Charting Our Embodied Territories: Learning Geometry as Negotiating Perspectival Complementarities

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If only geometry lessons enabled students to first encounter the content through their senses, it has been claimed, we would be better positioned to countervail enduring problems of poor engagement and low performance in the discipline (Freudenthal 1971; Thompson, 2013). To make this happen, however, we still need tighter grasps—both theoretical and actionable—on how informal situated *doing* becomes formal content *knowing*. Focusing on collaboration as catalyzing the doing-to-knowing learning process, here we envision and explore the process by analyzing how dyads negotiate contested situated perceptions into articulated content definitions.

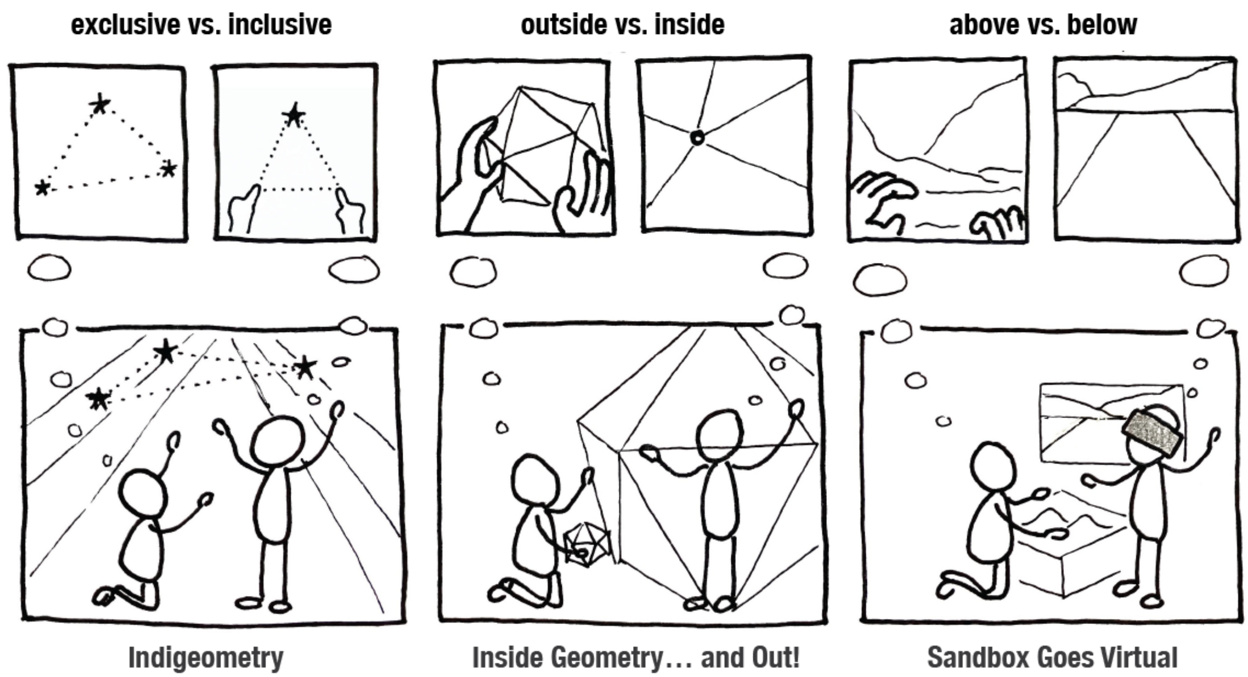


Figure 1. *The project's 3 learning environments viewed as perceptual complementarities. From left: (1) allocentric vs. egocentric views of a planetarium star constellation; (2) external vs. internal views of a geometric solid; and (3) hand-modelled vs. VR-immersed views of an adventure landscape.*

Looking across three geometry subject matter areas (see Figure 1), we present pilot-study results drawn from a multi-site design-based research project investigating socio-cognitive micro-processes of situated geometry learning as collaborative problem solving. The design architecture aims to distribute over each social unit two different, yet conceptually complementary, cognitive orientations toward a single sensory manifold. The design rationale is to stage, motivate, and encourage dialogic reconciliation of these differing perspectives by way of assigning the dyad a shared collaborative task. In coming to recognize and tune toward *each other*'s perspectival orientation, each participant is to engage in forms of reasoning and expression that solicit reflection on *their own* tacit perceptual mechanisms mediating their situated inferences and actions. In a sense, thus, this project is a phenomenological endeavor.

