LEARNING TO LISTEN: A STUDENT'S UNDERSTANDING OF POWERS OF TEN[1]

JERE CONFREY

Cornell University

I. INTRODUCTION

Basic to applying the constructivist epistemology to mathematics education are two claims:

1. When students genuinely engage in solving mathematical problems, they proceed in personally reasonable and productive ways.
2. Researchers and teachers must learn to listen and to hear the sense, and alternative meanings in these approaches.

Although necessary for the application of constructivist epistemology to research or teaching in mathematics education, acceptance of these two claims is not sufficient to ensure constructivist practices; for, they can be interpreted in multiple ways.

In this paper, I will:

1. contrast alternative interpretations of these claims as they might be understood in the traditions of discovery learning, problem solving and misconceptions with a constructive interpretation.
2. summarize some of the basic assumptions for conducting an investigation of students' conceptions within the constructivist framework, and
3. provide an illustration using interview excerpts from a student who was asked to represent on a number line a series of historical events given in scientific notation.

Examining only briefly the similarities and contrasts between constructivism and other perspectives on student learning in mathematics has certain advantages and hazards. It can assist us in learning how constructivism complements or contrasts with other widely held approaches (as a puzzle piece fits in a jigsaw puzzle) in preference to presenting, in isolation, its own tenets and claims. The contrasting theories were selected because they share many assumptions with constructivism, and yet, their treatment of the epistemological character of mathematics or of the socio-psychological genesis of ideas is different. An inherent hazard of a brief presentation is that the constrasting theories may appear oversimplified and too easily dismissed without an appreciation for their role in the progress of educational thought. To compensate for this, references to fuller treatments and reviews of the theories are provided.

E. von Glasersfeld (ed.), Radical Constructivism in Mathematics Education, 111–138.

II. DISCOVERY LEARNING

Many people will read the first claim for applying constructivisim as parallel to one of the basic assumptions behind “discovery learning” (Bruner, 1960, 1966). Discovery learning as it was advanced in the 1960’s referred to “methods that permit a student to discover for himself the generalization that lies behind a particular mathematical operation” and it was contrasted to *“the method of assertion and proof* in which the generalization is first stated by the teacher, and the class is asked to proceed through the proof.” (Bruner, 1960, p. 21) The methods of induction (drawing generalizations from an array of examples) and deduction (drawing conclusions from premises) formed the basis of the contrasting positions, with discovery learning relying on induction. As a proponent of discovery learning, one might interpret the first claim to mean: if a concept is inductively demonstrated through the use of mathematical problems, the students will actively participate in the process of “uncovering” that generalization in resourceful and effective ways.

Discovery learning certainly rested on some assumptions that constructivists share. It stressed the importance of: 1) involving the student actively in the learning process; 2) emphasizing the process of “coming to know” over the rapid production of correct answers; and 3) extracting and making increasingly visible the structure of a concept. Its basic claim, that students could be guided to discover rather than be told, was a radical departure from the assumption that mathematics could only be taught through direct instruction following by drill and practice.

However, the statement in the first claim, “genuinely engaged in solving a problem”, entails, for the constructivist, more than reasoning inductively to conclude with a predesignated generalization carefully manufactured by using a set of examples. Advocates for discovery learning assumed that behind the set of examples lies a generalization, awaiting discovery. That generalization, logically required by the structure of the concept, was assumed to be retrievable relatively automatically and uniformly across students by their application of inductive technique. The argument was that through such a process, the student would be more likely to internalize the results. Discovery learning was a model for promoting more effective learning - the epistemological content, (the claims about the mathematical knowledge to be learned) remained relatively untouched.

Most often, the applications of discovery learning did make the generalization itself evident as a repeated pattern. But, making such generalizations still did not ensure any deep understanding, for they left the content undisturbed. In striving to have students make inductive generalizations, the educators often neglected to assist students in developing a sense of the functionality of the concept - its purpose and usefulness as an explanatory construct, its epistemic character. For example, one can conclude inductively that changing a percent to a decimal requires one to move two spaces to the left without gaining any insight into the advantage of place value (even if the process includes use of the algorithm for dividing by 100). However, if the students do not see the efficiency and elegance of the multiplicative structure of place value (as opposed to the awkwardness of the cumulative structure of systems like Roman numerals), the inductive generalization will be only minimally more meaningful to the student. In discovery learning, revisions in the mathematics itself were not considered necessary; only the logical process used to learn was altered, as induction replaced deduction.

For the mathematicians and cognitive psychologists working on the ideas of discovery learning, mathematical ideas were seen to lie within a larger conceptual field whose structure was hierarchical, consistent and logically necessary. "Big ideas" such as sets, functions, deductive systems and properties were thought to provide a powerful and parsimonious structure into which individual generalizations by student would be slotted. Thus, "discovery learning" entailed a commitment to a form of Platonic idealism. The truth value of a mathematical claim was derived from outside the framework of human experience. When students "discovered" a mathematical necessity, there was no need to consider the epistemological status of such a claim - its certainty, necessity, unformity and stability were assured by the membership in the class of mathematical truths.

III. A CONSTRUCTIVIST VIEW OF MATHEMATICS

The view of mathematics underlying discovery learning is at variance with that of the constructivist. For the constructivist, mathematical insights are always constructed by individuals and their meaning lies within the framework of that individual's experience. Students' explanations, their inventions, have legitimate epistemological content and are the primary source for investigation (other potential sources include the beliefs of teachers and mathematicians). For the constructivist, mathematical ideas are created and their status negotiated within a culture of mathematicians, of engineers, of applied mathematicians, statisticians or scientists, and, more widely, in society as a whole, as it conducts its activities of commerce, construction, and regulation.

The constructivist does not deny those who study or practice mathematics the authenticity of their experience of profound certainty when they have uncovered a mathematical claim. The sense of profound certainty, the documentation of concurrent inventions, the unpredicted convergence of apparently disparate fields, the tenacity of certain mathematical constructs all require explanation. But, equally important, the constructivist does not disregard numerous examples of mathematical claims by courageous individuals, denied at the time as impossible, which eventually led to significant advances; nor does s/he wish to ignore the evidence of views, widely held as self-evident at one, time, which later were viewed as in error or outside the mainstream of important mathematical thought. (Klein, 1980; Davis and Hersh, 1981; Trudeau, 1987).

In rejecting the idea of Platonist truths whose existence is independent of humanity, the constructivist relies on explanation based in the interplay between social negotiation of meanings and individual creativity and genius. In constructivism, the development of mathematical ideas is explained through their cultural history of negotiation (Lakatos, 1976; Toulmin, 1972); particular attention is given to examining how multiple systems of representation, symbolism and tools create occasions for the convergence of meanings - a convergence that is created through the ways in which those who practice mathematics weave together notational, linguistic, manipulative and operational forms of description. Regrettably, the Platonic view of mathematical truth (as existing outside of space and time and yet accessible through the purest forms of human abstraction) is reinforced by the scarcity of historians and mathematicians capable of "rational reconstruction" (Lakatos, 1976; Unguru, 1976). The work demands individuals capable of unraveling rich and complex chains of reasoning winding over long periods of time of time (Elkana, 1974; Toulmin, 1972; Lakatos, 1976; Feyerabend, 1978; Tymoczko, 1986).

Relying on the philosophers of science to provide the historical analysis, educational constructivists engage instead on a pursuit of documenting and describing the course of development of mathematical ideas in children, adolescents, and adults. It is this "genetic epistemology" (Piaget, 1970), a description of how people come to understand the epistemic structure of mathematical or scientific ideas, to which the constructivist programme is committed. The constructivist view of development of mathematical knowledge, which is assumed to entail similar influences whether it be studied as historical (across communities over significant spans of time) or developmental (among individuals participating in society over life spans), is taken as an assumption which undergirds the constructivist programme. It can be stated as follows:

Assumption One: Constructivists view mathematics as a human creation, evolving within cultural contexts. They seek out the multiplicity of meanings, across disciplines, cultures, historical treatments, and applications. They assume that through the activities of reflection and of communication and negotiation of meaning, human beings construct mathematical concepts which allow them to structure experience and to solve problems. Thus, mathematics is assumed to include more than its definitions, theorems and proofs and its logical relationships - included in it are its forms of representation, its evolution of problems and its methods of proof and standards of evidence.

