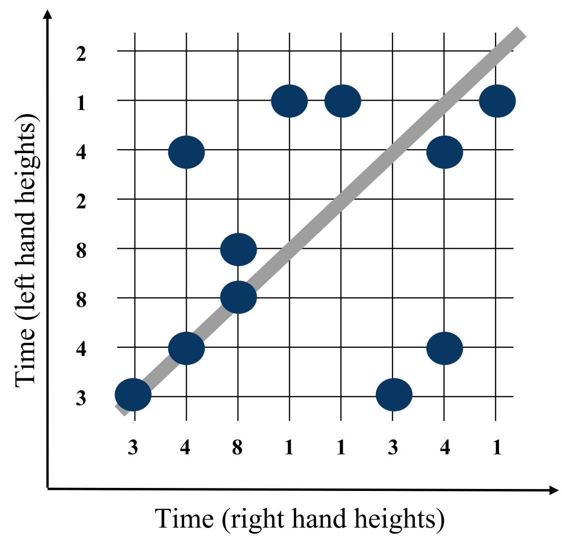


**Fig. 2.** Example cross-recurrence plot of right and left hand heights in a hypothetical time series. *Note.* In this example RQA plot, each of 8 positions recorded for the right-hand height is compared to all left-hand heights, and reciprocally, each of the 8 positions recorded for left-hand height is compared to all right-hand heights. Height states that align are indicated with a point on the plot.

RQA metrics quantify features of the recurrence plot, including the density of points and structures such as diagonal and vertical lines. Diagonal lines appear when a sequence of several changing states is taken up by both systems: for example, the 348 at the start of both the left and right-hand time series creates a diagonal line of length 3. This line appears on the line of synchrony (marked here in gray) because the sequence happened at the same point in both series. Meanwhile, the other diagonal line on the plot in the upper right is slightly below the line of synchrony because the left hand moved through these height states just before the right hand. Thus, diagonal lines show how many states in a row align for both systems, and their position reveals the lag between when each time series moved through these states. Vertical lines reflect times when the second time series maintained the same state over time, while horizontal lines reflect times when the first time series maintained the same state over time.

Recurrence plots reveal the degree of repetition, the length of repetitive sections, and the level of stability and structure in a time series' trajectory, and can reveal shifts between different modes of behavior. RQA metrics permit quantification of these patterns and comparison of such dynamics over time or across conditions or participants. Typical summary statistics such as means and standard deviations can tell an incomplete or misleading story about complex behavioral dynamics; two time series with the same mean might show different patterns in their regularity or degree of structure. RQA can provide a more complete picture by treating variation over time not as noise but as a meaningful dynamic of the system of study. Whereas statistical variability measures treat data as if each sample is independent, RQA captures and characterizes sequential dependence, that is, the dependence of each measurement in a time series upon those that came before. Further, RQA does not assume that the contributions of variables to the system are linear, offering a systems-level trajectory view that allows for interaction-dominant dynamics. This makes it an apt methodology for studying time-evolving complex systems.

### *0.5. The current study*

We analyze embodied design touchscreen data using cross-RQA. Building upon prior work conceptualizing changes in movement patterns in embodied design environments as phase transitions, we offer a quantitative evaluation of the distinct dynamics of previously qualitatively identified phases. The objective of this paper is to document changes in the nonlinear dynamics of perceptuomotor coordination associated with hypothesized phase transitions in the human–technology dynamical system. This paper analyzes the distinct RQA metric profiles associated with each phase of embodied design learning across all 39 participants. We then present a pair of contrasting case studies, analyzing dynamics within and between phases by connecting the participant's RQA plot and metrics to the video of their interactions with the task.

### *0.6. Research question*

How do the nonlinear dynamics of bimanual coordination patterns differ between Exploration, Discovery, and Fluency phases of an embodied-design learning task?

We will first present the methods used, including an overview of the dataset, the quantitative regression analysis methods, and the qualitative case study methods. We will then present the results of the regression analysis and case-study pair.

## 1. Methods

We conducted a secondary analysis of data collected through task-based semi-structured interviews with the MIT-P (Duijzer et al., 2017).

### *1.1. Materials and instruments*

The MIT-P task was implemented using a touchscreen tablet app. Researchers recorded the touchscreen position for each finger, the top of each bar, in y coordinates. Eye-tracking and video data were collected throughout the interview using the screen-based Tobii x2-30 model. Tobii Studio software overlaid the gaze data on the video such that it was possible to see where on the screen a participant was looking throughout the task. Transcripts were written and translated from Dutch to English by the original research team.

### *1.2. Participants*

Forty-five students ages 9–11 attending grades 5 and 6 at primary schools in the Netherlands participated in the study. Technical issues caused data to be incomplete for eight omitted participants. The 39 participants with full datasets for touchscreen hand position coordinates were included in this analysis.

### *1.3. Procedure*

Participants read instructions before the task inviting them to find a way to make the bars green and to keep them green while moving them (Duijzer et al., 2017). A researcher–tutor offered augmented information (Thelen & Smith, 1994) and guidance such as "find as much green as you can" (Thelen & Smith, 2006).

### *1.4. Data processing and variables*

We included the following variables in this analysis: left and right-hand height, rolling average proportion of time in green, and rolling average proportion of time moving both hands while in green (Table 1).

To compare phases in a consistent way across participants, we set quantitative criteria reflecting key MIT-P task milestones (see two examples in Fig. 3). We defined the beginning of the Discovery phase as the first time the discovery marker variable took on a value above 0.5, meaning a participant made green for at least 10 of 20 consecutive seconds. The window size of 20 s captured that the participant is substantially engaging with green rather than passing through it briefly without returning. We defined the beginning of the Fluency phase as the first time the fluency marker variable reached a value that was 80% of their maximum fluency marker value for that individual for 10 s. This threshold was set to mark when participants achieved moving-in-green approaching their personal best. The window size for the Fluency marker variable was shorter than for Discovery because it was intended to capture a local dynamic of close-to-peak performance rather than a general dynamic of frequently engaging with green locations.

We overlaid the transition points determined through this algorithm on a graph of the right- and left-hand heights over time (Fig. 3) to check for outliers where automatically-determined phase marker placement did not show face validity. The visual inspection revealed one outlier participant who triggered their Fluency phase by moving in green in a restricted area of the screen only before engaging in more exploratory behavior again. To check for undue impact of this outlier on the analysis, we ran the statistical analyses once with the RQA metrics for the original transition point and again with an alternate transition point set by increasing rolling average windowsize to 20.

