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ORIGINAL ARTICLE

Seeing chance: perceptual reasoning as an epistemic resource for grounding compound event spaces

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Abstract The mathematics subject matter of probability is notoriously challenging, and in particular the content of random compound events. When students analyze experiments, they often omit to discern variations as distinct events, e.g., HT and TH in the case of flipping a pair of coins, and thus infer erroneous predictions. Educators have addressed this conceptual difficulty by engaging students in actual experiments whose outcomes contradict the erroneous predictions. Yet whereas empirical activities per se are crucial for any probability design, because they introduce the pivotal contents of randomness, variance, sample size, and relations among them, empirical activities may not be the unique or best means for students to accept the logic of combinatorial analysis. Instead, learners may avail of their own pre-analytic perceptual judgments of the random generator itself so as to arrive at predictions that agree rather than conflict with mathematical analysis. I support this view first by detailing its philosophical, theoretical, and didactical foundations and then by presenting empirical findings from a design-based research project. Twenty-eight students aged 9–11 participated in tutorial, task-based clinical interviews that utilized an innovative random generator. Their predictions were mathematically correct even though initially they did not discern variations. Students were then led to recognize the formal event space as a semiotic means of objectifying these presymbolic notions. I elaborate on the thesis via micro-ethnographic analysis of key episodes from a paradigmatic case study.

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1 Introduction

The road to modeling a probability experiment is paved with good intuitions. Yet these intuitions often lead the traveler to inferences that conflict with mathematical theory and empiricism, so that the traveler must abandon all intuition and trust the well-traveled road. This essay is about the road not taken. Traveling this other road, learners may be able to sustain their intuitions as a means of grounding mathematical analysis.

Underlying the well-traveled road to compound events is a common cognitive–pedagogical assumption that learners will tend to modify their erroneous theory in the face of empirical evidence that contradicts their inferences. The didactical utility of this approach notwithstanding, the objective of this paper is to bring into question its alleged status as the sole pedagogical road to combinatorial analysis. Perhaps learners could instead ground the logic of combinatorial analysis in naïve perceptual reasoning?

I will attempt to support my thesis by discussing a design-based research project that investigated the cognition and instruction of probability. Previous publications argued for the didactical potential of the design used in that project and explained the cognitive–semiotic mechanisms by which that potential was realized by a tutor and students via discursive interaction (Abrahamson, 2009a, 2009b, 2012a, 2012b; Abrahamson, Gutiérrez, & Baddorf, 2012). In this essentially theoretical paper, I assume the role of a reflective practitioner, and in particular a "designer [who] reflects-in-action on the construction of the problem, the strategies of action, or the model of the phenomena, *which have been implicit in his moves*" (Schön, 1983, p. 79, my italics). More accurately, I step back "*post* intentionally" to deconstruct and reconstruct the project's "fleeting intentional meanings" (Vagle, 2010, p. 405). In particular,

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I aim to elucidate the cognitive–pedagogical position underlying the project's design rationale and implicit to "the discursive construction of mathematical thinking in the research process" (Barwell, 2009, p. 255). As such, the epistemological tenor of this essay is not an intact empirical inquiry per se but rather a retrospective reflection on what may be the most broadly generalizable theoretical as well as practical conclusion from a decade of intensive studies that have been summarized in peer-reviewed articles. Therefore, the intellectual scope of this essay will be disproportionally far-reaching as compared to its empirical illustration, whose rhetorical function will be primarily to contextualize the reflection. To the extent that my assertions constitute a contribution to the field, they do as much by inviting the reader to reflect on the epistemological premises of their own work.

To remove any misunderstanding, let me emphasize that I am by no means proposing to abolish empirical activities from probability instruction. That would be absurd, because actual experiments are essential for creating meaningful contexts in which students experience randomness, variance, and sample size as these relate to Classicist probability. Rather, I propose to interrogate implicit epistemological assumptions regarding the pedagogical role of experimentation in the case of introducing students to combinatorial analysis. The significance of this interrogation is in the dire consequences of ignoring the epistemic capacity of tacit perceptual reasoning. As long as we persist in assuming the inherent frailty of primary probabilistic intuition, I caution, we will continue to design and research activities that only further confirm this assumption.

I am thus proposing to consider perceptual reasoning as an alternative or complementary epistemic resource for students to understand compound events. Immediately below, I will discuss theoretical foundations for this central idea of an "epistemic resource". I will then explain the particular instructional design that was developed and empirically researched in the project. Next, I will overview findings from the implementation of that design with students in Grades 4–6 (9–11 years old) and then focus on vignettes from a case study of one student. The paper concludes with an evaluation of the central thesis and its implications for probability education.

2 Theoretical orientation

The construct of an epistemic resource is central to my thesis on pedagogical design for probability. Below, I briefly elaborate on this construct from the perspectives of educational philosophy, theory, and practice. Once I have defined the construct of an epistemic resource, I will exemplify its theoretical utility for analyzing mathematics designs. In particular, the construct will enable me to revisit the role of actual experiments in the instruction of combinatorial analysis and explore whether perceptual reasoning might play a compatible or even preferred epistemic role. First, though, I lay out the specific educational terrain wherein this argument evolved.