Accepting this description of mathematics has implications for the constructivist examining students who are engaged in solving problems. The constructivist does not expect a student to produce textbook generalization - in form or content. s/he begins with the assumption that what a student does is reasonable and then seeks to describe it from the student's perspective. The investigation is assumed to have epistemological content - not just psychological content - in that recognizing, unearthing and giving validity to a student's method requires one to confer on it the status of genuine knowledge. If the student fails to ever articulate it, or to continue to demonstrate its usefulness and viability in solving his or her problems, then the interviewer may abandon this assumption. Such a situation is, however, surprisingly rare. Thus, unlike in discovery learning where the generalization sought after is presumed to be predictable prior to the investigation, the constructivist is engaged in a processs of invention - invention of his/her own models for explaining students' actions and words.

Assumption Two: In examining a student's understanding of a mathematical concept, a constructivist seeks to represent how a student approaches the mathematical content. S/He expects diversity - and idiosyncratic rationality. The interviewer's knowledge of the mathematical content, complete with multiple representations, competing interpretations, various applications - guides the inquiry, but his/her intent is to examine the student's use of examples, images, language, definitions, analogies etc. to create a model which may well transform the interviewer's own understanding of the mathematical content in fundamental ways.

III. PROBLEM SOLVING

A second interpretation of the first claim (p. 1) might be that it endorses problem solving approaches (Polya, 1945, 1962; Goldin and McClintock, 1979; Silver, 1982; Schoenfeld, 1985) Problem solving advocates argue for explicit instruction in the solving of mathematical problems. Problem solving is described through its use of heuristics and metacognitive strategies frequently set within Polya's four-step process (understanding the problem, devising a plan, carrying out the plan and looking back.)

The paradigm for problem solving shares with constructivism some basic assumptions that include stressing: 1) the value of a problem as "a means of finding a way out of a difficulty, a way around an obstacle, attaining an aim that was not immediately obtainable." (Polya, 1972, p.v.); 2) the importance of problem clarification. "The worst may happen if the student embarks upon computations or constructions without having *understood* the problem". (Polya, 1945, p. 6); 3) the significance of elucidating strategies that are typically tacit but effective; and 4) the value of looking back, i.e. reflecting on one's solution path.

These shared assumptions constituted a profound shift in our perspectives on mathematics education. Identifying heuristics involved mathematicians and educators in a process of self-reflection which led to the expression of personal epistemological assumptions concerning the structure of mathematical problems. Moreover, by claiming that problem solving could be taught, problem solving theories weakened the insipid belief that mathematical ability was a holistic character trait conferred (or not) at birth. Remarkably, problem solving approaches began with the assumption that the bastion of mathematical creativity, the ability to solve problems, could be taught. Thus, the tradition of problem solving introduced talk-aloud methods of examining a person's talk during the activity of problem solving. Finally, by focusing on problem solving, advocates clearly demonstrated teachers' premature concern with proof, public standards of evidence while at the same time they ignored the process of invention - one which was to rely on intution, insight and heuristic.

Just as proponents of discovery learning assumed a specific structure inherent in a concept, problem solving advocates tended to assume a well-defined structure inherent in a problem. Heuristics, (try a simpler problem, consider special cases, use auxiliary information) supported the perspective that mathematical systems were well-organized structures in which problems were suspended. In Polya (1945), we witness this formalization of a problem in his characterization of a student's initial understanding of a problem as "incomplete". He states, "Our conception of the problem is rather incomplete when we start the work; our outlook is different when we have made some progress; it is again different when we have almost obtained the solution." (p.5) If an underlying structure is presumed to be inherent in a problem, then teaching problem solving can be viewed as mastery of a skill, as Polya did when he wrote that problem solving is "a practical art like swimming, or skiing or playing the piano: you learn it by imitation and practice." (Polya, 1962, p. v.)

Problems play an important role in the evolution of mathematical knowledge; their role in the teaching of mathematics is even more crucial. The common inclination to confer on them independent structure needs explanation. Even Lakatos after providing a rich and wonderfully social portrayal of the evolution of the Euler conjecture in *Proofs and Refutations* (1976) gives way to this inclination when he wrote:

> Mathematical activity is human activity. Certain aspects ... can be studied by psychology, others by history. Heuristic is not primarily interested in these aspects. But, mathematical activity produces mathematics. Mathematics, this product of human activity, 'alienates itself' from the human activity. It becomes a living, growing organism which *acquires a certain autonomy* from the activity which has produced it; it develops its own autonomous laws of growth, its own dialectic. The genuine creative mathematician is just a personification and incarnation of these laws which can only realize themselves in human action. Their incarnation, however, is rarely perfect. The activity of human mathematicians as it appears in history is only a fumbling realization of the wonderful dialectic of mathematical ideas. But any mathematician, if he has talent, spark, genius, communicates with, feels the sweep of, and obeys this dialectic of ideas.
>
> Now, heuristic is concerned with the autonomous dialectic of mathematics, and not with its history, though it can study its subject only through the study of history and through the rational reconstruction of history.

A footnote from Lakatos' students suggests that the Hegelian influence on him, apparent in this statement, was lessened by the influence of Popper, and thus this statement might have been significantly revised had Lakatos not died at such a young age. Nonetheless, his statement illustrates the strength of the human inclination to attribute to mathematical problems a timelessness and an independence from humanity.

IV. A CONSTRUCTIVIST VIEW ON MATHEMATICAL PROBLEMS

However, to confer on problems an independent ontological status as Lakatos seems to do in this passage and is so often done by problem solving advocates runs counter to the constructivist perspective. Problems, to problem solving advocates, are described as situations which require the solver to put together information in a new way. Polya describes problems as possessing an unknown (that which we seek), a given (that which we know as data which allow us to recognize the unknown) and a condition which will eventually link the unknown and the data. The problem in this description appears to be located in the very structure of the task.

The model that emerges appears to be as shown in Figure 1

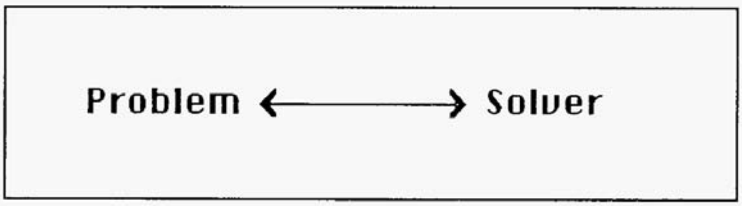

Figure 1.

Constructivists deny independent existence to problems. An example may demonstrate the point:

> Suppose a bank advertises 6% interest, compounded monthly, on savings accounts. If you deposit $100 on March 1 and withdraw it on April 1, how much interest will you have earned?

Without knowing that interest is, by convention, always quoted on an annual rate, one might easily answer $6.00. However, familiarity with this social convention allows one to deduce that *6% per year* is equivalent to 0.5% per month, and that the correct answer is 50. The structure is not in the problem - it is in the socially and contextually defined meaning of the words as interpreted by the listener. As von Glasersfeld reminded us.

> As seasoned users of language, we all tend to develop an unwarranted faith in the efficacy of linguistic communication. We act as though it could be taken for granted that the words we utter will automatically call forth in the listener the particular concepts and relations we intend to "express" We tend to delude ourselves that speech "conveys" ideas or mental representations. But words be they spoken or written, do not convey anything. They can only call forth what is already there." (p. 485)

For the constructivist, the problem is only defined in relation to the solver. A problem is only a problem to the extent to which and in the manner in which it feels problematic to the solver. When defined this way, as a *roadblock to where a student wants to be,* a problem is not given independent status. In order to differentiate this approach to a problem from the typical use of problems in mathematics classrooms, I have chosen to use the term, *the problematic,* to refer to a student's "roadblock". In learning to listen to students, the constructivist devotes considerable time to imagining how the student views the problem.

In discussing the relationship between a problem solver and the problematic as s/he sees it, one still needs to explain why problems are so crucial in mathematics. If they do not carry "content", why are they so compelling? Most everyone will admit that it has been the examples, the problems, not the definitions and proofs through which they have learned mathematics - and this needs explanation.