Our project's three learning environments exemplify what we propose to call *conceptually generative perspectival complementarities* (CGPC)—orientation pairings believed to induce conceptual synergy via reconciliation (qv. Abrahamson & Wilensky, 2007). Informed by the construct of CGPC, we conjecture, educational designers could create collaborative activities that foster and resolve tension between differing views on shared environments.

The CGPC addressed in our research project have been selected so as to occasion opportunities to examine how mathematics learning is impacted by, yet could avail of, a variety of perceptual differences. The CGPC we target are cultural (Barton, 2008), figural (Dimmel & Milewski, 2019), scalar (Herbst et al., 2017), and ontological (Bamberger & diSessa, 2003). We do not, and *could* not, argue for typological coverage of the CGPC construct—our project cannot exhaust all theoretical let alone empirical facets of the CGPC multi-dimensional ontological landscape. These would need to include intergenerational, intersensorial, interlinguistic, inter-situative, inter-conceptual, inter-media, and other CGPC all potentially relevant to our investigation.[1] Rather, we argue, our three designs offer a thematically coherent and topically non-redundant set of proof-of-concept pioneering studies that peg some landmarks and chart some prospective trails in the larger conceptual landscape of CGPC and their relevance to mathematics studies in geometry and perhaps beyond.

Perspectival Complementarities

We assume that formal mathematical notions can be grounded in informal, pre-symbolic, situated activity (Pirie & Kieren 1989). As such, features of the environment of relevance to technoscientific disciplines, including geometrical properties of objects such as an angle or a pair of parallel lines, may remain tacit to our immersed engagement until they emerge as necessary discursive means of coordinating collective goal-oriented practice; in turn, appropriating these cultural forms of speaking and reasoning transforms perception (Bautista & Roth, 2012). Building on the literature, we assume that: outside of school, learning can be understood as the enculturation of perception through joint enactment of situated activities; inside of school, too, perhaps, coordinating perspectives could serve as a powerful activity architecture for content learning; in so doing, particular care must be taken to avoid supplanting ancient perspectival practices with hegemonic perspectival practices.

Learning via Social Enculturation of Perception

Philosophers and theorists of perception, cognition, and action from diverse intellectual strains agree that *how* we experience the environment, that is, what meanings we glean from our sensory input, is related to who we are, what we are trying to do, and what we know. What we know, in turn, is constituted also by the cultural–historical forms we have appropriated through participating in the social enactment of cultural practice, so that perception can be said to be culturally constituted (Howes, 2019).

Within the cognitive and learning sciences we find strong resonance with this anthropological thesis of cultural visuality (e.g., "the domesticated eye," Radford, 2010). Curiously, the perceptual skills one develops as one engages in the enactment of cultural practices may, by and large, remain inaccessible to one's own reflection (Horsey, 2002). Indeed, heritage skills may be fostered without direct instruction but, rather, through apprenticing novices in authentic task-oriented activity settings and offering them feedback on their interactions with the environment (Ingold, 2000).

Macro-cultural evolutions of perception can be staged and simulated in micro-cultural settings of educational activities. In particular, *contested* perceptions of sensory displays could mobilize conceptual change. Productive contestation ideally takes the form of collaborative argumentation, whereby interlocutors: articulate their thoughts; attend empathetically to each other's points of view; experience conceptual growth; deepen their topical understanding via the discursive effort necessary for coordinating the enactment of their respective roles; and develop collaboration skills, socio-mathematical norms, metacognitive control skills, and mathematical epistemology—both as individuals and as groups.[1]

Learning Through Reconciling Perceptual Differences

Dialoguing individuals may enter a conversation with different senses of a shared referent (see Wegerif, 2011, on a Bakhtinian 'epistemic gap'). When they interact with students, teachers may deliberately use ambiguous sensory displays designed to evoke either naive or scientific construals, depending on which features are highlighted (Abrahamson, 2012; Newman et al., 1989). Teachers can change students' perception of objects, for example, shifting a student's attention from a triangle's three sides to its three vertices (Sfard, 2007). Objectively, it is the same shape on paper, but subjectively it is now construed anew so as to constitute a perceptual resource for engaging productively in the classroom's mathematical activity (Foster, 2011). Still, we submit, perhaps students need not mute and relinquish their own perspectival understandings.