2.1 An enduring design problem and a critique of prevalent solutions

The construct of an epistemic resource emerged from my attempts to design a compound-event random generator that responds to empirical findings in the mathematics education research literature (Abrahamson, 2009a). When analyzing compound-event random generators, students typically do not appreciate the relevance of order among singleton events (Batanero, Navarro-Pelayo, & Godino, 1997). For example, students who are guided to list all the possible outcomes of flipping a pair of coins, or are presented with the complete list, do not appreciate why HT and TH should be differentiated. Consequently, the students perceive the experiment's event space to be [HH, HT, TT] rather than [HH, HT, TH, TT] and thus infer that a head-and-tail event will occur 1/3 of the time, whereas the event should be expected to occur 1/2 of the time (Abrahamson & Wilensky, 2005).

A plausible pedagogical response is to create for students situations in which empirical experimentation conflicts with predictions based on erroneous combinatorial analysis (Pratt, 2000; Wilensky, 1995). Upon acknowledging conflict between predicted and actual experimental outcomes, students are expected to respond rationally by reflecting on their theory and adjusting it until it fits the ineluctable facticity of empiricism (see Karmiloff-Smith, 1988; von Glasersfeld, 1987). Indeed, the human capacity to build and refine theoretical models that generate deductive inferences in line with unanticipated empirical evidence is a hallmark of inquiry and innovation (Koschmann, Kuuti, & Hickman, 1998; Shank, 1998). Creating for students opportunities to experience and overcome confusion may thus bear the supplementary benefit of fostering important meta-cognitive skill. Notwithstanding, perhaps educators should reserve those learning experiences for concepts that are beyond intuitive grasp. Outcome distributions of compound random events, I submit, are within intuitive grasp, given appropriate design. Let me elaborate.

Implicit to the design rationale of pitting students against their own erroneous expectation so that they adopt mathematical analysis is the assumption that complex stochastic phenomena are inherently counterintuitive (Kahneman, Slovic, & Tversky, 1982), so that students cannot readily develop what Fischbein (1975) calls "secondary intuition" for these concepts. And yet, as a pedagogical design-based researcher holding a constructivist epistemological worldview, my *modus operandi* is to hesitate before announcing that a mathematical concept is intrinsically unfathomable (Núñez, Edwards, & Matos, 1999). As such, I am committed to investigating the roots of students' intuitive reasoning and nurturing those roots toward the targeted didactical content rather than deracinating and supplanting intuitive roots with formal procedures that consequently become subjectively arbitrary (Borovcnik & Bentz, 1991; Bruner, Oliver, & Greenfield, 1966; Gigerenzer, 1998; Smith, diSessa, & Roschelle, 1993; Streefland, 1984; Wilensky, 1997). Specifically, this radical-constructivist worldview assumes that if we dig deep enough, we will find intuitive roots whose situated enactment resonates with inferences drawn from mathematical analysis of compound random events. If we found such roots and nurtured them, then students' informal inference from "primitive" perceptual reasoning would constitute an epistemic resource—alternative to experimental empiricism—for grounding compound event spaces and formal combinatorial analysis.

As an educational designer, I surmise that eliciting from students mathematically viable intuitions that *are* aligned with compound event spaces may require the construction of suitable interaction contexts, and in particular an innovative random generator that affords this intuitive reasoning. As a learning scientist, I am conscious that intuitive roots may not grow directly into formal structures, so that fostering an understanding of combinatorial event spaces may involve grafting, mapping, or molding cultural forms upon intuitive roots. Just how this negotiation might be accomplished is of great interest and concern for design-based researchers of mathematical learning.

Having explained the content terrain and philosophical orientation of my thesis, I can now elaborate on its central construct, an epistemic resource.

2.2 Epistemic resources, pedagogical artifacts, and guided learning

When I evaluate the epistemic role of a pedagogical activity, I consider learners' subjective sensations and inferences arising from that activity and I refer to the potential capacity of this cognitive content to facilitate learners' appropriation of relevant formal analysis. For example, consider the notions that empirical experimentation evokes. Once empirical results are obtained, replicated, and validated, the rational response is to view these results as bearing on the truth-value of prior hypotheses pertaining to anticipated results. As such, any erroneous beliefs that had been held as true are re-tagged as tentative, imprecise, or incorrect—their epistemic status is altered in accord with the activity's results. On the other hand, learning activities may result with a student bolstering their beliefs. In both cases, the activities played an epistemic role by either supporting or refuting an existing notion.

In this paper I am interested specifically in the conceptualization and engineering of epistemic resources for grounding formal mathematical analysis inherent to the structure of a compound event space. I apply the proposed construct of an epistemic resource as a design–researcher's mid-level analytic lens, that is, a heuristic means of building and evaluating activities vis-à-vis grand learning theory (Ruthven, Laborde, Leach, & Tiberghien, 2009). More broadly, by developing the construct of an epistemic resource, I wish to stir discussion among mathematics-education theoreticians and designers over the relative merits of learning activities that either conflict or cohere with presymbolic notions and inferences.

Students' notions and inferences that constitute prospective epistemic resources for learning mathematical content need not be couched in formal semiotic register. For example, these subjectively veridical notions might be visual/imagistic (Arnheim, 1969; Barwise & Etchemendy, 1991), auditory (Bautista & Roth, 2012), kinesthetic (Nemirovsky, 2011), somatic (Damasio, 2000), or diagnostic (Braude, 2012). As such, I treat pre-articulated, informal, embodied notions as epistemically *bona fide* cognitive content bearing directly on inferential reasoning. Theoretical antecedents of this epistemological perspective can be found in cultural–historical psychology (Vygotsky, 1962), cognitive linguistics (Lakoff & Johnson, 1980), philosophy (Polanyi, 1967), and cognitive science (Barsalou, 1999). Also well established is the pedagogical sequence of first evoking learners' presymbolic notions and then fostering these notions as personal resources for learning formal cultural procedures (Diénès, 1971; Freudenthal, 1983; Froebel, 2005; Montessori, 1967; Skemp, 1983). In particular, developmental psychologists have demonstrated pre-articulated notions bearing quantitative information relevant to understanding mathematical ideas (see Carey, 2011; Dehaene, 1997). For example, some *a/b* mathematical objects are perceptually privileged (Gelman, 1998): we experience the slope of a line / as a gestalt intensity, not as piecemeal rise | over run _.