The constructivist responds to this in two ways: by reconceptualizing what knowledge is and by describing the process of problem solving in relation to its affective character. In redefining knowledge, the constructivist draws on the fundamental processes described by Piaget: assimilation and accommodation as they act to restore equilibration. Human beings are living systems that seek equilibration and problems disrupt that equilibration. Once we admit of a problematic, (i.e. notice a disturbance), we work towards reestablishing equilibration. In this sense, problems while acting as roadblocks to where we wish to be are also "calls to action" - they get us poised to operate on a system.

We act through sensory-motor and cognitive operations. We use tools and previously familiar systems of representation. Then, we monitor the results of our actions to see if the problematic has been resolved and equilibration restored. This may end the sequence, lead to a reconsideration and perhaps alteration of the problematic, and subsequently a new cycle of action and reflection. When an action or operation is seen as repeatedly successful in our progress towards resolution, we set apart that action or operation by various processes of naming it, discussing it, objectifying it (Confrey, 1985) or creating it into a tool or representation for further action. This is the process of reflective abstraction (Piaget, 1971).

These intellectual processes are, for the constructivist, the source and the content of knowledge. Knowledge is not the accumulation of information; it is the construction of cognitive structures that are enabling, generative and proven successful in problem solving. Thus, problem solving becomes an essential intellectual act - not an enrichment, an occasional pastime. Figure 2 provides a simple model to describe this process of construction.

The objectification of problems, seen in these terms, reveals a human tendency which is itself a part of the problem solving process. We act as though problems (and their solutions) are out there. Michael Polanyi in *Personal Knowledge* (1958) emphasized that a person (a student or a creative mathematician) needs to believe a problem capable of solution and be "looking for it [the solution] as if it were there, pre-existent" in order to consider it a problem. Thus, inherent in our establishment of a problematic is a belief that it is capable of being solved. We may not possess the confidence or arrogance to presume that we will be the successful ones - what we are expressing is our belief that we will recognize the solution if we were to see it, though we may not be able to articulate anything about it. Our assurance is only in its *possibility* of solution - an assurance which easily leads us to talk as though problems were out there.

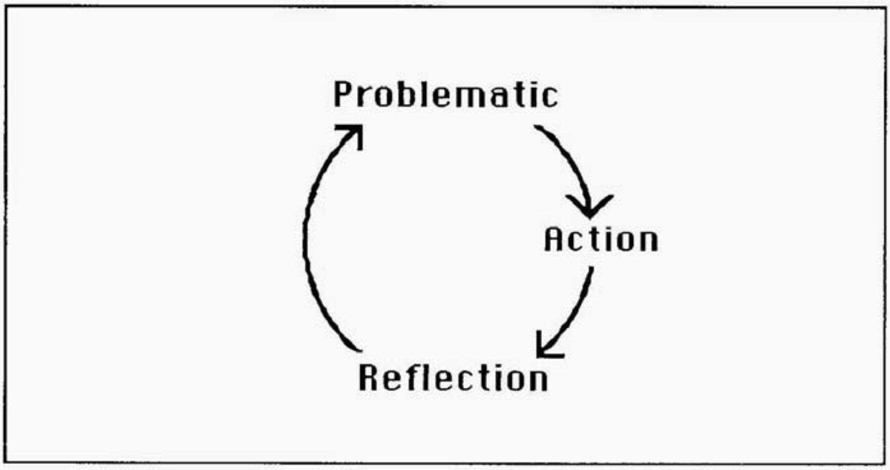

Figure 2.

His characterization of the emotional dimensions of problem solving explain further how our investment in the process encourages us to confer on the problem an independent existence. Our belief that "we can't create something from nothing" makes us wish to assume the problem solution came from outside. However, with a revised perspective on knowledge, we accept the recursive quality of knowledge construction and the emotional intensity helps to explain why problem solving is so successful in generating knowledge. As Polanyi put it, "obsession with one's problem is, in fact, the mainspring of all inventive power .... And the intensity of our preoccupation with a problem generates also our power for reorganizing our thoughts successfully . . . (p. 127).

Polanyi summed up the power of the problem solving activity in his definition of a problem: "A problem is an intellectual desire ... and like every desire it postulates the existence of something that can satisfy it .... As all desire stimulates the imagination to dwell on the means of satisfying it, and is stirred up in its turn by the play of the imagination it has fostered, so also, by taking interest in a problem we start speculating about its possible solution and in doing so, become further engrossed in the problem."

This leads to a third assumpton for the constructivist position:

Asssumption Three: Problems serve a crucial role in the construction of knowledge. Problems reside in the mind of the student - not in textbooks or in the mathematics. Problems are felt discrepancies, roadblocks to where a student wishes to be and therefore catalysts for action. To accept a problematic an individual must believe that it is capable of being solved - and act as though the problem and solution were preexistent. The cycle of identifying (noticing) problematics, acting and operating on them and then reflecting on the *results of those actions is emotionally charged, motivating and demanding. It is this process of knowledge construction which is the critical site for constructivist researchers/teachers.*

Solely defining a problem in relation to the solver does not prove adequate, for the constructivist researcher/teacher, who uses problems and tasks as the means of examining students' conceptions, promoting their continued development and assessing their progress. Thus, when problem solving is no longer a solitary affair, it involves an act of communication, an interaction which must be considered. The witness to the student's problem solving activity cannot claim access to some ontologically independent world of problems, so that she too has a problematic, an individual interpretation of the problem. Thus, the model must be revised as illustrated in Figure 3.

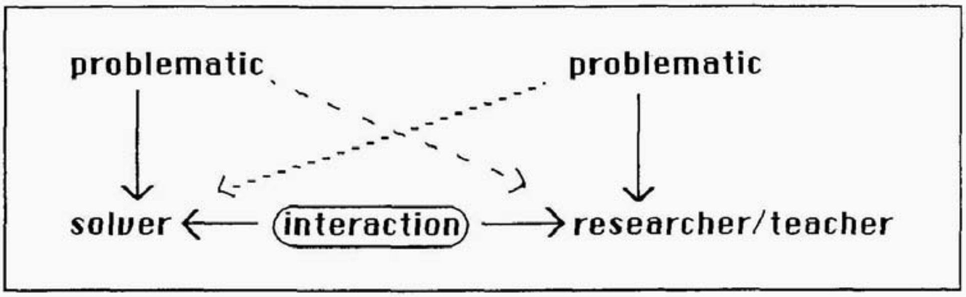

Figure 3.

Notice that lines drawn from each other's problematic are added to indicate that the researcher/teacher is trying to imagine what the student's problematic is like, while the student, in a classroom or interview, is trying to anticipate what the researcher/teacher wants. Over the course of the interview or the class, it is hoped that the two will come to believe that they understand better each other's expectations, conceptions and forms of solution –butthis is best done when the student is engaged in his/her problematic and its resolution, and the researcher/teacher, guided by those responses, forms a powerful model and tests it while challenging the student with more sophisticated and yet relevant problems. Much of the success of the constructivist instructional or research model depends on how willingly the teacher 1) seeks to imagine how the student might be viewing the problem; 2) hears mathematical notions which differ from her/his own but possess internal consistency; 3) examines his/her own mathematical beliefs and 4) witnesses and describes the student's choice of operation (action) and method of evaluation and recording (reflection).

Assumption 4: Problem solving as enacted in interviews or constructivist instruction is an interactive process. The interviewer selects a task for its potential to invite students to engage with a particular mathematical idea. The task will yield to multiple interpretations and resulting approaches. The interviewer must seek out an understanding of the students' *problematic, choices of actions and means of reflection. The interview setting will itself promote more self-reflection and a stronger approach to knowledge construction. The definition of the problem, of what concepts are related and of what constitutes an appropiate answer will evolve over the course of the interview.*

V. STUDENT ERRORS AND MISCONCEPTIONS

A third interpretation of the first claim, that when students genuinely engage in solving mathematical problems, they proceed in personally reasonable and productive ways, would be to interpret it to refer to reasoned error patterns. Research on the diagnosis of student errors has indicated that overly quick and localized responses by teachers to errors can result in a failure to see how individual mistakes can be linked together and remedied more effectively as classes. Though this research has proven relatively effective in remedying errors on simple arithmetic computations (Van Lehn, 1983) its effectiveness in more complex circumstances has been limited.