### *1.5. Regression analysis of RQA metrics*

We conducted continuous cross-RQA on left- and right-hand height data using the crqa package in R (Coco & Dale, 2014). We selected recurrence parameters for each participant using an optimization routine based on average mutual information and false nearest neighbors methods (Coco & Dale, 2014), applying rescale and z-score normalization transformations. These transformations align the positions of each hand according to frequency of visitation; frequenting green positions establishes positions where the right hand is roughly twice that of the left as aligned. The overall level of recurrence detected through RQA is shaped by the parameters (delay, embedding dimension, and radius) used. For example, one could increase or decrease overall recurrence by manipulating the radius parameter, which defines the maximal distance between points to be considered recurrent: a small radius yields few points, a large radius many. Following conventions, RQA parameters for this analysis were set such that recurrence rate for each participant's overall plot was between 2

**Table 1**
Summary of variables used in Recurrence Quantification Analysis (RQA).

| Variable | Type | Scale | Description |
|---|---|---|---|
| Hand height | Continuous | 0–1065 | The y-axis position of each finger was recorded using the touchscreen at a variable frequency of approximately 50–120 Hz. The average position was taken every 10 Hz for this analysis. Missing data was forward-filled with the last recorded value. |
| Discovery marker | Continuous | 0–1 | A proxy for bar greenness was coded as 1 if the left-hand y coordinate divided by the right-hand y coordinate was between 0.4 and 0.6 for a given 100-ms interval, and 0 otherwise.<br>The two-sided rolling average of this variable was then taken with a window size of 20 s. |
| Fluency marker | Continuous | 0–1 | Moving-in-green was coded as 1 if the left *and* the right fingers changed position since the last measurement *and* the green variable was equal to 1 for a given 100-ms interval, and 0 if any or all three of these conditions were not met. The two-sided rolling average of this variable was then taken with a window size of 10 s. |

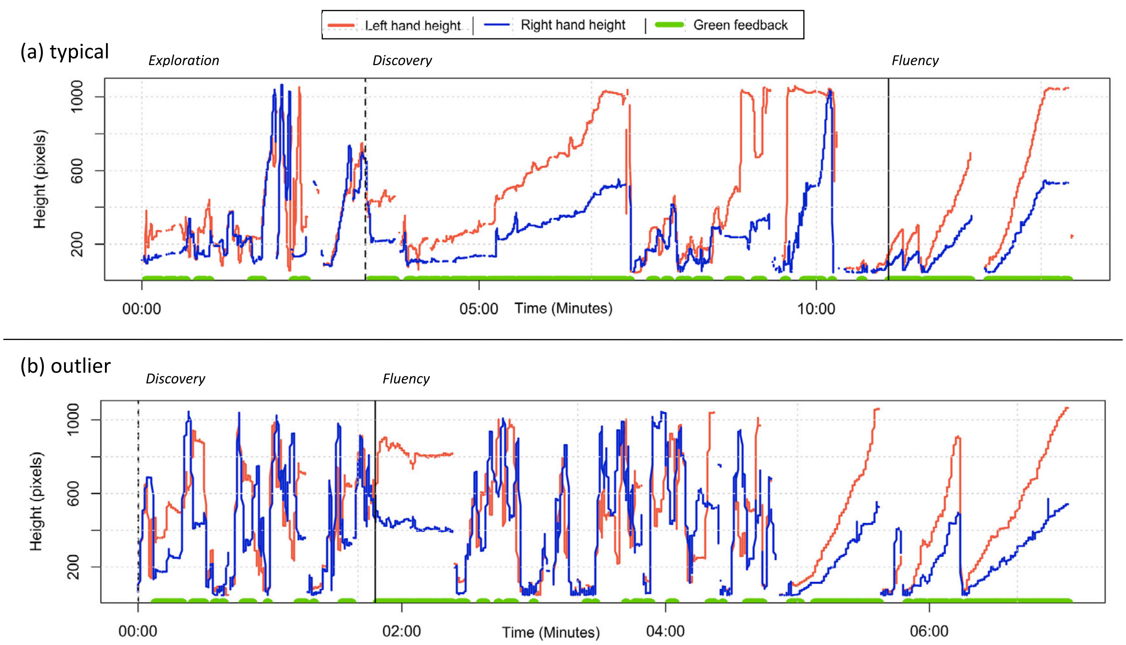


**Fig. 3.** Example hand position time series' transition points. *Note.* The dotted and solid lines indicate the start of the Discovery and Fluency phases respectively. (a) Typical participant: Fluency phase begins when the participant moves smoothly in green. (b) Outlier participant: Fluency phase marker is triggered before the smooth movements at the end of the time series. At the marker, their hands move over only a small fraction of the screen, followed by a long phase with very little green.

and 2.5%.[1] Recurrence metrics are meaningful moreso as a tool for comparison than as a raw characterization of the data due to their dependence upon the parameters used.

We regressed a panel of five RQA metrics resulting from this analysis on learning phase to explore the impact of phase on dynamics. Each RQA metric describes a different aspect of the bimanual coordination system. Broadly speaking, recurrence rate reflects degree of repetition, percent determinism the degree of coupling and predictability, mean line length the predictability and stability, normalized entropy the level of disorder, and trapping time the degree of lingering of the left hand (Table 2). For more details on these and other RQA metrics, see Marwan, Romano, Thiel, and Kurths (2007).

The following statistical model was used five times, once to address each RQA metric:

$$\mathrm{y_i} = \beta_0 + \beta_1 \ \mathrm{d_{phase2}} + \beta_2 \ \mathrm{d_{phase3}} + \varepsilon_\mathrm{i}$$

The regression coefficients $\beta 1$ and $\beta 2$ (for the dummy variables $\mathrm{d_{phase2}}$ and $\mathrm{d_{phase3}}$) represent the mean difference in each RQA metric of being in Discovery or Fluency, as compared to the reference phase, Exploration. Regression coefficients were tested against a significance level of 0.05.