However, mathematical ideas are difficult to come by without guidance, so that learners generally need to participate in social activities by which knowledge is mediated (Newman, Griffin, & Cole, 1989). Often these pedagogical activities involve engaging a problem as well as available media proposed by an instructor as means of exploring and pursuing solutions to the problem. In so doing, participating students are encouraged to construct and interact with artifacts bearing semiotic potentiality of constructing pedagogically targeted cultural forms; instructors steer students to adopt these cultural forms via nuanced discursive orientating (Mariotti, 2009) and by highlighting the forms' contextual utility for students' situated purpose (Pratt & Noss, 2010; Saxe, 2004). Generally speaking, education practitioners seek to create conditions under which children will endorse mathematical analyses of phenomena under inquiry as a means of satisfying an ecologically authentic need, be it enactive (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011), epistemic (Harel, 2012), or meta-discursive (Sfard, 2007).

We are left with a paradox. On the one hand, we assume the epistemological position that presymbolic notions and mathematical analyses are disparate forms of knowledge (Piaget & Inhelder, 1969, pp. 46–48). On the other hand, we assume the pedagogical position that presymbolic notions are necessary for grounding formal knowledge (Bruner et al., 1966). And yet, formal analysis cannot always be direct articulation of tacit judgment, because formal analysis often requires one to parse source phenomena in ways that are very different from naïve perception (Bamberger & diSessa, 2003). How, then, might we theorize the "dialogue" of tacit and formal knowledge? What is the nature of the "membrane" between these epistemically disparate forms of knowledge by which grounding occurs?

2.3 Radford's semiotic–cultural theory as informing a design approach

The role of presymbolic notions in mathematical learning is elegantly modeled via the semiotic–cultural approach put forth by Luis Radford (2000, 2003). In Radford's neo-Vygotskian theoretical perspective, learners appropriate a mathematical form if it occurs to them ad hoc in some situated context as serving a cognitive–discursive function. Specifically, students engaged in problem-solving activities will use a new form, such as a symbol, a diagram, or an icon, if they are able to construe the form as a semiotic means affording the objectification of a presymbolic notion they wish to express—a thought they are trying to articulate, capture, grasp, anchor.

The semiotic–cultural theory, I maintain, can be interpreted as bearing implications for pedagogical design (Abrahamson, 2009a). In the remainder of this section I briefly overview a general design framework derived from this theory, "perception-based design" (Abrahamson, 2012a). In later sections I will exemplify the framework via presenting the probability design used in this project.

Radford's theory informs the creation of materials and activities for supporting the construction of new mathematical signs as coordinations of spontaneous and scientific resources. The rationale of this pedagogical design framework is that educators can and should take measures to facilitate students' objectification of relevant presymbolic notions in mathematical forms, and the charge of design-based researchers is to identify and elicit these notions as well as to create "bridging tools"—appropriate versions of standard forms that have been tailored to resonate with these notions.

As such, in order to foster student construction of a particular sign, two substantive pedagogical resources are necessary: (a) a source phenomenon that evokes presymbolic qualitative notions and inferences in agreement with formal quantitative analysis of the phenomenon; and (b) a mathematical model of the phenomenon, where the model is a customized isomorph of its standard configuration that renders it conducive to relevant perceptual reasoning. Yet how might we approach the engineering of these resources? We return to cognitive science.

Humans have the cognitive capacity of sensing holistically certain magnitudes that science models as relations between two magnitudes. For example, object constancy is established by the brain tacitly comparing between retinal images of shapes on the basis of their aspect ratio (i.e., height/width, Suzuki & Cavanagh, 1998). This affords education researchers auspicious epistemological tension between percept and concept. We can investigate whether and how tacit knowledge might afford epistemic resources for conceptual development and how social interaction might facilitate this process.

Accordingly, the targeted mathematical notion in designs deriving from the perception-based framework is typically an *a/b* intensive quantity, such as slope, density, or velocity, and this quantity is embodied in the source phenomenon as the property of a perceptual stimulus. Students' attention is oriented toward this specific property by engaging them in a presymbolic problem-solving activity in which determining a qualitative sense of this property bears utility for accomplishing an objective. Next, the child is asked to consider a mathematical resource that is introduced into the problem space, for example a measuring instrument, as bearing further utility for refining notions evoked by the phenomenon. The child is guided to apply this mathematical resource and, through engaging in this process, to create a product, such as a set of numerical values, a diagram, or an event space. During this process, or upon completing it, the child is to "discover" a way of seeing the mathematical product as a means of objectifying their presymbolic notion that had been evoked by the source phenomenon, and the child might articulate a general principle. Only then might the child retroactively accept the analysis process by which the product was constructed.