Building on this line of research, we propose to create activities where participants come to perceive geometrical ontologies through conferring not just with teachers but with fellow participants, who are orienting differently onto the same sensory display. Coordinating collaborative activity across two different perspectives on some would-be objective reality compels one to confront one's own perceptual bias and reason from another person's perspective, which can contribute to learning content as well as to cognitive and even moral development (Schwarz & Baker, 2017).

Situated Knowings as Heritage Embodiments of Alternative Epistemologies

A program to promote geometry learning through interpersonal coordination of perceptual perspectives is *ipso facto* a program to promote the negotiation of mathematical identity (Heyd-Metzuyanim & Sfard, 2012). Becoming mathematically fluent demands grounding formal propositions in everyday language. And yet, by entering academic discourse, one necessarily resigns oneself to a discipline's hegemonic routines of reifying situated practice in culturally specific technoscientific forms; one subjects oneself to perceive within everyday situations a variety of specialized forms exogenous to one's traditional unreflective ways of being in the world (Gutiérrez et al., 2010). These foregign forms stand to destabilize Indigenous cultural–historical ontologies and, with them, heritage practices (Urton, 1997).

Mathematics as a formalized domain is grounded in Western perceptual–linguistic orientation to the sensory manifold, which has historically strived to recast egocentric subjectivity as allocentric objectivity (Jay, 1988). As such, a geometrical form apprehended in a natural environment is stripped of its perspectival idiosyncracy to offer, instead, a standard, generalized, and definitive shape plotted on an infinite plane. Western epistemology thus imposes a disembodied allocentric relation to the environment by promoting a view of points and lines represented on the Cartesian plane. As such, a schooling in Western geometry is predicated on subjugating perception to Western ontology and epistemology; and with the subjugation of perception, one subjugates ecological relationships of inclusive orientation and identity (Cajete, 1997). A culturally-sensitive geometry education would re-embrace ecological dimensions of immersed phenomenology, while steering individuals to reflect on the situated contingencies of their perceptual knowings. The construct of CGPC could frame a design for reconciling differing views empathetic to embodied, social, and cultural–historical experiences.

Three Vignettes

Our project is, thus, grounded in an interdisciplinary pedagogical conviction that *geometry learning should begin from tacit spatial phenomenology situated in goal-oriented collaborative activity*. The project is distributed across three design contexts, with populations of multi-dimensional diversity, and so far includes preliminary empirical data: (1) *Indigeometry Planetarium* (Tohatchi, New Mexico, USA): 1 Navajo student; (2) *Inside Geometry… and Out!* (Jerusalem, Israel): 9 university students in the course "New Ways to Think, Learn, and Move" and 60 highschool students in an enrichment activity; and (3) *Sandbox Goes Virtual* (Oakland, California, USA): 16 middle school students from a multicultural urban secondary school and an afterschool program. All contexts were designed to bear epistemic, affective, and social CGPC. The vignettes, below, present episodes of interest to the project's line of inquiry on CGPC.

Vignette 1

In the *Indigeometry Planetarium* (IP) learning environment (Figure 2), constructed as a canvas-covered dome, students enact essential perspectival qualities of Navajo archaeoastronomical practice in negotiation with Euclidean geometry. Navajo phenomenology of Euclidean angle is not an absolute allocentric feature of the environment ontologically independent from the viewer. Rather, Navajo angle intrinsically encompasses the viewer as the apex of an egocentric perspectival triangle, whose projected base subtends the extent of the percept.