This dataset is clustered: we have up to three data points per participant, one per phase. In clustered data, error terms are often correlated with each other by participant. We noted such heteroscedasticity in the determinism variable as well as the entropy

[1] Although the recurrence rate is 2.5% for the whole time series, the recurrence rate for segments or windows of that time series may be up to 100%. For example, if we were to model the location of two cats over 100 min, we would set our radius parameter to get a total amount of togetherness of around 2%. This might come out to defining recurrence as being within a radius of 5 ft of each other. However, if we split our plot into a pre-dinnertime phase and a dinnertime phase, we might find that during the pre-dinnertime phase, recurrence rates were extremely low (0% of the plot is shaded) and during dinnertime, recurrence rates were extremely high (100% of the plot is shaded). That is, they were close together during 0% of pre-dinnertime, 100% of dinnertime, and 2% of the overall measured period.

**Table 2**
Summary of variables used in regression analysis.

| Variable | Type | Scale | Description |
|---|---|---|---|
| Recurrence rate | Continuous | 0–100 | Recurrence rate is the percentage of the plot consisting of recurrent points. It reflects the overall degree of repetition across the two sequences. |
| Percent determinism | Continuous | 0–100 | Determinism is the percentage of points on the plot that fall on diagonal lines (length >1 point). It reflects the predictability and degree of coupling in a system. |
| Mean line length | Continuous | 0–½ length of time series | The mean line length is the average length of diagonal lines (length >1 point) on the recurrence plot, corresponding to the average length of coupled state sequences. It also reflects the predictability of the system. |
| Normalized entropy | Continuous | 0–1 | Shannon entropy is the distribution of line lengths in the plot. It reflects the stability of coupling structures. An increase in entropy reflects a decrease in the level of order in the system. Normalized entropy is Shannon entropy normalized by the number of lines in the recurrence plot. |
| Trapping time | Continuous | 0–½ length of time series | Trapping time is the average length of vertical lines in the recurrence plot. It reflects the average duration of a connected state. |
| Phase | Dummy | 0–1 | Phases were defined using the performance-based transition criteria described above. Discovery and Fluency were compared against Exploration, then against each other with post hoc analysis. |

variable. To ensure standard errors were not underestimated, we used cluster-robust standard errors (Cameron & Miller, 2015), an adjustment to standard error estimates in regression models for grouped data, for all our models. We also ran the model for entropy with and without data from two outlier participants heavily affecting homoscedasticity of variances to check if these outliers were affecting the power of findings. Another assumption of the linear model is that residuals are normally distributed. We applied a logarithmic transformation to the recurrence rate and trapping time variables to better meet this assumption. Trapping time, meanline, and determinism variables' residuals did not pass the Shapiro–Wilk normality of residuals test; however, with the current sample size, this was not likely to have impacted results (Ernst & Albers, 2017; Minitab, 2014).

### 1.6. Illustrative cases

We selected one prototypical and one contrasting case participant to examine the evolution of their dynamics in more detail using recurrence plots. For the prototypical participant, we generated a recurrence plot for each phase in their fluency development. This was not possible for the contrasting participant since their latter two phases were so brief. To compare how dynamics evolved for each participant across the whole time series, we graphed the change in each recurrence metric over time for each participant using windowed recurrence plots. Windowed plots are generated by calculating each of the recurrence metrics repeatedly for a sliding "window" segment of the data, calculated repeatedly at regular intervals across the time series.

Selection criteria for the representative participant were: (1) *representative trends* trends in each RQA metric reflected statistically significant effects of phase found in the linear regression model (N = 7); (2) *competence reached:* green was maintained throughout the Fluency phase (N = 29); (3) *phase comparison possible:* all phases were long enough to generate recurrence plots using the participants' set of overall RQA parameters (N = 14); and (4) *full range of motion:* the participant engaged with positions all over the screen across task phases (N = 22). The former two criteria ensured the participant was illustrative of trends across students, while the latter two ensured that phases were readily comparable to each other. Three participants met all of these criteria, from which one participant (Nils, pseudonym) was selected randomly. To contrast with this typical case, we selected the participant who spent the greatest relative amount of time in the Exploration phase compared to the other two phases, Liam (pseudonym). To enrich our analysis of these cases, we included relevant information from the video, interview transcripts, and graphs of their hand positions over time.

## 2. Results

### 2.1. Descriptive analysis

The mean start time of the Discovery phase was 2 min 2 s, with a standard deviation of 1 min 58 s. The mean start of the Fluency phase was 5 min 45 s, with a standard deviation of 1 min 46 s. Some participants spent too little time in a given phase for RQA metrics to be calculable: 14 participants for the Exploration phase, 10 for the Discovery phase, and 4 for the Fluency phase. For example, some participants found green immediately upon starting the activity, or went straight from finding green to moving-in-green.

Boxplots of each RQA metric by phase show a general trend of higher median levels in the Fluency phase than in prior phases (Fig. 4). Data in each phase are generally normally distributed, with the exception of left skew in determinism data in the Exploration phase and normalized entropy in the Fluency phase, and right skew in the mean line length and trapping time data for the Exploration and Discovery phases, and recurrence rate in the Fluency phase.

### 2.2. Regression analysis

From the Exploration to the Discovery phase, there was a statistically significant estimated mean increase only in determinism, which grew by an estimated 6.31% ($t = 2.02$, d.f. = 84, $p = 0.043$) (Table 3). From the Discovery to the Fluency phase, there were statistically significant estimated mean increases in recurrence rate of 176% ($t = 3.37$, d.f. = 85, $p = 0.001$), meanline of 14.83 points ($t = 3.01$, d.f. = 85, $p = 0.003$), normalized entropy of 0.06 ($t = 2.41$, d.f. = 84, $p = 0.016$), and trapping time of 150% ($t = 1.47$, d.f. = 84, $p < 0.001$) (Table 3). The proportion of variance explained by phase was 17% for recurrence rate ($R^2 = 0.169$), 13% for determinism and meanline ($R^2 = 0.126$), 7% for entropy ($R^2 = 0.066$), and 27% for trapping time ($R^2 = 0.268$).