This heuristic design framework for embodied mathematical learning, which is elaborated elsewhere (Abrahamson, 2009b, 2012a, 2012b; Abrahamson & Wilensky, 2007), informed the development of a design for the binomial. I will now describe the design rationale, materials, and activities and then present overall findings as well as a paradigmatic episode from its implementation. I will interpret the episode as supporting my claim that presymbolic notions of chance can serve as an epistemic resource for grounding compound event spaces.

3 A design for the binomial: rationale, build, and results of the "seeing chance" activity

Chance, like slope or aspect ratio, is a perceptually privileged intensive quantity. On the one hand, humans are capable of perceptual judgments of random phenomena that elicit a gestalt sensation of likelihood (e.g., see Xu & Garcia, 2008). On the other hand, the chance of an event occurring as a result of operating a random generator is formally quantified as the quotient of two values: the total numbers of favorable and possible outcomes. Accordingly, in planning a design for probability I sought to create situations that would elicit primitive perceptual mechanisms relevant to the content of compound events, while concurrently reconfiguring the standard event space so as to accommodate perceptual reasoning. In accord with the semiotic–cultural approach and the notion of an epistemic resource, students were to construe this customized event space as semiotic means for objectifying their presymbolic sensation of likelihood. Below I present results of this iterative design process.

3.1 Design-based research question

As a researcher of mathematics learning, who was interested in building a design for compound events, I asked: (a) what subjective notions of chance could potentially ground its formal analysis; (b) what sensations could give rise to these notions; (c) what source phenomena might evoke these sensations; (d) what framing activities with the source phenomena would evoke these sensations and notions; (e) what variants on standard mathematical forms might best accommodate students' perceptual reasoning; and (f) what discursive interaction would best support student coordination of these informal and formal views?

3.2 Cognitive domain analysis

Apparently, babies are sensitive to relations between the color make-up of a population of objects and a sample from the population. When the sample is configured as a sequence of discrete singleton outcomes, such as [Green Green Blue Green], babies' statistical judgment indicates that they ignore this internal order but treat the color ratio to gauge the sample's representativeness vis-à-vis the population (Xu & Garcia, 2008). It thus appears that collections of colored objects can evoke in young children presymbolic qualitative notions that loosely correspond to formal measures of these collections' stochastic propensities. It further appears that these notions emanate from comparing two sensations, one evoked by a source and one by its sample.

These findings from cognitive developmental psychology indicate the activation of an endemic perceptual capacity—an evolved "enabling constraint" on perceptual reasoning (Gelman, 1998) or "ecologically intelligent" behavior (Gigerenzer, 1998). The findings also resonate well with conclusions in Kahneman et al. (1982). That is, people perceive HTHT as a more likely outcome of flipping a coin four times than the equiprobable HHHH, because HTHT has equal numbers of H and T, thus evoking a sensation more in accord with the random generator that produced it, a fair coin.

For education, the implications are a paradoxical relation between framing and perception. When students attend to "HTHT" in the context of comparing its likelihood to "HHHH", they mentally construct "HTHT" as "2H2T" without attending to order. And yet the aggregate event 2H2T truly is more likely than the event 4H—it is six times as likely! Thus people demonstrate mathematically correct inferences for mathematically incorrect models of phenomena (Abrahamson, 2009b). Indeed, in a controlled experiment we found that students who attended to "HTHT" in the context of a probability activity, as compared with a counting activity, were more likely to ignore or resist the order of singleton events (Mauks-Koepke, Buchanan, Relaford-Doyle, Souchkova, & Abrahamson, 2009). It follows that when students' perception-based probabilistic reasoning is evoked, they should become inclined to draw mathematically correct inferences about the likelihood of events-seen-as-aggregates even as they become disinclined to attend to particular variations on these events. What should a designer do?

3.3 Design rationale

The above cognitive domain analysis bears directly on design rationales for the domain. In particular, the cognitive domain analysis orients the designer in creating materials and activities for supporting students' grounded learning of formal views on compound-event experiments. To begin with, it turns out that the source phenomenon for our design might be a random generator composed of a large collection of objects (the "population"), small collections of objects (the "samples"), and an indication that the small collections came from the large population. For example, there may be some device that draws a small set of objects out of the population. When students are asked to comment on the likelihood of a particular sample, they would respond by activating schemas that evoke sensations of ratio representativeness vis-à-vis the population, and from these sensations they would infer notions of relative likelihood, such as a sense of a particular event's plurality as compared with other events. The students would then objectify these presymbolic notions of likelihood in mathematical form by utilizing dedicated semiotic means that are made available to them in the learning environment. We now discuss a customized configuration for the compound event space designed to afford the objectification of these notions.

Recall that students construe compound events without attending to the specific internal order of singleton outcomes. How, then, might a student ground a formal permutation-based event space in these informal, combination-based, holistic presymbolic notions? Practically, what semiotic means of objectification should a designer build so as to enable students to leverage their naïve perceptual reasoning as an epistemic resource for accepting compound event spaces? Per the framework outlined earlier, the designer should consider formatting the standard display of an event space using expressive media, representational elements, and spatial form that render the display conducive to perceptual reasoning and instructional discourse, such that students would be able to align and coordinate their informal inferences with formal analysis. The following explanation of the design built for our project will clarify the above rationale.