Amaya, an 8 years-old 3rd-grade female Navajo student, participated in a 30-minute semi-structured task-based interview conducted by Jessica (the first author). Sitting on a swivel chair in the center of IP, Amaya was asked, "Can you point at the *start* of the shooting star?" She raised her left hand and pointed to the left-side end of a shooting star ahead and above her. Further instructed to "use the other hand to point at the *end* of the shooting star," Amaya, still holding her left hand up, raised her right hand and pointed to the right-side end of the same star. Her two arm–rays now project from her body–origin to these stars to embrace their span (see Figure 2).

Figure 2. *Embodied measuring in the* Indigeometry Planetarium. *The left shows a sketch of the right inside the IP.* (left) *The icosahedron shaped dome is constructed from dowles and plastic tube joinings with black tent-like material covering the structure. The inside ceiling features appliqué stars and shooting stars. (*right) *Gazing toward a shooting star, Amaya's left- and right-hand index fingers pointing respectively to the left- and right sides of a shooting star.*

Jessica then encourages Amaya to apply her "little hand measurement" to the rest of the stars. Rotating around the center, Amaya becomes increasingly independent in initiating the task performance and then self-adjusting her armspan to subtend her sight of each star with her hands.

At the conclusion of the activity, the following conversation took place.

Jessica	As you were doing that, did you feel a difference between, like, the different stars?
Amaya	[*nods in confirmation*]
*Jessic*a	What was the difference you felt?
Amaya	[*quietly*] It was longer and shorter.
Jessica	[*nods*] How did it make your arms feel, when you were doing it?
Amaya	[*looks off to the left*] Hm, like it was, like, shorter, like, my arms, like, and it, like, made my arms, like move.
Jessica	A little bit, what?
Amaya	Well, [*clears throat*] more, it's, like, when I measure, it's, like, like, um, [*looks down, lifts hands to shoulders, arms bent at the elbows* (see Figure 3)] it, like, makes my arms [*taps her hands on her shoulders*] like, sure, [*lowers hands, looks at Jessica*] spread out…
Jessica	[*smiles excitedly*] Ooh okay
Amaya	…longer and closer [*on "closer" brings hands together briefly*]

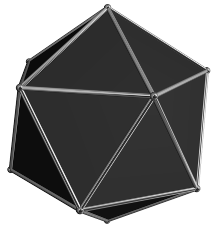
Figure 3. *Amaya lifts her hands to her shoulders, elbows bent out to the sides.*

The retinal image of a sighted object corresponds to the aperture of the viewer's stretched arms toward the object's extremities. This embodied egocentric apprehension of spatial magnitude, while endemic to authentic Indigenous cultural–historical practices and psycho-linguistic constructions, is conscientiously elided from formal Western geometrical allocentric discourse. That young Amaya could *embrace* a measure of optical spatial interval by enacting an embodied angularity suggests that geometry can be grounded in first-person multimodal phenomenology. We thus witness the conceptually generative potential of a staged synergy between two conceptually complementary perspectives on a situation. Future iterations of the IP activity design will encompass student–student collaboration.

Vignette 2

The empirical context *Inside Geometry… and Out!* explores collaborative learning for geometric solids. A distinctive feature of this environment is that students construct the same geometric objects at different scales using wooden rods and silicone joints. In one of the tasks (see Figure 4) students explore an icosahedron—a polyhedron whose exterior is composed of twenty equilateral triangular faces. In this activity, students are given a 2D-diagram and are to construct a relatively small icosahedron as well as a human-scale icosahedron. Once both models are built, students are asked questions concerning the icosahedron's geometric and topological properties, for example, "How many vertices does an icosahedron have?", "How many parallel edges?", "If the icosahedron were standing on its triangular base and filled half-way up with water, what would be the water's surface shape?" We analyze students' choices of small vs. large icosahedra to investigate each question.

Your team has to construct two three-dimensional models (one large, one small) of the following geometrical solid, a polyhedron.



The polyhedron has the following properties:

- All the faces are congruent equilateral triangles.
- The same number of edges converge at each vertex.