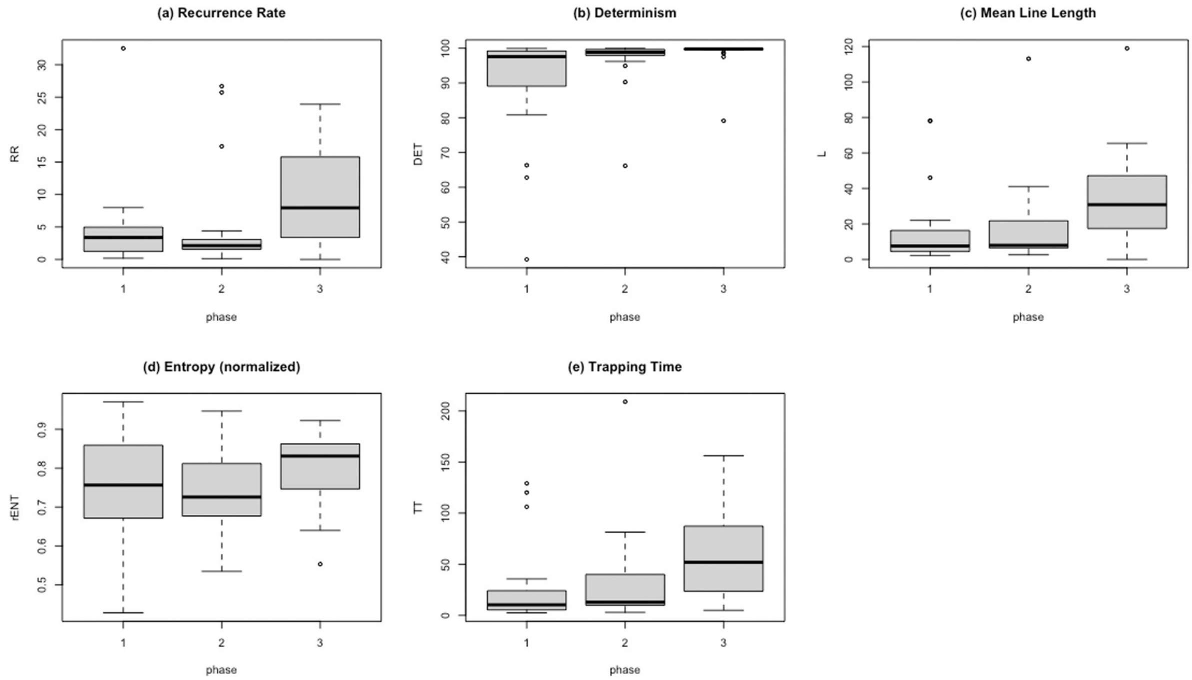


**Fig. 4.** Boxplots of RQA metrics by phase. *Note:* The boxplots did not show a clear change in median RQA metrics from the Exploration (1) to the Discovery (2) phase. The interquartile range narrowed upwards for determinism. All median recurrence metrics increased in the Fluency (3) phase.

**Table 3**
Results of linear regression.

| RQA Metric | Phase | Coefficient | SE | T | Pr (> \|t\|) |
|---|---|---|---|---|---|
| Log (Recurrence Rate) | Discovery | −0.02 | 0.29 | −0.07 | 0.941 |
| | Fluency* | 1.01 | 0.30 | 3.37 | 0.001 |
| % Determinism | Discovery* | 6.31 | 3.12 | 2.02 | 0.043 |
| | Fluency | 1.83 | 1.34 | 1.37 | 0.170 |
| Entropy (normalized) | Discovery | 0.00 | 0.03 | −0.11 | 0.915 |
| | Fluency* | 0.06 | 0.03 | 2.41 | 0.016 |
| Mean line length | Discovery | 0.91 | 4.12 | 0.22 | 0.825 |
| | Fluency* | 14.83 | 4.93 | 3.01 | 0.003 |
| Log (Trapping time) | Discovery | 0.38 | 0.26 | 1.47 | 0.142 |
| | Fluency* | 0.92 | 0.19 | 4.73 | 0.000 |

Note.
*Indicates statistically significant at the 5% level. Values in table compare each phase to the one directly prior (i.e. Discovery to Exploration, Fluency to Discovery).

### 2.2.1. Outliers

We reran these models with the adjusted transition points for the outlier participant with the atypical Fluency marker. Coefficients for the Discovery and Fluency phases and the post hoc comparison between Discovery and Fluency phases were 0%–16% of baseline levels higher with the exception of the coefficient for recurrence rate in Discovery, which was 1% lower. There were no changes to the trend or statistical significance of model estimates.

The model for entropy was run a second time omitting two outlier participants with Exploration phase entropy values below 25th percentile who were causing issues with homoscedasticity. With these participants omitted, the entropy variable met the homoscedasticity assumption. This analysis showed the same overall trends: no statistically significant difference from Exploration to Discovery ($t = -0.56$, d.f. $= 79$, $p = 0.578$) and a statistically significant increase in estimated mean entropy of 0.07 from Discovery to Fluency ($t = 2.63$, d.f. $= 79$, $p = 0.009$).

## 2.3. Case studies

We present the learning trajectory of a prototypical learner, Nils, alongside that of a learner who spent most of the time in Exploration, Liam. We begin with a summary of their interactions with the task before presenting the dynamics of these interactions captured by recurrence plots and how RQA metrics evolved for each participant.

### 2.3.1. Overview of learning trajectory

Nils' learning trajectory was typical of participants in that he steadily progressed from Exploration through Discovery to Fluency (Fig. 5a). In the Exploration phase, Nils tried several common bimanual exploration movement patterns: raising and lowering hands together, then in alternation. Once the latter solicited his first green position, he held one hand still while raising and lowering the other to seek additional greens, starting again at the bottom of the screen when this proved ineffective. Nils, now considered to be in Discovery due to engaging green positions, then began cycling through several approaches: (1) maintaining the distance between his hands while raising and lowering them (the *additive* strategy), (2) swapping hand heights, and (3) moving each hand one by one. The former two strategies reflect embodied hypotheses of what invariant gives rise to green, while the latter reflects targeted exploration, both hallmarks of Discovery strategies across participants. Of these strategies, only the latter