3.4 Design build, methods, and implementation results

Figure 1 shows images of the resources created for the design: (a) a concrete random generator; (b) media for building an event space via combinatorial analysis of the random generator; and (c) an innovative form of organizing the event space so as to make it more conducive to heuristic perceptual inference. In addition, I designed and built a suite of computer-based simulations of the experiment featuring schematic models of the random generator. As the simulated experiments run, the model aggregates cumulative results either in standard bar charts or in pictographs composed of iconic representations of the actual random outcomes (see in Abrahamson, 2006).[1]

As researchers developing both theory and artifacts, we operate in the design-based research approach (Collins, 1992). Our conjecture-driven interventional study was in the form of an explorative, task-based, semi-structured, tutorial clinical interview (Goldin, 2000). As such, our interview protocol can be viewed as part of the pedagogical design as much as it is an investigative instrument—it is both a means of eliciting student response and a potential contribution to educational practice (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Subsequent to collecting data, we employed extensive micro-ethnographic collaborative analysis to build coherent, consistent, consensual theoretical models that purport to explain across the data corpus how interactions led to conceptual outcomes (Schoenfeld, Smith, & Arcavi, 1991).

We worked individually with 28 students aged 9–11 years. First, we had students briefly examine the random generator's experimental mechanism (see Fig. 1a). Immediately after, we asked, "What do you think will happen when I scoop?" Importantly, *the students did not conduct any actual experiment at all*, so that their responses were based only on perception and reasoning.

In accord with the design rationale, students tended to offer likelihood judgments that agreed with mathematical theory. In particular, they predicted a plurality of 2g2b (2 green and 2 blue) samples, a rarity of both 4b and 4g, and, in between, the events 1g3b and 3g1b. When asked to support their responses, students referred to the equal number of green and blue marbles in the bin.

We next offered the students a stack of cards bearing a schematic iconic representation of the empty marbles scooper (see Fig. 1b) as well as a green crayon and a blue crayon, and we asked them to show us what we might get when we scoop. Confirming our cognitive domain analysis, the students were "blind" to order. They tended initially to create only five cards, one for each of the aggregate events 4b, 1g3b, 2g2b, 3g1b, and 4g. We guided them to create variations on each of these events (to expand the 1-1-1-1-1 distribution into 1-4-6-4-1). Whereas the students were generally able to produce these variations, they claimed that *these supplementary objects were irrelevant semiotic means to completing the task of showing possible outcomes or supporting their earlier predictions for what would happen when we scoop*. We then guided the students to sort the 16 cards according to these 5 event classes and

[1] Strictly speaking, the physical marbles-scooping experiment is hypergeometric, not binomial, because as each marble is captured by a concavity in the scooper, there is one less of that color in the bin. However, the fairly minute ratio of the sample size (4) to the total number of marbles in the bin (hundreds) enables us to think of this experiment as quasi-binomial and, for all practical effects, as actually binomial. In this paper I do not expand on the computer-based simulations, because my thesis here pertains primarily to students' perceptual judgments of the random generator itself and, in particular, how students coordinated these judgments with their guided perceptions of the event space. Moreover, my experimental design was such that students engaged the computer activities only after they had made sense of the event space, so that any observations of students grounding the actual outcome distribution in the event space are contaminated by the prior activities. That is, the study was designed as an experimental unit, not a comparison experiment.

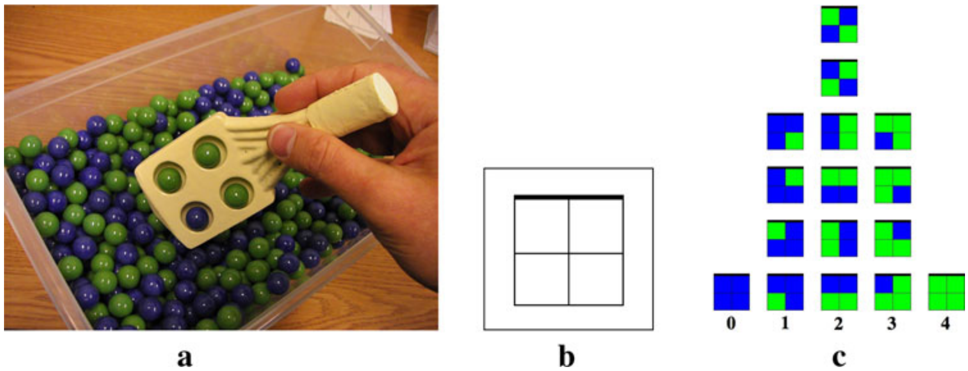

Fig. 1 Materials used in a design-based research project investigating relations between informal intuitions for likelihood and formal principles of the event space: **a** a "marbles scooper", a utensil for drawing out ordered samples from a box full of marbles of two colors; **b** a card for constructing the sample space of the marbles-scooping experiment (a stack of such cards is provided, as well as a green crayon and a blue crayon, and students color in all possible outcomes); and **c** a "combinations tower," a distributed event space of the marbles-scooping experiment, structured so as to render quantitative relations among the events conducive for heuristic perceptual inference

assemble them in a spatial configuration that highlights quantitative relations among these 5 sets (see Fig. 1c).

Once this tower had been fully assembled, a "strange" thing happened. Students tended to endorse *all* 16 cards as pertinent to the task. In particular, *they now were willing to accept the event space, stating that the five discrete sets explain their own notions of the five aggregate events' relative likelihoods*. Students' insight can be modeled as a heuristic semiotic leap via guided abductive reasoning (Abrahamson, 2009b, 2012b).