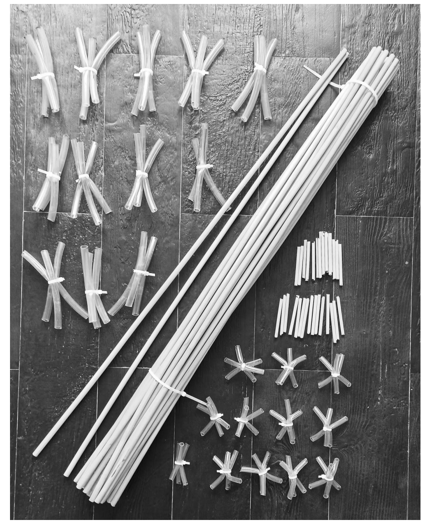


Figure 4. The icosahedron construction task and materials. Left: Worksheet for student teams; Right: The construction kit, with long dowels for the large form, small dowels for the small form. For each form, 12 sets each of three latched silicon pipes serve as vertices where 5 dowels are inserted.

The following account is from a pilot outdoor implementation of the activity with a group of high-school students. It describes how they worked with their constructed models to solve the questions (see Figure 5).

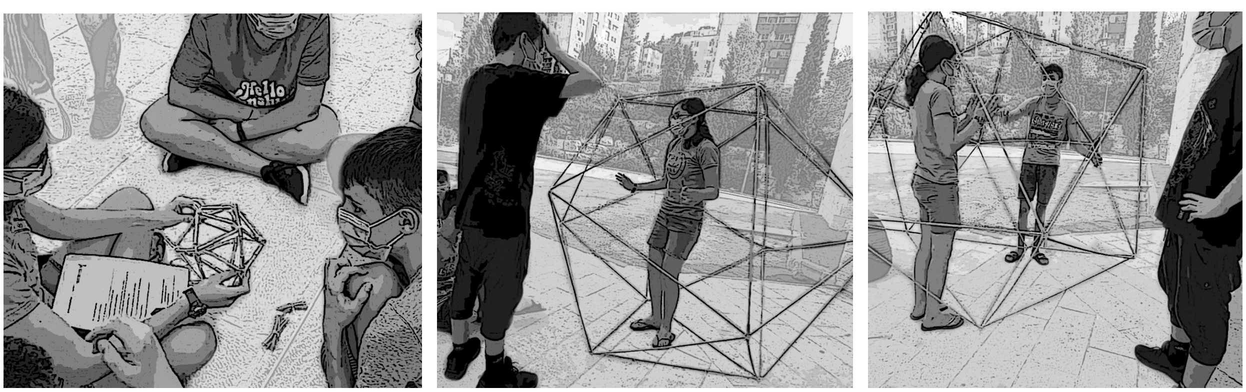


Figure 5. *The geometry activity* Inside Geometry… and Out! *combines construction, problem-solving, and justification tasks, where each task provides different CGPC-related coordination challenges. From left: (1) Students discuss a small-scale model; (2) Students' problem-solving inside and outside a human-scale model (standing on a triangular face); (3) Having tilted the structure onto a vertex, the students soon arrive at a critical breakthrough.*

Having constructed both the small and large models, the students used the small model to answer correctly that an icosahedron has 12 vertices. Next, they tackle the question of how many edges an icosahedron has. They will soon find it difficult to solve this problem using the small model. (Participants are referred to by the color of their t-shirts; transcription translated from Hebrew by AP)

Yellow	How many edges are there?
Black	Okay, that's tricky because they're shared. *[ie. each edge is shared by two triangles].*
Blue	I'll put a finger *[on the first edge that Yellow counts, to help her monitor the count].*
Orange	You just count the sticks.
Yellow	I'll go to the big one *[ie. the large-scale model].*
Black	The big one is just nicer.

Three of the six students rose and walked over to the nearby large-scale model. This larger model is advantageous for counting, because its edges are more perceptually distinct. But a model's greater size, while availing perceptual acuity, may come with a price that its figural elements in question (the to-be-counted edges) are never all in one's arm's reach—you cannot directly touch or clearly gesture to each edge as you tally it. Immediately, Yellow entered inside the model, from whence all edges are within her peri-personal reach. Still, when one is inside an object, part of it is always behind you..., and so you might lose track of your count! Indeed, Yellow's initial attempts to count failed. As the excerpt below demonstrates, she then attempted to use some of the icosahedron properties that the team discovered during construction, yet again she failed to develop a systematic approach.