A more promising interpretation of the phrase "personally reasonable and sensible ways" links it to the research on student misconceptions. Researchers have documented that students hold mini-theories about scientific and mathematical ideas that the theories and their forms of argument must be understood and directly addressed if students are to come to a more acceptable understanding of the concept. Unlike error patterns, these mini-theories relate formal scientific/mathematical meanings for terms with their everyday usage, examine how the theories relate to historical development and discuss how the theories reflect the child/student's view of science or mathematics as a whole. (For reviews of this literature, see Confrey, 1990; Perkins and Simmons, 1987; Driver and Easley, 1978)

Since many of the researchers in this area have contributed significantly to the constructivist program (Pope, 1985; Driver and Erickson, 1983; Novak and Gowin, 1984) no general distinctions can be offered. However, debate within this community over what to label the student's perspective is an indicator of the variation which exists in people's interpretation of constructivism. Labeling a student's model as a *misconception* fails to take in consideration the perspective of the student, for whom the belief may explain all instances under consideration and fail only in cases to which s/he is not privy. Another optional label has been that of alternative conceptions which stresses the possibility of multiple solutions, but subtly seems to give them a lower status, "alternative" as opposed to "normal" conceptions. Finally, others have chosen more simply conception, which omits any indication that the perspective may deviate considerably from the expert's position. For the constructivist, this difficulty can be resolved by using the term "conception" while always declaring a frame of reference (observer, expert, or participant) and indicating whether it seems adequate from that person's perspective.

VI. A CONSTRUCTIVIST VIEW OF ERRORS

In my interpretation for constructivism, the phrase "personally reasonable and sensible ways" needs to incorporate certain commitments. As we examine students' conceptions and encounter responses which deviate from the traditional presentation, we need to keep in mind that:

1) seldom are students' responses careless or capricious (Ginsburg, 1977). We must seek out their systematic qualities which are typically grounded in the conceptions of the student. If we **assume** this, we will increasingly understand the sensibleness of their approaches from their point of view. Thus, our first efforts must be towards encouraging students to describe their beliefs so that we might explore them.
2) frequently when students' responses deviate from our expectations, they possess the seeds of alternative approaches which can be compelling, historically supported and legitimate if we are willing to challenge our own assumptions.
3) these deviations provide critical moments for researchers to glimpse and then to imagine how students are viewing an idea. The role of the discrepant event in building up conceptual structures is as essential as refutation is in the conduct of science/mathematics. It is essential in an individual's progress in the development of the ideas and also provides an opportunity for us to describe the student's conceptions. When the student proceeds in accord with our own models and expectations - we assume agreement with our models, but as von Glasersfeld (1984) reminds us, conceptual frameworks must *fit* our experience, as any key with the appropriate appendages can trip open a lock. That key need not *match* our own. It is at points of contact, at moments of discrepancy, that we have the highest probability of gaining insight into another person's perspective.

Assumption Five: Students' responses which deviate from our expectations as researchers/teachers can appear to he reasoned and well-considered to the student. They may be entirely legitimate - as an alternative perspective, or be effective for a limited scope of application. We must encourage students to express their beliefs, keeping in mind that deviations provide precious opportunities for us to glimpse the students' perspectives.

VII. AN EXAMPLE: SUZANNE

In order to demonstrate how one learns to listen to a student in a constructivist tradition, I have written up an episode which occurred over two interviews with a college student. The questions listed below summarize the major issues raised in the previous sections and provide a framework for the reader to consider as s/he considers the example:

1. What mathematical ideas are under investigation and how can they be viewed as genetically developmental, functional for making sense of experience and viable in solving problems?
2. What problematic is the student undertaking and does she solve it adequately? From whose perspective?
3. How does she act to solve her problematic? What operations, representations and tools does she use?
4. How does she reflect on her actions in relation to the problem?

TIME-LINE PROBLEM:

The world is very old, and human beings are very young. Significant events in our personal lives are measured in years or less; our lifetimes in decades; our family genealogies in centuries; and all of recorded history in millennia. But we have been preceded by an awesome vista of time, extending for prodigious periods into the past, about which we know little - both because there are no written records and because we have real difficulty in grasping the immensity of the intervals involved.

- Carl Sagan, Dragons of Eden

The problem: Represent the following dates on a number line:

Period or epoch	Years ago	Development of life on earth
now		Development of science and technology
Renaissance	500	Voyages of discovery from Europe and Ming Dynasty, China
	1000	Mayan civilization; Crusades; Sung Dynasty, China
	1800	Zero and decimals invented in Indian arithmetic; Rome falls
	2000	Euclidean geometry; Archimedian physics; Roman Empire; birth of Christ
Holocene epoch	10,000	development of agriculture
	500,000	Domestication of fire by Peking Man
Pleistocene epoch	2×10^6	Modern human beings develop; mammoths and woolly rhinos flourish
Miocene epoch	2.4×10^7	apes, bats, monkeys, whales
Oligocene epoch	3.7×10^7	rodents, cats, dogs, elephants, early horses
Eocene epoch	5.8×10^7	birds, amphibians, small reptiles, fish
Paleocene epoch	6.6×10^7	flowering plants; small mammals
Cretaceous period	1.44×10^8	dinosaurs with horns and armor common
Jurassic period	2.08×10^8	dinosaurs reach their largest size
Triassic period	2.45×10^8	cone-bearing trees plentiful; insects; appearance of turtles, crocodiles, dinosaurs
Devonian period	4.08×10^8	the first forests; many fish and amphibians appear
Ordovician period	5.05×10^8	Trilobites, corals, and shelled animals
Cambrian period	5.7×10^8	fossils plentiful for the first time
Precambrian time	1.1×10^9	coral, jellyfish, and worms
	3.5×10^9	Bacteria
	4.5×10^9	no living things are known
Big Bang	1.5×10^{10}	

Figure 4.

The preceding example is drawn from a teaching interview which I conducted on students' understanding of exponential functions. The student interviewed was a college freshman at Cornell University who was majoring in nutrition. At the time of the interviews, she was taking a remedial algebra/trigonometry course. She volunteered to participate in a study of students' mathematical reasoning and was paid for her participation. She met with the interviewer twice weekly for one-hour sessions over a five week period. The task described took place during the second and third interviews. The task (reproduced in Figure 4) was to represent a series of dates of significant historical events given in scientific notation (for the most part) on a number line. Scientific notation refers to the system of writing numbers as a product of a number between 1 and 10 and a power of ten. The task was designed in order to examine how students understand scientific notation and, in particular, the relations among the exponents as orders of magnitude.

Suzanne was given the problem statement and asked to read the introduction out loud and "to represent the dates on a number line". She was provided a calculator, ruler, pens, pencils, a stack of computer paper and pads of graph paper. She chose to use the graph paper to make her number line and began by stating that she will "divide it up" and go from $1.5 \cdot 10^{10}$, $1.5 \cdot 10^{9}$ When asked what she is looking at to make that decision, she says "Well, um, let's say I put in 1.5 times 10^{9} here, if I was going by thousands of years or whatever it is. Um, I would have to fit in here 3.5 whatever it is.." She proceeds to mark off her intervals, $1.5 \cdot 10^{8}$, $1.5 \cdot 10^{7}$ etc.

This decision immediately meant that these intervals were spaced as in a logarithmic scale: as she moved right to left, each equal-sized interval spanned 10 times as many years as the previous interval. For example the interval from $1.5 \cdot 10^{1}$ to $1.5 \cdot 10^{2}$ covered from 15 to 150 years ago whereas $1.5 \cdot 10^{2}$ to $1.5 \cdot 10^{3}$ covered from 150 to 1500 years ago or 10 times as long (Figure 5).

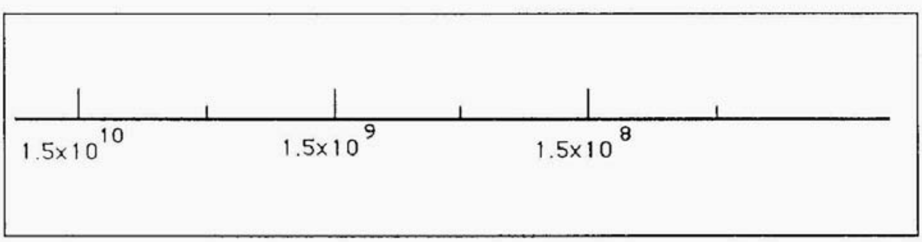

Figure 5.