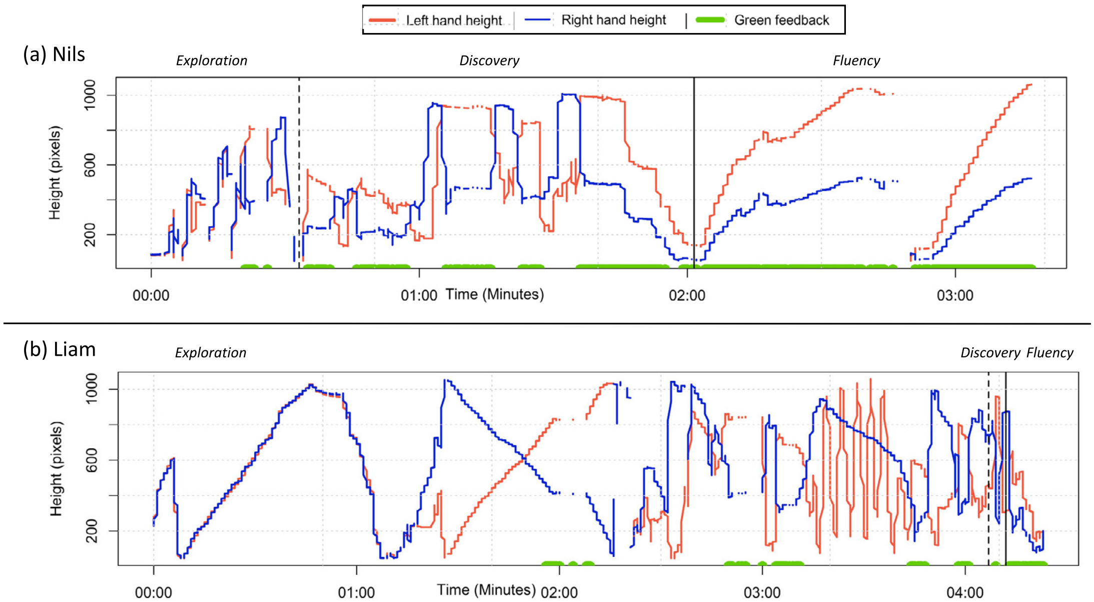


**Fig. 5.** Time series of left and right bar heights and resulting green feedback. *Note.* The dotted and solid lines indicate the start of the Discovery and Fluency phases respectively. (a) Nils began with rapid, varied Exploration strategies, followed by more systematic strategies during the Discovery phase, and finally, coordinated movement-in-green spanning the full range of the screen in the Fluency phase. (b) Liam's more slow, less varied Exploration phase lasted the majority of the time, making green consistently only during the final 12 s of the task.

yielded a new green. Nils then described his strategy to the tutor: "try to make half". After several more inversions, he noted that "high on the right [...] will turn green". He then lowered his hands incrementally, making green as he went. Entering the Fluency phase, Nils, with encouragement from the tutor, then moved from the bottom of the screen to the topin coordination, with occasional stops for correction.

In contrast to Nils and most participants, Liam did not engage in substantial Discovery or Fluency phases during this interview segment (Fig. 5b). Like Nils, Liam began exploration by moving his hands together at the same height. Unlike Nils, he persisted in this strategy for the first 80s of the interaction until prompted by the tutor, "You can move your hands independently". Following this comment, Liam moved his fingers in opposite directions, yielding his first green location two minutes into the task. Prompted by the tutor to look for "more places on the screen" and to "move the bars up and down so that they turn green again", Liam engaged several idiosyncratic strategies, including moving the fingers in opposite directions and moving the left hand down and the right hand up and down. Intermittently, these strategies yielded green positions very close to the first one he found. Towards the end (the beginning of Phase 2 in the graph), the tutor asked Liam, "have you noticed some things?" and Liam responded, "I have to put this [right hand] above, compared to that [left hand] while moving the hands". Liam then lowered his hands simultaneously, finding three new green locations along the way.

### 2.3.2. Recurrence plots by phase

Nils' recurrence plots showcase distinct dynamics for each phase. In Exploration (Fig. 6a), the plot is mostly white, indicating low coupling between the hands and low repetition overall. Exploration exhibits low self-similarity; Nils tried a variety of movements to seek the desired green feedback. In Discovery (Fig. 6b), the plot shows a loosely dotted diagonal line through its center, illustrating intermittent moments of green. The plot also shows a line of points running parallel to the line of symmetry

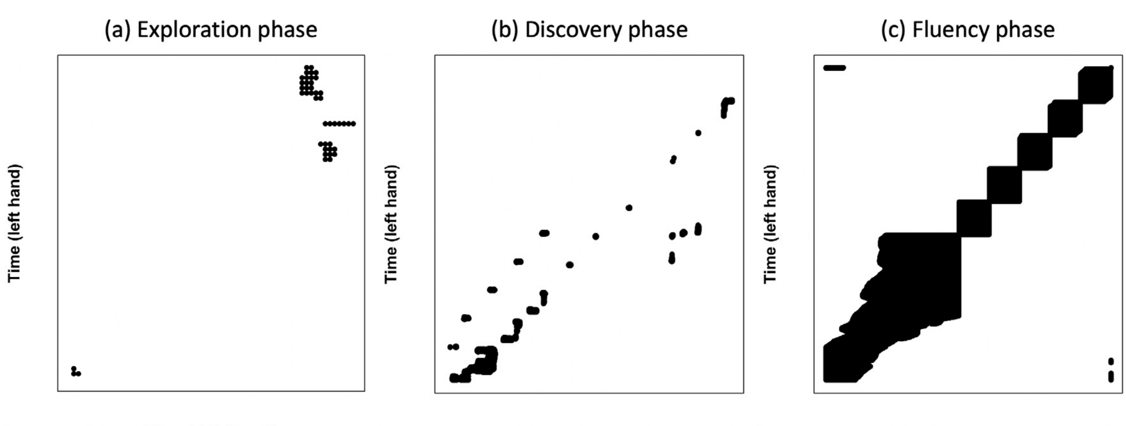


**Fig. 6.** Exploration, Discovery, and Fluency Phase Recurrence Plots for Nils. *Note:* During the Exploration phase, the mostly empty recurrence plot shows low coupling of the hands. In the Discovery phase, the plot shows intermittent points falling along the line of synchrony, reflecting coming in and out of the target ratio. In the Fluency phase, the plot shows a series of thick squares populating the full line of synchrony, showing coordinated movement in the target ratio punctuated by a series of regular, short pauses.