Yellow	There are five from each vertex. One should be subtracted, then there are four. Two should be subtracted here, it's three. It doesn't work that way. …3, 4, 5. I can't count this. How many sticks did we use [*during the construction stage*]*?* Three and another three, and another three, and another three, and another three, it's 12, another three, 15, another three…[*referring to triangular faces*]
Black	We need a formula for this…

From a mathematical point of view, it does not matter how the icosahedron is positioned in space—the polyhedron's mathematical properties remain the same. In a material gravitational world, however, the model usually lies on one of its triangular faces, making it difficult to perceive certain structural symmetries. The next excerpt shows how by tilting the model onto a vertex (see Figure 5, on the right) suddenly these inherent symmetries became apparent as tripartite: two opposing "bases," each comprising 10 edges, and a connecting "belt," also with 10 edges.

Gray	It will be easier to count like that [tilts the model so it stands on a vertex [*holds the model in place*]. 1, 2, 3, 4, 5 [*counts the edges diverging from the base vertex*]*;* 1, 2, 3, 4, 5 [*counts the edges of the pentagonal base*]*;*
Yellow	1, 2, 3, 4... 1, 2, 3, 4… [addressing Grey] Put your hand here. 1, 2, 3, 4, 5 *[continues to count silently]...* ten, ten, ten, ...thirty!

This vignette demonstrated that some of the students preferred to answer geometry questions using the small model and some—the large model; at least some questions were answered utilizing material, non-mathematical affordances of the model: entering it (large), holding it (small), and changing its physical orientation in a shared space (large). The fluidity with which students moved (both physically and in their reasoning) from one model to another suggests that a multitude of perspectives helped them to generate the concept of the polyhedron.

In sum, the different physical sizes of the two models of a single geometric structure afforded students different modes of interaction, which in turn generated shifts in their perceptual perspectives on the models, revealing their invariant geometrical properties. Thus, the activity's designed perspectival complementarities were conceptually generative in constituting solutions to a set of mathematical questions.

Vignette 3

Sandbox Goes Virtual is a hybrid Spatial Augmented Reality (SAR) sandbox and Virtual Reality (VR) system that we developed to support children's collaborative design processes involving geometrical solids and topographic projections. Using a depth-sensing camera installed above the physical sandbox, the system scans the surface of the sand in real time and generates a correlating 3D, VR rendering of the sandbox topology that is constantly changing as one child physically sculpts the "sandscape" (see Figure 6). In the corresponding VR world, a second child wearing a head-mounted display (HMD) can virtually walk through the mountains, valleys, etc. that were physically crafted in the sandbox, with a first-person point of view. We intentionally employ only one HMD as we want children to take turns being the physical landscape manipulator at the sandbox and the immersed explorer in the VR world. Thus, one child could see the terrain from an "outside" and broader perspective while another child experiences the same terrain in VR in real-time from an "inside" perspective (where scale is 1:1). For example, a child in VR might suddenly see a gaping canyon appear in front of her because her partner in the physical world just scooped up a handful of sand. The sandbox is augmented with color projections from above to visually emphasize topography such as lakes, peaks, etc. The virtual model is using the same colors as the projection.

In our preliminary study, we have asked middle school children to work in pairs to design a maze that has three mountains to climb anywhere along the path, where Mountain A must be two times taller than Mountain C, Mountain B has to be three times taller than Mountain C, and Mountain C can be any height.

In the following example, Sam wears an HMD and explores the model in VR while Ruth physically sandscapes the maze model. A nearby LCD shows the VR user's view.

Sam [*talking to Ruth*] However we are trying, just make sure that, it's, um, big enough to be considered a mountain. And small enough to... make sure it could be three times as large as... for the mountain B.

Ruth Uh huh. I'm also making a path while I'm making this mountain.