In order to decide on the spacing in terms of boxes between the intervals, she counted the number of boxes across the graph paper and the number of events which are listed in the chart (19 events) and divided number of events into the number of spaces (47 spaces). She indicated that "..I'm probably going to need a calculator because you mark this at the proper divisions between each one."

From this, she decided to space her labels two squares apart.

In one respect, it seems that Suzanne is muddled. She has not proceeded in the textbook fashion of discerning the range and then choosing a scale, labelling it, and then placing her points. Her method of deciding on the spacing seems flawed and her decision to make the intervals in increments labelled $1.5 \cdot 10^n$ seems to demonstrate a failure to understand the impact of exponents on the size of numbers. It seems unlikely that she will complete the task successsfully.

However, if we struggle to listen to her as a competent problem solver, she has also provided us a rich opportunity to glimpse what constructs she is using in creating such a number line. Her first concern was with fitting the data on her paper. She was aware that “I can use the whole room if I want to, because this is such a long period of time,” and she recalled an exercise in ninth grade earth science where they created a timeline with pictures. However, she decided to limit her task to one piece of graph paper, and worked towards mapping the data within those restrictions.

The interviewer who is charged with hypothesizing a model of her problematic listens carefully to her words to guess at her task - she saw her task as “dividing up” the page (creating intervals) and “fitting in the points” (locating all the points on that page). She stated her own problematic as “getting them all on, from first to last.” It seems that her immediate task was to place the points on the scale. The concern for representing the relative passage of time was never mentioned early in her activities.

Within the constructivist paradigm, one would seek to imagine how these concepts, range, interval labelling and spacing within the interval are understood within the context of her problematic. We see that our “conventional wisdom” on how to construct a number line does not apply. She does not proceed in textbook fashion by: 1) identifying units (the variable in the one-dimensional scale), 2) determining the range, 3) choosing an origin and 4) deciding on an appropriate scale. Her problematic is to get the points on the line, i.e., to find their locations.

To do so, she selected the Big Bang, the first point in the set, and hence ended up with a marker of $1.5 \cdot 10^n$ a choice which she will revise later. She chose her markers to proceed consecutively, but the counters she relied on were exponents; she exhibited no evidence that she recognized this implied a change in interval size as she moved right to left. It felt orderly and it allowed to cover her range of points systematically. From her later actions, it seems reasonable to assert that in the early stages, she was unaware of the discrepancy between her scale and a linear one.

An important insight for the researcher is that it seems that her construction of the number line (on which she later must place her points) is not clearly differentiated from locating the set of numbers (data) she must place. We witness this in her decision to use the Big Bang as her origin, despite its somewhat awkward value ($1.5 \cdot 10^{10}$), rather than to choose a value like $1 \cdot 10^{10}$. Another example is evident in her assumption that the events themselves will be equally spaced, rather than the intervals on the scale. Recall how she decided how many spaces to put between each label of a marker; she counted the number of events, divided 44 by 19, and spaced by twos to "fit them all on."

For the researcher, these deviations from our expectations raise interesting questions about the conception of the number line, since she must create this representation in order to place the points. We explore our own understanding of the interrelationships among scale, range, unit and interval (Goldenberg, 1988). Her initial lack of distinction can draw our attention to ways in which our own mathematical language makes such a distinction difficult to articulate. For instance, we refer to the representation of the number line without points labelled as a number line, yet, possess no language for referring specifically to a number line with the data points marked. In contrast, in Cartesian graphing (a representation of two perpendicular number lines), we ambiguously use the term, "graph" to refer to both the particular curve drawn (the relation) on the coordinate axis (and can put multiple graphs on the same axes) and to the combination of the particular curve and the coordinate axis. We do not refer to a naked coordinate axis as a graph.

As we consider these as two related tasks, we begin to view the problem as the coordination of two kinds of representation - numbers represented in scientific notation and numbers represented as locations of points on a number line. By not differentiating these systems of representation initially, she uses particular points and intervals from her list of events to build her number line and unknowingly breaks some of her tacit assumptions about numbers. Interestingly enough, as the task proceeds, and she most locate a variety of points in relation to her interval labels, she progressively solves local problems and ends up with a practical solution.

Her second attempt has her using two sheets of paper, and she increased her interval size to four boxes. She did so in response to beginning to try to plot the three events in Precambrian time and suspecting that she lacked the necessary spacing. As she began to imagine plotting points, she decided she has made a mistake. She moaned, "Oh, I totally messed this up, because when I counted I didn't take into account that there are four, five, six at 10^8 and so many at 10^7 so that's not going to work .... I'm going to end up with a bunch of extra marks with nothing to put there." She continued to examine the chart and then struggled to create a language to describe her error. She said, "So when I counted the total [number of events] I included them [multiple events with the same exponent] in the total not thinking that they are not necessarily in the total but they're in the **group** .... So, what I have to do is, say, count the exponent as the group and not each individually."

The significance of this event for the constructivist is twofold: 1) she was easily convinced to change her mind on this, because she encountered conflict and acted to resolve it; and 2) her development of a language by,which to convey to the interviewer her insight required significant effort and probably helped to stabilize that insight. In this regard, the interview setting constitutes a different setting for learning than would likely occur if she had been alone. The use of the term group, is an interesting one, for it establishes a way of reducing the size of her problematic of mapping from scientific notation to the number line - each point is a member of a **group** as defined by its exponent, and that exponent corresponds to a single interval on the number line. She now must only determine how to map within the intervals. Thus, her problematic has changed as a result of her own actions.

In the next exchange, Suzanne acted on her insight to determine how many intervals she needed. Since the final seven events are not presented on the table in scientific notation, she transformed them. She proposed to change 500,000 into 5 • 105 counting the zeros but expressed a lack of security with scientific notation. When the interviewer responded by asking, "What is 10^5?", she multiplied it out to get 100,000 and checked it.

Her encounter with the label "now" provided her with another opportunity to differentiate between the representation of the number line and the placement of her particular data. It also drew her attention to her units. She stated, "And I guess I am going to just put that on the graph [the word now]. I can't really. I can put it to the zero. I mean, I guess, Oh, I can just, if I put 19[87]. Wait, this is actually zero, isn't it, because it's years ago? It's right now, it's because its zero years ago, so will go, um, I'll have to put something in here, as 101 just so the graphs will stay even ... it will be 1.5 times 10^1 so that, no, that may not be important as far as the data, but I need it to keep the intervals in my graph." This is a significant step for her for she makes the claim that the number line must meet a certain criterion, completeness, that may not be directly required by her data.

Notice that the placement of "now" cannot be represented using her geometric progression of exponents since 1.5 • 100 = 1.5 • 1 = 1.5 and were she to continue her pattern of intervals, $1.5 \cdot 10^{-1}$, $1.5 \cdot 10^{-2}$ $1.5 \cdot 10^{-3}$ which yield .15, .015, .0015. No extension of such a sequence reaches the value of zero. Because she is still primarily concerned with placing points, she sees her task as finding a place to put "now". She puts it on the point one, calling that "now".

This example illustrates how a student can simply dismiss an anomaly (and fail to "see" it), leaving intact the operating conception. The interviewer asked her if zero can be written in the same form as the other numbers, challenging her to see that it provides a challenge to her framework. She responded, "Um, I don't know. As 1 . 10^0 or $10^0$is 1, so it would just be one. I guess I could call it [now] one if I want to . . . since it's my graph." On hindsight, it is unfortunate that the interviewer did not pursue this either immediately or later.

Suzanne now set about creating the third version of her number line. She counted to see she needed 11 intervals and decided to try a spacing of 10 boxes per interval. When that came out to be too large for her graph paper, she decided to use 8 squares. She drew this fourth version and labelled the intervals, $1.5 \cdot 10^{10}$, $1.5 \cdot 10^9$, $1.5 \cdot 10^8$ ... and wrote in zero for her last entry. She then changed her mind again as she faced the task of locating points and said "Actually, I shouldn't have used eight, because its hard to break that up. Probably 5 or 10. If I stayed at 10, it would have been a lot easier." Her choice of using ten may have signaled either an awareness that she was facing the task of placing decimals or that she was accustomed to seeing ten spaces in previous work.