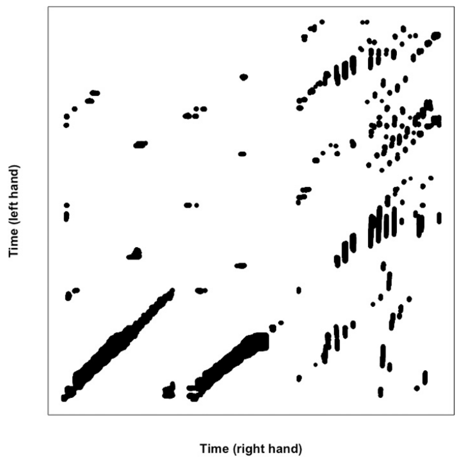


**Fig. 7.** Exploration phase recurrence plot for Liam. *Note:* Liam's Exploration shows several distinct dynamics (thick diagonal lines, sparse plot, and vertical structures) reflecting changing exploratory approaches.

caused by Nils swapping hand heights at different heights on the screen. In Fluency (Fig. 6c), the plot shows a thick diagonal line along the line of symmetry featuring a series of rectangles. This plot reflects continuous synchrony between the time series for each hand, here interpretable as moving-in-green. The rectangular structures reflect lingering in the same state. The plot reveals that lingering behavior is higher at the start of the Fluency phase (the large initial rectangle) and then decreases. Some degree of rhythmicity appears in the lingering pattern, as rectangles repeat with even spacing and size.

Liam functionally remained in the Exploration phase throughout this interview segment. His recurrence plot (Fig. 7) shows an initial diagonal line of synchrony between the hands, here reflecting when he moved his hands at the same heights for 80s, in contrast to the varied strategies Nils engaged in at the start of the task that yielded a blank plot. Liam's plot becomes sparser through the middle, corresponding to his diverging-finger movements. Vertical structures appear towards the upper right quadrant as Liam moves his left hand faster than his right.

### 2.3.3. *Evolution of RQA metrics*

Windowed RQA plots visualize how our focal participants' bimanual coordination dynamics evolved over the time series. Were changes gradual or abrupt? Focalized around the phase demarcations we defined, or elsewhere?

Nils' windowed RQA plots (Fig. 8) detail how his evolving dynamics affected RQA metrics across phases. Rather than gradual increases in RQA metrics, the plots show *abrupt and synchronous change across metrics* at the onset of the Fluency phase (Fig. 8). Nils' meanline and trapping time grew to 8 times their prior levels, and his recurrence rate grew to more than 20 times its prior level. Entropy also increased, and percent determinism flattened near its ceiling value of 100%. These increases reflect newfound predictability and stability in his coordination pattern. This point in the interview corresponds to Nils expressing to the tutor his rule for finding green. The dynamics of Nils' performance transformed in an abrupt and permanent way when he expressed and demonstrated the rule that he had noticed. Although finding green and engaging with green (Discovery) might seem like the critical breakthrough in this task, we see that it is this conceptual breakthrough of how Nils thinks about the task that yields transformed dynamics.

In Liam's case, we do not see the abrupt increases observed for Nils; Liam's recurrence metrics generally decrease over the course of the time series, with the steepest drop-off arising when he stops testing the hands-together pattern at the 80s mark (Fig. 9). This reflects the low degree of fluency observed in his movements; Liam has not figured out the coordination dynamics to solve this task.

## 3. Discussion and conclusion

This study used RQA to model the nonlinear dynamics of learning in an embodied-design mathematics learning environment. We found distinct coordination dynamics across different phases of movement-based mathematics learning. Linear regression of RQA metrics on phase showed an estimated mean increase in determinism when students progressed from Exploration to Discovery, and an estimated mean increase in recurrence rate, mean line length, normalized entropy, and trapping time when learners progressed from Discovery to Fluency. A prototypical case study participant showed RQA metric increases onsetting abruptly at the beginning of the Fluency phase, while a contrasting participant who did not reach fluency showed RQA metrics decreasing over time.

The results of the linear regression analysis support differentiating learning phases of Discovery and Fluency as featuring distinct dynamics. In the MIT-P task, participants learned to move in a new way that grounds mathematical conceptual understanding of proportionality. Prior qualitative work has identified the landmarks of engaging green feedback (Discovery) and performing the mathematically-critical *moving-in-green* multiplicative pattern (Fluency). The statistically significant differences in RQA metrics across phases corroborate the distinctiveness of these phases. Discovery showed higher predictability (determinism) than Exploration; Fluency showed greater coordination (recurrence), stability and predictability (meanline), lag (trapping time), and level of order (normalized entropy) in participants' bimanual movements. The case studies corroborate the presence of such qualitative differences between phases: for the prototypical case, Nils, RQA metrics increased abruptly at the onset of moving-in-green, suggesting a break in dynamics. That such a pattern was not observed in the contrasting case learner who maintained low fluency suggests that these dynamics are not an inevitable by-product of participating in the MIT-P activity but, rather, a feature of fluency development.

A complex dynamical systems view of conceptual change predicts that phase changes, ubiquitous in biological systems, would manifest in learning data. Our results are consistent with this hypothesized presence of phase transitions in the learner–technology system. They suggest that learning, at least in an embodied-design environment, is not a linear progression towards a target. Learners' dynamical interactive attempts to solve the control problem under predetermined task- and environmental constraints bring forth an adaptive reconfiguration of sensory perception regulating the increasingly refined motor enactment of the multiplicative movement pattern, thus satisfying the activity's primary pedagogical objective. MIT-P discovery offers a case of learners transitioning from one (pre-adapted) working model of an interactive-inquiry situation to another (adapted) (Karmiloff-Smith, 1988). The dynamics of motor behaviors reflect and constitute the implicit cognitive dynamics of conceptual learning. This study shows the promise of a research agenda approaching conceptual learning as a complex, nonlinear, dynamical process. In particular, we provide preliminary support for the traction of a dynamical, enactivist view in a context often thought to be resistant to this perspective: mathematical conceptual learning (Hutto et al., 2015). Our results offer empirical fodder for debates around the still-contentious philosophical stances on the nature of embodied cognition (Wilson, 2002),