[...]

Sam Where do you think we should put the other mountains? [*looking around in VR*]

Sam Do you see where I am looking?

Ruth Yeah. [*Ruth goes back and forth between Sam's perspective provided by a nearby LCD and her own perspective of the physical sandbox*]

Ruth I think we should put them fair distance apart. So that they are not clumped up in one location.

Sam Do you see where I am looking? Somewhere over there? [*speaking from her VR perspective*]

Ruth Here. [*points to a location in the sandbox. Now Ruth's hand is represented as a part of landscape Sam can see in VR*]

Sam Right there? OK. [*Sam sees what Ruth is pointing at as a part of landscape in her VR view*]

Sam Um, maybe Mountain B there.

Ruth Mountain B goes here. [*Ruth is looking at the sandbox and LCD, and starts sculpting Mountain B*]

Sam Yeah. So the one that's three times as big as this one. [*speaking from the VR view*]

Both Ruth and Sam communicate their effort to fulfill the task requirements as well as aesthetic concerns for their maze design. Practically, it would be simpler to have the mountains close to each other so that the heights could be compared easily, yet Ruth wishes the mountains to be "fair distance apart" for aesthetic reasons. Likewise, making Mountain C small enough would keep Mountain B's height manageable. Sam communicates this point to Ruth while also assuring that it should be "big enough to be considered a mountain" from the perspective of an actual maze user. Through accessing such multiple views available to them (i.e., view of physical sandbox, VR view, view of each other), Ruth and Sam negotiate their perspectives and set up their own goals in achieving a design that meets the task specifications.

[*Ruth is at the sandbox measuring their small mountain with a physical ruler (see Figure 6, on the left)*]

Ruth Two and a half inches. [*Ruth reads the ruler*] So then, like... around seven inches. [*Ruth looks at the ruler, and now measures the taller mountain*] OK. Cool. [*Ruth recognizes that the taller mountain is not tall enough. She then puts the ruler away and makes the tall mountain even taller.*]

Sam I just saw the mountain. [*laughs*] [*Sam is seeing the mountain being created by Ruth in the VR perspective*]

Ruth This mountain is really steep. [*Ruth finishes the tall mountain in the sandbox*]

Ruth There we go. [*takes a physical ruler and measures the height of the tall mountain she just created in the sandbox*]

Ruth Yup. Still not tall enough. Alright. [*Ruth puts the ruler away and sculpts the mountain to be even taller*]

Sam Remember, you can make the other one shorter. [*laughs*] [*Sam in VR view*]

Ruth Duh! [*laughs*]