I have now presented the design and overviewed results from its implementation. In particular, I have offered empirical evidence to support the plausibility of a thesis by which presymbolic perceptual judgment can serve as an epistemic resource for grounding the logic of a compound event space, given appropriate design. I now focus on an episode from these data that contextualizes the thesis with authentic, idiosyncratic circumstances and interactions. In the interest of communicating the pedagogical entailments of this design approach, the analysis highlights the tutor's pivotal and complex role in scaffolding the student's learning toward initial insights as well as beyond them.

4 A paradigmatic episode: the case of Tamar

Tamar (pseudonym) is a sixth-grade middle-school female student characterized by her mathematics teachers as "middle achieving". We will discuss only the first 25 min of Tamar's hour-long interview, because after that point she engaged in computer-based simulations that go beyond the scope of this paper. This particular episode was selected from the data corpus as paradigmatically demonstrating all participants' struggle to coordinate tacit and analytic views, albeit Tamar's particular resolution of this struggle was unique.

Asked what would happen when we scoop, Tamar singled out the 2g2b event as most likely as compared with each of the other four aggregate events. Asked to support her prediction, Tamar alludes to a perceptual judgment of the source phenomenon:

> It's like a 50–50 chance of getting two-green-two-blue…because it kinda looks like there's an even amount of them [green and blue marbles in the container], so if you scoop, it's, like, yeah…[2]

The researcher encouraged Tamar to explain further or be more rigorous, but she could not offer any more insight on this subject.

Soon after, once Tamar had created the expanded sample space, so that the cards were spread out on the desk in loose order, the researcher asked her whether the cards had any bearing on her earlier prediction. She responded:

> I'm not…I think that…I'm not sure…I just…yeah…

Thus whereas Tamar was able to conduct combinatorial analysis per se, she did not intuit the practical objective of this activity—neither its process nor its product. In particular, Tamar had yet to discern any relation between the number of variations per event and the relative likelihood of events.

The researcher guided Tamar to organize the sample space cards by the number of green singleton events, and Tamar assembled the 16 cards into the tower (see Fig. 2).

[2] Square brackets communicate indexical information with respect to speech referents, which can be gleaned quite unequivocally from the agent's gestures.

Fig. 2 The "combinations tower", a compound event space, that Tamar was guided to generate and assemble. The vignette will culminate with Tamar's reasoning as she compared the likelihoods of two possible outcomes represented by *cards on the bottom row* of this structure: the all-green outcome on the far right and a three-green-and-one-blue outcome immediately to its *left*

The researcher then asked Tamar whether she had any new observations. Tamar surveyed the assembly and offered that she had overestimated the chance of 2g2b:

> It actually seems like it could be more…like it's not exactly 50–50 chance of getting two-and-two [as compared to] getting something totally different, because there are more…There're a lot more combinations and stuff…Now I think there's actually more chance of getting something different.

The researcher asked Tamar whether she knew how to express this idea otherwise, perhaps with numbers. Tamar said she does not—"It's just, looking at it, it *seems* like that." Thus whereas Tamar's perceptual reasoning was proportional, her explicit reasoning was not, possibly because she was not sufficiently fluent in rational numbers.

To the extent that the above transcription is of interest to researchers of probability education, I would like to suggest that what is interesting about it is what Tamar did *not* say more so than what she said. Namely, I am referring to Tamar's facile endorsement of the compound event space concurrent with its stochastic implications. Tamar, who only 10 min prior was unable to suggest any rigorous means of supporting her prediction for the plurality of 2g2b beyond referring to the color ratios in the box, and who still could not offer an explanation once the event space was completed yet scattered on her desk as 16 discrete items, immediately assumed mathematically appropriate analytical reasoning once the event space was reconfigured so as to make salient the number of outcomes per event. Per semiotic–cultural theory and the perception-based design framework, once the event space was more conducive to perceptual reasoning Tamar availed of these material means so as to objectify and modify her qualitative notions. She linked sensations of differential representativeness in the random generator with differential discrete quantities across the five event sets. This heuristical anchoring of qualitative sensation in an enumerable display is striking in its educational significance precisely due to its discursive insignificance.

Still, heuristical anchoring of presymbolic holistic notions in articulated analytic structures does not imply conceptual understanding. In fact, there is much work to do in order to render this implicit reasoning explicit and available for reflection. In particular, such anchoring may engender struggle over contrasting meanings of ambiguous objects—informal and formal meanings (Abrahamson et al., 2012; Abrahamson & Wilensky, 2007). For example, does Tamar see a particular 3g1b card as 1 of 16 equiprobable elemental events or as 1 of 5 heteroprobable aggregate events? Is she conscious of how she is seeing the outcome and why she sees it as such?

The researcher (Dor) lifts up two cards from the completed combinations tower and holds them side by side, well within Tamar's visual field. Looking at Fig. 2, these are the single 4g card on the far right and the 3g1b card immediately to its left[3]:

Dor: Is one of these patterns more likely to show up than the other?

Tamar: I actually think that this one [3g1b] is more likely to get, because it seems like it's harder to just get four of one color than to have it more mixed.