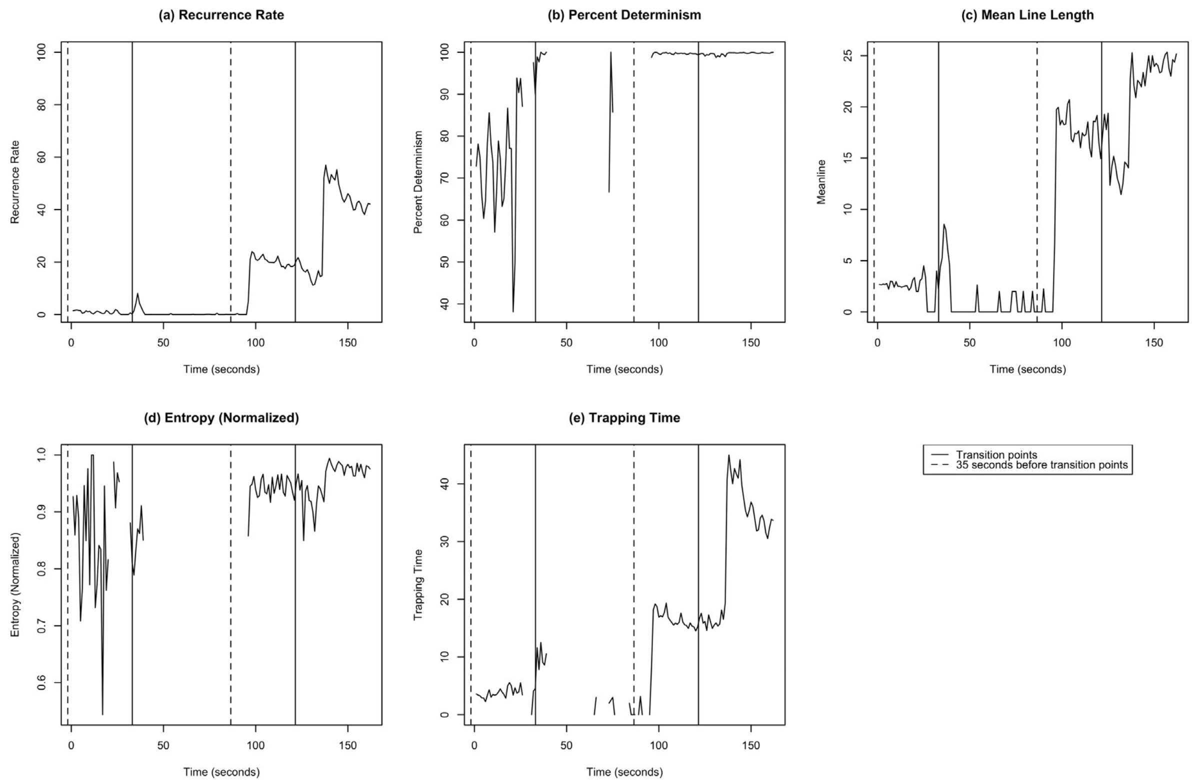


**Fig. 8.** Sliding window recurrence plots for RQA metrics across phases for Nils. *Note.* For all windowed plots, window size = 35 s, window step = 1 s, lag width = 5 s. Blanks on the plot correspond to phases with no recurrent points. Nils' RQA metrics showed a sudden increase at the onset of the Fluency phase.

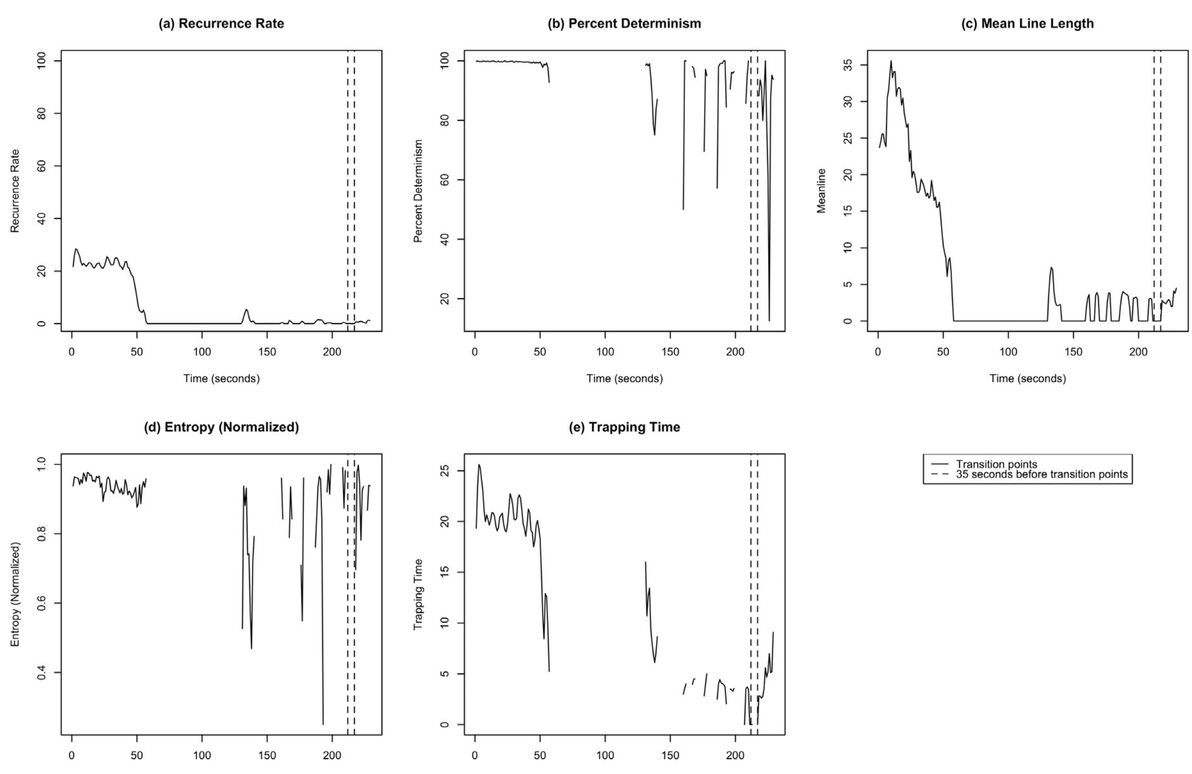


**Fig. 9.** Sliding window recurrence plots for RQA metrics across phases for Liam. *Note.* Liam's RQA metrics decreased over the course of the time series.

supporting the viability of a radical enactivist approach (Hutto & Myin, 2013).