She decided to continue to work with eight squares and encountered her next obstacle. The final resolution of this obstacle provided her with a consistent, albeit alternative, representation for this problem. She tried to decide how to divide the interval up. Her first suggestion was to divide the interval next to $1.5 \cdot 10^{10}$ into three parts and label tentatively from left to right the first section $1.0 \cdot 10^9$ and the next $.5x\ 10^9$ (Figure 6).

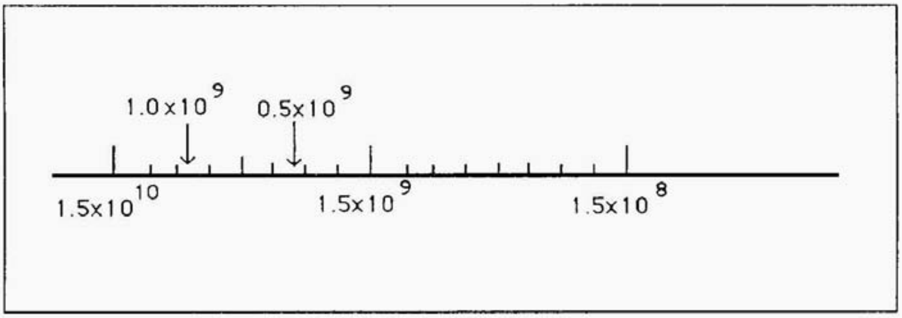

Figure 6.

She quickly stated her confusion. After a short pause, she concluded, "I am going to make this one [the marker] $1 \cdot 10^{11}$ and this, $1 \cdot 10^{10}$ so I can use "1" and have some place to put this [$1.5 \cdot 10^{10}$]. So, I can put $1.5 \cdot 10^{10}$ here." She pointed to the middle of the interval (Figure 7).

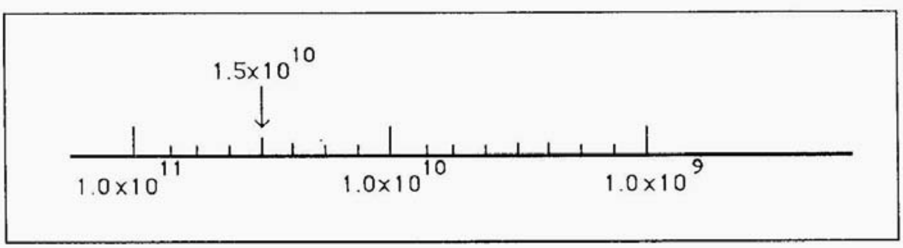

Figure 7.

This change in labelling the intervals can be viewed as another step in differentiating the formal representation of the number line from the particular data. It also provides an opportunity to glimpse some of her assumptions which are in operation. She assumes that .5 must represent a "middle" point, perhaps because it is so frequently the middle in number lines of consecutive integers; perhaps because she is attending to the consecutive values of the exponents. If she had considered her intervals as 10^n and 10^{n+1} (rather than as $1 . 10^n$ and $1 \cdot 10^{n+1}$ she might have claimed that the middle is $10^{n+.5}$, a claim which might have led towards a logarithmic scale. However, given her revised problematic, locating points within the intervals, she is most likely to be attending exclusively to the decimal number (n.nn) rather than the exponential (10^n).

Because she remained dissatisfied with having eight rather than ten squares, she redrew her number line with the interviewer's assistance. She then had the number line shown in Figure 8. It started with $1 \cdot 10^{11}$ and proceeded downward by powers of ten till $1 \cdot 10^1$ next to which was placed a final interval with a marker labelled NOW. She chastised herself stating, "My biggest problem is I don't think things through enough before I start."

Her first inclination was to split each interval in half, which later she explained was to label it 1.5, but she caught herself and said, "no, I'm not going to do that. I am going to ... since our 10 ...". She counted off the spaces between her markers, starting from her right marker at 1 and moving to the left counting 2, 3, 4, 5, 6, 7, 8, 9, 10 .... This caused her to reach 10 on the ninth square and to recalculate that value and say, "no, if I do 10 times 10 is . . ."

Suzanne was experiencing the impact of a variety of competing frameworks. Her markers read $1 \cdot 10^8$ and $1 \cdot 10^7$ They seemed well-ordered to her, but she needed to figure out how to divide up the interval. Her first framework was to hypothesize that $1.5 \cdot 10^7$ would be halfway in between - a theory based on the consecutive values of the exponents. Her other framework guided her to rely on her choice of ten spaces and its resemblance to the decimal system and it led her to count off the values. This method, however, produced the surprise result that she ended up with an extra space since her count begins at one.

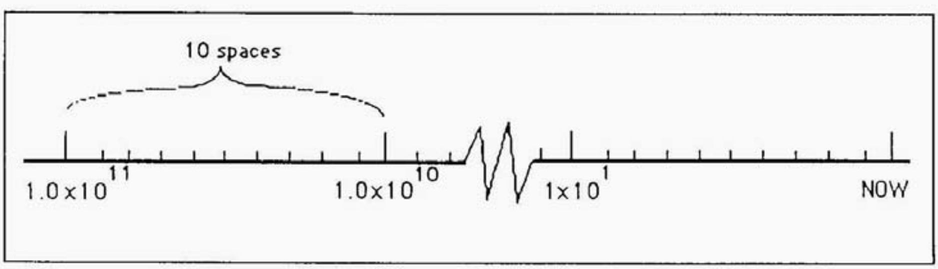

Figure 8.

She admits "Um, I am confused now as to what I'm doing. Um.." and then chose to write out the numbers in standard form below the scientific notation. On the interviewer's request, she restated her problem, this time using the standard form of the numbers. "I don't know how to go from 100 million to 10 million. It's just 10, 10 million times 10 is 100,000. The scientific notation is messing me up. This would be 10 million and 500 thousand, right? [She pointed halfway in-between 100 million and 10 million.] It would be halfway in-between, right? So it would be 1.5 right here?"

The interviewer repeated her claim asking if 10,500,000 is halfway between 10 million and 100 million and she quickly responded that it's halfway between 10 million and 11 million. Although she was confused, by writing the numbers in standard form, she has provided herself a third representation (in contrast with her segmented number line and her scientific notation) and she used this one to resolve her dilemma (Figure 9). She said, "All right, then I have too many spaces ... since this is 10 million and this is 100 million, I have to go by 10 millions. So, it would be - if this is 10 million - 20 million, 30 million, 40 million, 50 million, 60 million, 70 million, 80 million, 90 million. Then I'd have an extra box." Then she switched back to scientific notation and said, "then it would be $2 \cdot 10^7$, $3 \cdot 10^7$, $4 \cdot 10^7$ - like that. On interviewer's request, she confirmed her method on the interval from 1 million to 10 million counting off by millions. As the interview ended, she expresses her need to redraw the number line and the interviewer agrees to provide one with nine divisions per interval for the next session.

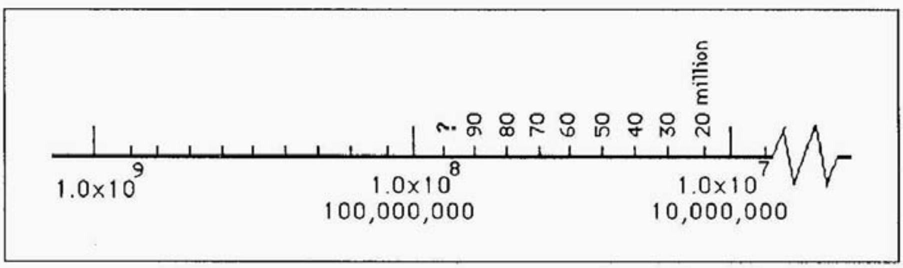

Figure 9.

To this point, Suzanne has created a number line which resolves all her outstanding dilemmas. She has created a number line which involves two mapping procedures - the first maps the events within a group by their decimal numbers and to place them on a scale ranging from 1.0 to 9.9 ... within that interval. In more formal terms, her proposal creates a complete mapping from the set of all real numbers greater than zero to the number line representation. It is a combination of a logarithmic and a linear scale.

The interviewer began the next session by asking her to review what she had done to which she replied: "I tried to make a number line with the numbers given here [in the chart], and I put them in scientific notation, just to make it easier ... I tried to make even divisions so I could divide them up, I did them by tenths. I started 10^{11} and 10^{10} 10^9, and I put in finally, once I figured them out, put the proper numbers and space in between so that each could be labelled evenly, like $1 \,.\, 10^{11}$, $1 - 10^{10}$, down like that. From there it would be easier to divide them up like into the decimal point and things like that."