Tamar views the particular 3g1b card as "more mixed" than the single 4g card. Her assertion would be mathematically correct if she had qualified the "more mixed" as the collective property of *all* the 3g1b cards. Indeed, it is four times as likely to sample any one of the 3g1b cards than the single 4g. Only that Tamar's speech utterance explicitly indexed not the collective of all 3g1b cards but a specific 3g1b card that is in fact equiprobable to the 4g card. As such, Dor and Tamar share a referent—the particular card—but they construct it differently, with Tamar seeing it as 3g1b per se and Dor seeing it as the 3g1b card with blue in its bottom-right-hand corner (Abrahamson et al., 2009). *Tamar's interaction with the 3g1b card is analogous to seeing HHTH as 3H1T and inferring that it is more likely than HHHH.* Yet this finding is more striking

[3] An accompanying video clip of 2′15″ min duration can be viewed online at http://tinyurl.com/dor-tamar.

than the Kahneman and Tversky work, because here the entire event space is explicitly available for inspection. The constructivist tutor's challenge becomes to help Tamar sustain these order-blind presymbolic notions of likelihood while guiding her to re-map these notions onto card *sets* rather than individual cards, in accord with mathematical analysis. In a sense, the tutor has to help the student re-wire a sign:

Dor: Ok, so that's interesting—what you're saying is…

Tamar: It's like a 50–50 [the two patterns are of equal likelihood] but…it's just…to me it seems like that [3g1b] would get more.

Tamar fluctuates between a view of the 3g1b card as a heteroprobable aggregate event and as an equiprobable elemental event. But she is becoming conscious of this tension.

Tamar may not be able to resolve this tension on her own. It is Dor's role, in his capacity as tutor, to facilitate and encourage Tamar's awareness of her competing interpretations, while negotiating language and forms by which she may own, accept, and further articulate *both* interpretations. What Dor chose to do is guide Tamar toward realizing that she is sometimes seeing the particular 3g1b card as an order-less event:

Dor: Now, I want for us to be careful with the definitions here, because you said this [4g] is all green and this [3g1b] is mixed. So…[3 s silence]

Tamar: Well, it's actually harder to get just that pattern [the particular 3g1b card], I guess, so it's, like, even [i.e., equivalent chances of getting the particular 3g1b card as compared with the 4g card].

When Tamar says, "It's actually harder to get just that pattern", she appears to be comparing the specific 3g1b card not to the 4g card but to the entire group of 3g1b cards. Dor asks for clarification:

Dor: Oh, ok. Can you explain to me now what just went on in your mind when you made that observation? 'Cause that's important for me. [He places the two cards back on the desk in their respective locations.]

Tamar: Well, I just, like, saw all of them [the event space] and just…At first I thought that if you got [the particular 3g1b card]…It could be anyone of those that…[any of the four different 3g1b cards], and then I like just stared at that one [the particular 3g1b card], and I knew that it was, like, just as hard, because you have to get that exact pattern, so…

Tamar is reflecting upon her construction of the particular 3g1b card. Initially, she had construed it intuitively as an aggregate event, one of five in the entire space, but then she attended to it analytically as a specific pattern whose likelihood is equiprobable to the other cards in the space. Dor explores how robust this new awareness may be by orienting Tamar to other event columns in the tower and essentially reiterating the previous question. As we will see, Tamar's awareness was not too robust:

Dor: Is there any exact pattern in this field…this space, or collection [the event space]…that is…I don't know…easier or harder to get than any other particular pattern?

Tamar: No, I don't think so.

Dor: Interesting. So…

Tamar: Well, I think that this [a particular 2g2b card at the bottom of its column] might be a little bit easier [than 4g], because it's…well, I don't know! It just seems more difficult, to me, to get four of one color than to get them mixed.

We see that Tamar, upon attempting to generalize her embryonic awareness about objects in the event space, from the 3g1b column to the 2g2b column immediately adjacent to its left, "regressed" to the holistic view of individual cards. Tamar is in transition:

Dor: Ok…but do you have the sense of what you're flipping between? On the one hand, you're saying "to get mixed", and you're kind of referring to the whole thing [the entire 2g2b column], but…

Tamar: Yeah…

Dor: …then, when you stare at one [the particular 2g2b card at the base of the column]…

Tamar: Yeah…I, I think it's even…

Dor: Do you recognize the little confusion…

Tamar: Yeah.

Dor: …that there is here between the specific pattern and the group?

Tamar: Yeah.

Dor: Ok, that's a confusion I think we have to sort out, I think, in order to, like, understand this stuff.

The dyad continues to clarify terms, and Tamar achieves stability within under 2 min. Next, they move on to work with the computer simulations of the experiment.

In sum, when Tamar was able to make sense of the event space vis-à-vis the experiment it analyzes, she did so by drawing on her notions of relative likelihood, which she had gleaned from scrutinizing the random generator. She had not run any experiments with the device beyond several introductory scoops that demonstrated its mechanism, and so she could not have drawn on empiricism. To emphasize, if Tamar had not established any notion of likelihood prior to seeing the event space, she could not