By substantiating the effects of an enactivist approach to the design of environments for teaching and learning mathematics, the present study carries implications for child–computer interaction design. We found that learning environments geared to embrace and guide perceptuomotor activity elicit changes in motor dynamics. Students bring their implicit perceptuomotor schemes to bear on tasks, then figure out new ways of moving concordant with feedback elicited through engagement with the technological resources. As such, students keep grounding the meaning of new mathematical concepts in enactive knowing. Our case study on Nils shows these changes to be associated with concept development: Nils' breakthrough into fluency occurred immediately following his articulation of a rule of making half. Thus, the present study's results corroborate the potential to foster conceptual change as an adaptive bimanual exploration for a better grip on the world (Abrahamson). Bimanual-interaction technological systems like the MIT-P can instantiate a broad spectrum of mathematical relations beyond proportionality, such as quadratic (Shvarts & Abrahamson, 2019) or trigonometric (Bongers, 2020) functions.

The observed dynamics can also inform the design of pedagogical virtual objects, including artificially intelligent tutoring systems. Our results suggest that recurrence metrics may be a useful means to characterize students' intrinsic dynamics and identify, predict, and thus cater to key moments of transition during a learning task. For example, RQA metrics evidence the conceptually contrasting rules articulated verbally by our case-study learners. Nils' arithmetic rule to make half conceptualized a multiplicative relation between his hands and elicited a sharp increase in RQA metrics; Liam's qualitative rule to keep the right hand higher did neither. Finally, our results bolster the importance of task and environment design and instruction in cultivating new perceptual orientations. At their broadest, the findings support a renewed interest of learning scientists (Manches & O'Malley, 2016) and cognitive developmental psychologists (Allen & Bickhard, 2015) in the role of educational manipulatives as mediating the enculturation of action into concepts.

Limitations of the present analysis include the modest sample size of 39 participants, further impacted by the fact that not all participants' learning trajectories spent ample time in all three defined phases. From a technical perspective, limitations such as inconsistent contact with the touchscreen sometimes yielded order gaps in the data. To accommodate this, we used a relatively low sampling rate of 10 Hz and forward-filled missing data points from the last recorded position. It is possible that this strategy may have somewhat unevenly inflated recurrence or trapping time measures if contact with the screen was more inconsistent during certain phases of the task. Another technical challenge of this dataset was that target (green) positions on the screen do not reflect hands being at the same height, but rather one at half the height of the other. In this study, we z-score normalized the data from the two hands, but future research seeking to compare between participants may want to rescale left-hand heights and right-hand heights according to the target ratio. Finally, we note that because determinism data did not meet the homoscedasticity of variances assumption, the finding that the increase in determinism at the onset of the Discovery phase was statistically significant should be retested in future work. It will be important to test for the patterns in RQA metrics identified through this exploratory analysis in other embodied design learning contexts without these limitations.

Putting the present study in dialogue with prior studies suggests several avenues for future research. Stephen et al.'s work on problem-solving observed a sharp rise and fall in entropy just prior to discovery (Stephen, Boncoddo et al., 2009). Since the present study focused on characterizing phases, it was not poised to identify such a phenomenon except in the case studies, where lack of recurrence obscured this metric. Across participants more broadly, we saw an increase in entropy reflecting a decrease in degree of order in the bimanual system. One possible interpretation of this result could be that fluent movement actually requires dynamic fine-tuned adjustments that yield higher variability than more exploratory movement. The entropy RQA metric warrants closer attention in future research on embodied problem-solving.

Additionally, Kostrubiec et al. (2012) have shown that the routes to learning new coordination patterns depend upon learners' intrinsic dynamics, that is, their prior predispositions and capabilities. The embodied design environment appears equipped to accommodate learners' idiosyncratic intrinsic dynamics, with all learners attaining some degree of fluency by the end of the task. However, further characterization of intrinsic dynamics could further inform instructional design, for example in relation to shift vs. bifurcation learning pathways. In our case studies, Nils embodies the abrupt qualitative change of bifurcation with his abrupt dynamic shift into fluency. Liam, meanwhile, shows a slight gradual increase in trapping time and meanline in the last few seconds of his performance. Might he, with more time, have exhibited a gradual, smooth shift into greater fluency? In future work, it would be of value to characterize the nature of transition across participants. Charting the impact of intrinsic dynamics could improve differentiated support for learners in embodied design environments.

We close with a reflection on the broader utility of the RQA method for interaction design researchers. RQA offers several unique advantages. One is that it supports modeling the overall dynamics of a system even without access to all components in that system through phase space reconstruction (Coco & Dale, 2014). For example, we know from prior MIT-P research that gaze patterns participate in learners' growing fluency. We can study the evolution of the hands-gaze-technology-tutor cognitive system by using touchscreen data. The capacity to model dynamics of an emergent system from some of its components makes RQA a versatile research tool. Although not the focusal modality of this analysis, it is worth noting that changes in visual fixation have been implicated in the emergence of new perceptual structures that organize movement in the MIT-P (Abrahamson et al., 2016). One meaningful frontier in embodied design is that RQA can provide a means to study new perceptual structures in populations where eye-tracking is not appropriate, such as in the learning of blind students. This would allow for movement-based research designs to be replicated with learners using nonvisual modalities. Thus, RQA can support a more inclusive research agenda to better investigate the microprocesses of embodied learning and efficacy of embodied design for a greater diversity of learners. Another affordance of RQA is its capacity to quantify lead–follow dynamics based on the concentration of structures on the upper-left or lower-right quadrants of the recurrence plot. This could, for example, be used to examine whether the lead–follow dynamics between hands and gaze evolve as fluency develops. Prior studies have shown that the eye movements of tutors anticipate those of learners (Shvarts & Abrahamson, 2019). RQA could provide a means to test the presence and dynamics of these effects at scale and compare contexts, such as between dyads performing at different levels of fluency. RQA affords methodological traction on phenomena that have previously been only qualitatively depicted.

This study provides a proof-of-concept for the utility of RQA in tracking and analyzing the microprocesses that constitute conceptual learning from a complex dynamical systems perspective. Embodied-design research can use RQA for tracking conceptual learning in the context of any interactive system where motor actions are conceptually relevant and can serve to characterize and potentially predict learning breakthroughs. Beyond embodied design, any research context that centers embodied interaction where time series can be collected may use RQA to identify the dynamics of that evolving system, including the system's stability, degree of order, and degree of coupling across components of the system (parts of the body as in this example, data from two different conditions, or partners in a team). Modern multimodal data collection provides a wealth of complex time series such as gaze, movement, or categorical data. RQA allows us to embrace the complexity of these data rather than treat their variation as noise to reveal new insights into the microprocesses of learning and discovery.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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