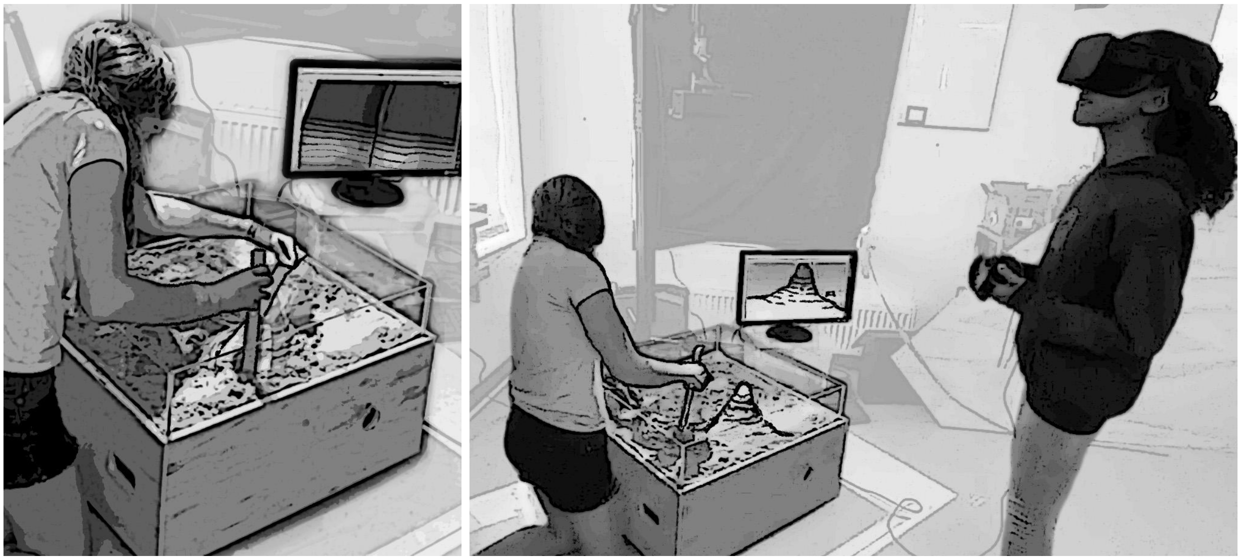


Figure 6. *Ruth is measuring the height of the tall mountain using a physical ruler, while her partner Sam looks at the model from the VR perspective.*

In sculpting sand mountains physically, Ruth struggles to make Mountain B three times taller than Mountain C. Sam, with her VR HMD, has an "inside" 1:1 scale perspective on the sculpted mountains, which enabled her to notice and communicate with her partner that Mountain C could be made smaller to facilitate the construction of Mountain B. As such, two children take turns being the creator of the physical model and the evaluator of the same model from the VR perspective, collaborating simultaneously at two different scales. It is this turn-taking between the two roles, creating and evaluating a sensory manifold from multiple perspectives, which provides an opportunity for dialogic reconciliation of these different views. This, in turn, could surface conceptually productive differences relevant to the study of geometrical solids and topographic projections.

Discussion: Substitution, Mutuality, and Synergy

The three studies surveyed above exemplify our collective efforts to develop an explanatory process model of geometry learning as dyadic collaborative negotiation across different perceptual perspectives on shared situations. The vignettes suggest the validity and robustness of our thematic construct, conceptually generative perspectival complementarity (CGPC), by demonstrating similar interaction patterns across variable phenomenal dimensions.

Across the vignettes, the discursive negotiation of perspectives played a crucial role in mobilizing the participants' actions and insights. Bearing different perspectives on the activity's focal objects, the collaborating children were compelled to articulate what they see as a condition for engaging in pragmatic discourse. Not only did they surface and reify the objects' geometric properties from their subjective perspective—they also identified and "conserved" *universal*, pan-perspective, scale-free, and, thus, mutually intelligible features, per the design's objectives.

Comparative analysis of the vignettes suggests a new elaboration of CPGC by negotiation *type*: substitution, mutuality, and synergy. In *substitution*, one of the perspectives replaces the other, because it is tacitly evaluated as contextually advantageous for attaining information relevant to the task at hand (e.g., when Amaya utilized arm aperture as a qualitative measuring tool). In *mutuality*, the viewing perspectives are both sustained, with participants retaining their initial perspective while sanctioning the alternative view (e.g., when Sam and Ruth each kept experiencing the landscape from their respective scale). In *synergy*, a new perspective emerges that is greater than the sum of its parts (e.g., counting together the edges of an icosahedron from within and without it).

Conclusion

When geometry activities deliberately summon from students different perspectival orientations on a shared task space, this difference can be leveraged as a means of transitioning from intuitive to disciplinary practices and understandings. As they figure out together how best to collaborate across their perspectival gap, students may either substitute, maintain, or combine their views, with varying consequences for learning outcomes. We are only beginning to identify which sociomaterial circumstances result in each form of negotiation, and how these negotiation outcomes bear on the emergence of mathematical ontologies. As educational designers, we look to understand the role of different media in facilitating productive negotiations, and we seek to investigate challenges and opportunities of integrating CGPC activities into classroom settings and mainstream curriculum.

More broadly, we aspire to delineate heuristics for creating activities that optimize for learning across perspectival differences. Ultimately, an approach to the learning of mathematics grounded in reconciling perspectival complementarities, we surmise, could bear on broader ideological and socio-political issues of diversity, inclusiveness, and cultural identity.

Endnote

[1]For citations from the literature, please see Authors (in preparation).

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