Curious to see if the solution she has proposed will satisfy the overall expectations Suzanne has for the task (recognizing that it does solve her local problematics), the interviewer asked Suzanne what would constitute a successful completion of her task. She identified three criteria: 1) to be able to locate the points on the chart; 2) to be able to look at the number line and "clearly see what was being, was trying to be said, or trying to be explained by things in the number line"; and 3) to see the years passing, "the difference between them. When you look at the line, you can see clearly that this is obviously closer. This one point is closer to this point than this one is. So there is a longer period of time between here and here."

Her answer is striking, because her number line only satisfied her first criterion completely. To this point, we see no reference by her to the context of the problem, the passage of time; and yet, when asked she states this as a criterion. Later in the interview, she learns to "see" on her number line and to calculate the years passing - but the placement of numbers remained of primary concern.

She then located the events listed on the number line with little difficulty. Her representation allowed her to map the dates for the events listed in an algorithmic fashion. The interviewer began to raise questions to cause her to consider the impact of a changing unit size across the intervals. The question posed is "did the dinosaurs live for a longer period than humans, or have humans lived longer?" They decided to assume that the dinosaurs died out during the Triassic period. Thus it turns out they lasted approximately 100,000,000 years and people, having entered the timeline at 2,000,000 years ago, have survived a much shorter length of time. On her number line, by distance considerations alone, it appeared that humans have outlived the dinosaurs (Figure 10).

She answered, "They began a longer time ago, but they lasted a shorter period of time". This is the response one would expect her to have given, based on the appearance of her number line.

The interviewer asked to be convinced, and she responded at first by saying, "Each exponent is a million years. I guess that's what we were doing. Is um, what did we say. I have to think about this for a second." She wrote out the standard notation for all of her intervals. Then she said, "OK, each of these, each of the divisions signifies a multiplication by 10 um? each of the large division. This is 10 years multiplied by 10. 10 million multiplied by ten is 100 million. With considerable effort, she pondered the length of the interval for the dinosaurs which we had marked as from $2.45 \bullet 10^8$ to $1.44 \bullet 10^8$. She quickly said it's one and with a long pause added, "I guess it would be one . . . hundred million." To work this problem, she stayed within her number line representation, switching from scientific notation to expanded decimal form. In then deciding how long humans had survived, she also used her number line and began to add intervals. She added the first interval correctly (from $2 \cdot 10^6$ to her marker, $1 \cdot 10^6$) as 1 million but then called the next interval 100,000 (rather than 900,000) and then simply concluded that dinosaurs were around longer.

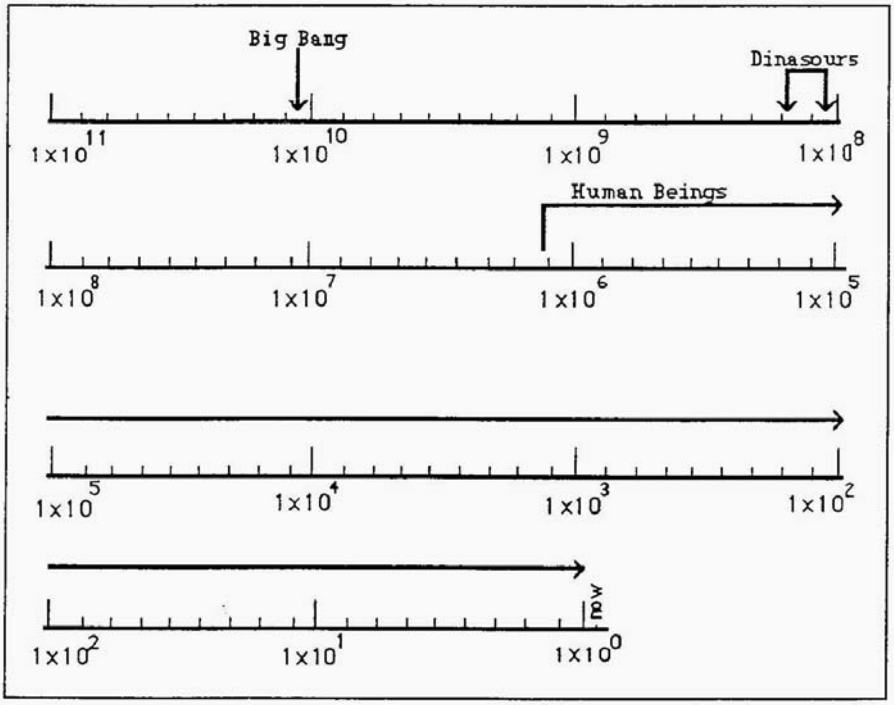

Figure 10.

By asking this question, the interviewer hopes to stimulate Suzanne's thinking about interval size. Whereas before we have heard Suzanne express her awareness that the value of numbers between intervals increase ten times, "each of these, each of the divisions signifies a multiplication by 10" we saw no evidence that she was interpreting this in relation to the passage of time as represented by the length of the line. By working across her forms of representation, the length of the line, subtraction of scientific notation and subtraction of expanded format, she gains some insight into these connections. It is at this point that she begin to apply her last two criteria to her number line, to "clearly see what was being, was trying to be said" and to see the years passing. Once again, when presented with a problem, she acts to solve it in an appropriate way.

Her decision to determine how long humans had been around by adding her intervals, 1 million + 900,000 + 90,000 + 9,000 + . . . (which would be an infinite process in her representation if she had arbitrarily located one as now) rather than by reading off the number 2 million provides an interesting insight into her understanding of the meaning of the labels on her intervals. It might have also led to an interesting inquiry into her awareness of the differences in her number line from the traditional linear one. However, it was not pursued.

Suzanne seemed to be reasonably capable of using her graph and seemed to understand what changes occurred in the value of the boxes. She summarized by saying, “Each box is worth a lot more over here than it is over here; you multiply by ten.” To pose a final test, the interviewer asked her how long it took for a human being to discover fire, a calculation which required her to calculate between two intervals. She responded, “Oh, um, 500,000 right here [counting the units in the smaller interval]. So it’s 500,000 thousand right here. And then right here is a million so it will be 1,500,000.” She used her representation quite successfully by breaking the problem into two pieces. She calculates each interval piece separately using its correct unit and adds the total.

It appears that Suzanne has created a logically consistent solution to the problem. Once she learns to use it, it satisfies all her criteria and allows her to complete the task. The question for discussion is how legitimate is her solution?

VIII. DISCUSSION

The episode described is not unusual. Nearly every extended teaching experiment yields inventive solutions such as the one that Suzanne has demonstrated. Regardless of the extent to which one finds her solution mathematically acceptable, it seems reasonable to point out that she has certainly engaged vigorously with the problem and demonstrated the ability to create sensible solutions to each obstacle she encountered.

It seems also reasonable to conclude that Suzanne seemed relatively effective in her use of her representation. She recognized the changes in the intervals and could adjust to account for those changes. Furthermore, it seems fair to claim that she does solve most of her problematic. From her perspective, what she has created is a perfectly adequate solution.

This is not to suggest that no further work might be done with Suzanne on these types of problems. She does not appear to have engaged deeply with the implications of the absence of a point for zero - an implication which might be important for contextualized problems. The tension between seeing the label as a magnitude from zero (as in the two million years of humanity) and as a sum of an infinite geometric sequence is unresolved. And, in future work with all negative exponents in a comparison of small-sized objects, we witness an entirely different approach. Finally, she does express her own uneasiness with having different sized units and her wish that she could have found a solution which did not demand that “concession”.

Secondly, her solution path provides us with interesting and useful information about the psychogenesis of a number line. It suggests that in coordinating two representations, scientific notation and a number line, we witness a process of progressive differentiation. She does not treat the scaling of the number line as her prior task and the placement of the points as its successor. The two acts

process despite its inefficiency from the expert's perspective is that she begins to create a functional need for ideas like origin, range, units and scale. With one experience, it is doubtful she could articulate these constructs in any sophisticated way, but it might turn out that she could use these ideas successfully in future tasks similar to this one.