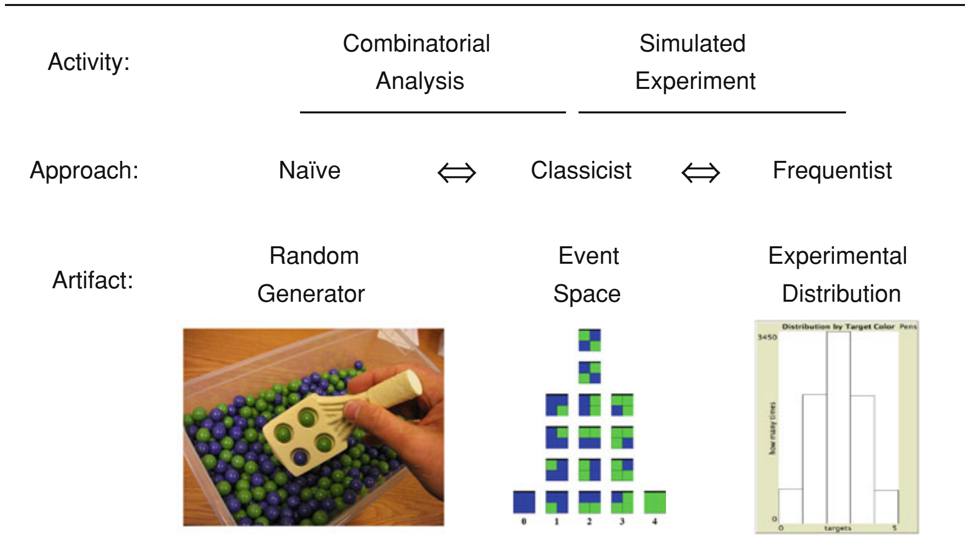

Fig. 3 Learning resource trialogue in the seeing chance design

have made sense of the space *in its totality*—the event space would not have appeared to signify the experiment's probabilistic propensities.

5 Conclusion

Steinbring (1991) recommends that probability designs interleave classicist and frequentist activities: students should learn to approach randomness experiments by determining their theoretical propensity, analyzing actual experiments, and aligning inferences across these two activities. My thesis did not challenge this view and in fact drew on findings from a project that emulated this view. Rather, I have examined whether experimentation is the only pedagogical means of enabling students to make sense of classicist theory. Drawing on the learning-sciences literature as well as on findings from an empirical intervention, I have argued for the viability of perceptual reasoning as an epistemic resource—alternative to experimentation—for students to ground the logic of combinatorial analysis as the disciplinary means of modeling random compound events. In particular, I have demonstrated that under auspicious design and steering, naïve perceptual reasoning can lead students to accept the differentiation of variations, which has been a chronic challenge in probability education (Jones, Langrall, & Mooney, 2007).

The pedagogical agenda of enabling students to ground mathematical knowledge in tacit perceptual knowledge builds at once both on cognitive developmental psychology (Gelman, 1998) and sociocultural theory (Newman et al., 1989). On the one hand, Piaget and Inhelder (1969) recognized that concepts are not direct articulations of primitive perceptual capacity, so that interaction and reflection are requisite. On the other hand, developing logico-mathematical models of phenomena depends on opportunities to appropriate cultural forms through participating in social practice (Rogoff, 1990), typically educational activities. In reform-oriented pedagogy, these activities characteristically involve experiences with dedicated artifacts that augment informal experience, a tradition that in modern times can be traced back to the first kindergarten created in 1837 by Froebel (2005). Earlier, Rousseau wrote in his 1762 treatise on education, "As a general rule—never substitute the symbol for the thing signified, unless it is impossible to show the thing itself" (Rousseau, 1972, p. 132).

As a design-based researcher with dual commitments to radical constructivism and sociocultural theory, my framework for building pedagogical artifacts resonates well with the semiotic–cultural theory of objectification (Radford, 2000, 2003). In the case of random compound events, this paper has demonstrated that learners can appropriate mathematical analysis by objectifying presymbolic notions of probability using a customized event space designed for this project.

Whereas naïve perceptual reasoning and empirical experimentation each may constitute an epistemic resource for accepting the compound event space, these activities operate differently and bear different conceptual entailments. Figure 3 asserts a didactical realization of these dual resources by highlighting their analogous structures and function: each constitutes a conceptually critical coordination that is embodied in an activity across two artifacts in a design for the binomial. The structural analogy and pedagogical complementarity of these activities is largely constrained by the event space itself, which the activities share as a common artifact. Double arrows indicate that learners need to interpret a new artifact they encounter along the activity sequence as signifying meanings they had established for a previous artifact: the random generator suggests its own stochastic propensity, the event space models this propensity, and the experimental distribution exemplifies the propensity (see "trialogue" in Wilensky, 1996).

Both perceptual and experimental approaches have the potential to evoke notions commensurate with the theoretical outcome distribution; in both cases, making sense of the event space is contingent upon and mediated by heuristic mapping of notions and perceptions; and both activities may result in implicating the event space as explaining the random generator's propensities. Thus both naïve and frequentist conceptualizations ground the classicist artifact.

Notwithstanding their compatible epistemic role, naïve perceptual judgment of random generators is different from experimental activity. Perceptual judgment has the capacity to directly evoke presymbolic notions of distribution, the mathematical property in question, as tacit inference from sensations of ratio representativeness. In comparison, experimental outcome distributions evoke these notions only indirectly and as contingent on some minimal fluency with this representational form. This difference between immediate and mediated notions may confer on perceptual judgments a unique advantage over experiments as candidate introductory activities for learning the mathematics of compound events.

Working with, rather than against, a student's perceptual biases presents instructors with challenging dilemmas that appear to trade off favorably with pedagogical gains. Thus, curriculum developers who are interested in building on the design presented in this study should bear in mind the pivotal role of reflective classroom discourse around these activities. Elsewhere, I have demonstrated the curricular extensibility of these introductory activities along two dimensions: incorporating symbolic displays and considering cases of non-equiprobable outcomes (Abrahamson, 2009a).

Students possess natural capacity to perform powerful presymbolic perceptual reasoning pertaining to the study of probability. Designers, teachers, and researchers may greatly avail themselves by leveraging this power so as to support the learning and continued investigation of this chronically challenging subject matter. Along this road rarely taken, there are yet miles to go.

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