The question which remains is whether to consider the work she has done is legitimate mathematics. Many would deny its acceptability, because it is a hybrid between logarithmic and linear scales. Some would argue that it does not show the passage of time comprehensibly. However, if this objection were taken as valid, one would have to reject the logarithmic scale as well. Furthermore, if she were to create a fully linear scale with one inch equal to 10,000 years, such a representation would be fully fifteen miles long.

Others would deny her solution legitimacy by claiming that her solution is a distortion of a logarithmic scale and therefore wrong. If one were to develop a purely logarithmic scale given her method, one would change all the numbers into a form of 10n.nn with no decimal number up front and then place them on a scale based on the value of the exponents. Thus, $10^{.5}$ would be halfway between 100 and 101 and thus the midpoint between 1 and 10 could be 10 or approximately 3.16 rather than 5.5 on her nine point scale (Figure 11).

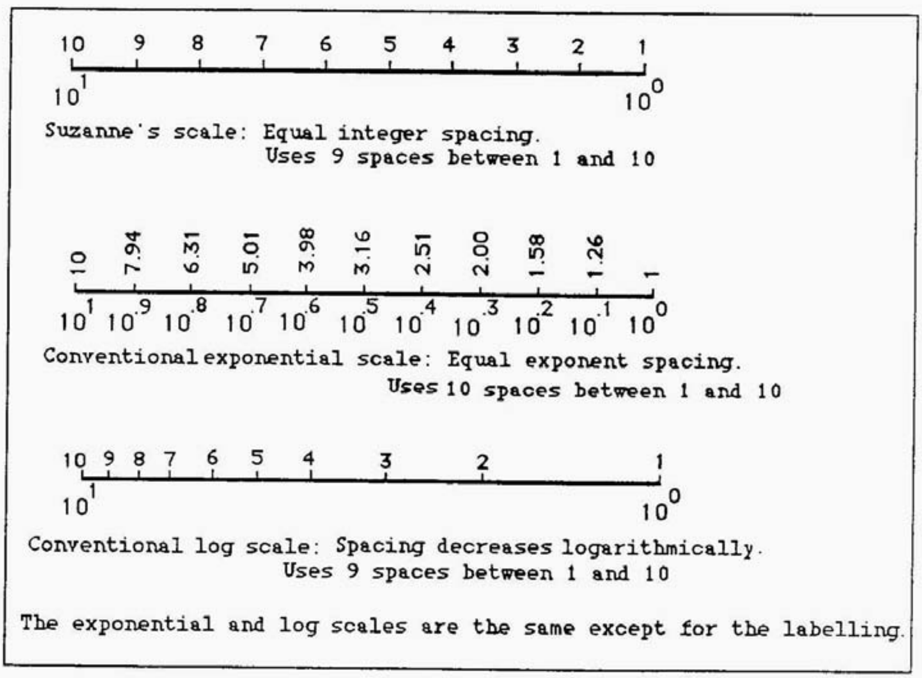

Figure 11.

Her scale is a distortion of the logarithmic scale, but its rejection as legitimate mathematics does not follow. Just as one would not agree to reject the logarithmic scale because it is a distortion of a linear scale. Perhaps we would be better off to refer to the changes in scale as **transformations** and examine their advantages and disadvantages. For the constructivist, however, no rejection of the legitimacy of her method can be made simply on the basis that it does not mirror the conventionally accepted solution. This may be a disadvantage in communicating her solution to others, but it does not undermine the valdity of her solution.

I would claim that Suzanne's solution is: 1) mathematically valid within a specifiable system of mathematics, 2) preferable to a logarithmic scale for displaying data in scientific notation, and 3) reveals a fundamental tension between multiplicative structures and counting structures which is built into our systems of numeration in fundamental ways, and hence her solution provides us as researchers with an opportunity to reconsider our own understanding of numbers.

1. *Mathematical validity.* Suzanne has created a system with a one-to-one mapping between the set of all real numbers greater than one and the number line. On her mapping, she identifies one as her origin (and decides to call that now - a relatively small quirk when we are spanning 10^{10} years.) She develops a set of algorithms for working, identifying the units in any given interal as 10^{n} and for adding and subtracting within an interval. Her algorithm for crossing intervals is to split it into two parts and treat each one separately. Such a system lacks a constant rate of change either additively (as in linear scales) or multiplicatively (as in logarithmic scales). It does possess a predictable rate of change, which is a combination of the two.

2. *Betterfor Scientific Notation.* If one wishes to claim that Suzanne's scale is distorted (in a negative sense), one would have to ask distorted from what? For the constructivist, there is no "real" number system, preferable to all others. Number systems are created to allow us to accomplish certain things, to compare, represent, measure, order etc. Fractional form is easier when dividing up quantities; decimals allow easy comparisons of size. Thus, if her number system is a distortion of a logarithmic scale, then scientific notation would be also a distorted system. Few would make such a claim.

In fact, I would claim that her system of representing those numbers on a number line creates the best imaginable fit with scientific notation. As with scientific notation, she is able to represent a huge span of time in an efficient manner. Her two part mapping allows her first to identify the interval and then to locate the decimal number. Scientific notation is designed to do exactly the same thing. By forcing the decimal portion into a standard form between one and ten, the scientist can compare the orders of magnitude. S/He can quickly reduce the problem to one of working with integral values, our counting numbers and our most familiar system.

There is a functional elegance to scientific notation. It is a transformation of a logarithmic scale, but a useful one. Suzanne has simply created a number line with the same advantages and disadvantages. In order to add and subtract, you must have like exponents in scientific notation. In Suzanne's system, you must break the problem into like units. By creating her system, Suzanne has given us an opportunity to consider thoughtfully the assumptions behind scientific notation and behind the construction of the number line.

If mathematics is viewed a functional, the emphasis is not with mirroring some unknowable reality, but in solving problems in ways that are increasingly useful in one's experience. In mathematics that means one must weave together multiple representations in order to solve interesting problems. Effective uses of multiple representations possess two characteristics: 1) they display enough variance across representation to allow insight, and 2) they allow enough convergence among the representations to give one confidence in one's solution. Suzanne's coordination of the representations of number lines, tabular data, scientific notation and expanded form illustrate these two characteristics quite nicely.

IX. MULTIPLICATIVE AND COUNTING STRUCTURES

Suzanne's treatment of this problem provides insight into a fundamental tension in mathematics: the relationship between multiplicative and counting (additive) structures. Confrey (1988, 1990) has argued for an existence of two relatively independent primitive structures in young children: counting and splitting. She has suggested that when these primitive structures are assimilated to each other (when we define multiplication prematurely as repeated addition) and can cause certain conceptual obstacles (powerful ideas which are difficult to learn) emerge. Place value is an example of a conceptual site in which students must coordinate their use of *values* ranging from 0, 2, ..., 9, and *placement* into powers of ten which are multiplicatively increasing.

Suzanne's solution and the insight which it gives into the structure of scientific notation provides another illustration of the uneasy but useful coordination of multiplicative structures (her intervals) and counting structures (within intervals).

X. CONCLUSIONS

The paper began with the statement of two assumptions concerning the application of constructivist theory to mathematics education. It made the claim that these two assumptions needed to be interpreted in a particular way in order to be compatible with constructivist epistemology. These included the claims that the mathematical content must be viewed as a human creation designed to serve human purposes; that the mathematics created inspired confidence because of its interwoven representations and that in any circumstance one should seek multiple mathematical solutions, diversity rather than uniformity.

In the second section, I discussed the idea that within the constructivist paradigm, problems must undergo individual interpretation and labelled this the problematic. I sought to argue that to understand a student's actions, one must seek to model their problematic and not presume it is identical to one's own. I further argued that apparent errors, discrepancies from the observer's expectations, provide particularly useful opportunities for one to imagine what a student's problematic might be like. Finally, I argued that in examining students' problems and methods of solutions, one has an opportunity to reconsider the mathematics involved.

Suzanne's solution to the number line problem provided such an example. She created and solved a series of problematics each of which was reasonable though somewhat atypical. Her solution was unconventional but inventive and original. She demonstrated adequate insight into it. For the most part, deviations which appeared erroneous from our position of expertise turned out to be surmountable conceptual obstacles for her. When given an opportunity to engage in her own cycle of problem solving, she demonstrated considerable progress, insight and experienced, by her own report, a sense of satisfaction.

NOTE

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