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UNDERSTANDING CHANCE: FROM STUDENT VOICE TO LEARNING SUPPORTS IN A DESIGN EXPERIMENT IN THE DOMAIN OF PROBABILITY

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Six middle-school students participated in pre-intervention interviews that informed the design of learning tools for a computer-enhanced experimental unit on probability and statistics. In accord with the PME-NA XXVII conference theme, we elaborate our methodological frameworks and design principles. Our design was in response to students' failure to solve compound-event problems. We characterize student difficulty as 'ontological fuzziness' regarding the stochastic device, its combinatorial space, and individual outcomes. We conclude that students need opportunities to concretize the combinatorial space. Also, we conjecture that, given suitable learning tools, students could build on their comfort with single-outcome problems to solve compound-event problems. To those ends, we designed the '9-block,' a mixed-media stochastic device that can be interpreted either as a compound sample of 9 independent outcomes, a single independent event, or as a sample out of a population. We explain activities around the designed tools and outline future work on the unit.

Introduction

This paper reports on an empirical study of student mathematical cognition in the domain of probability (see the '*Connected Probability*' project, Wilensky, 1995, 1997). In accord with the PME-NA XXVII conference theme, "Frameworks That Support Research and Learning," the paper foregrounds the methodology employed for this study. Specifically, we discuss student difficulty with problems involving compound events, e.g., three coin tosses. We explain how a *design-research* framework enabled us to respond to student difficulty with tools designed to support students in building from what they know towards fluency with this class of problems.

We begin with the methodological frameworks and design principles of our research, then lay out the theoretical background and data resources of the study. Next, we describe a set of pre-intervention interviews, in which students worked on probability problems. We explain how our interpretation of student difficulty in these problems shaped a design rationale for innovative tools and activities, which we developed and then implemented in an experimental classroom unit. In this unit, students: (a) analyze stochastic devices that produce compound events; (b) classify the combinatorial space of these devices into subclasses and produce and assemble this space into a physical structure (a *combinations tower*); (b) interact with computer-based simulations of probability experiments, which draw from the same sample space and stack outcomes by the same subclasses; (c) compare products of these activities, i.e., students compare the shapes of the 'theoretical' combinations tower and the 'empirical' outcome distribution; and (d) participate in statistics activities in which samples are items from the same combinatorial space of the original stochastic device. The paper ends with future directions for this research.

Methodological Frameworks

At the Center for Connected Learning and Computer-Based Modeling at Northwestern University, we study student mathematical cognition and develop learning tools and activities. Design research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) is a useful paradigm for our research work, in that it affords us immediate and rich feedback on the efficacy of our tools in supporting student learning. Our goals are both pragmatic and scholarly: (a) our theoretical perspectives, methodologies, and design principles are all aimed at probing student domain-*specific* challenges, to which we attempt to respond by engineering, creating, and field-testing educational artifacts; yet (b) we take from our studies domain-*general* insights—"humble theories" (Cobb et al., 2003)—which we share with the community of education researchers and practitioners and which we incorporate as new theoretical lenses upon data harvested in future studies. In sum, our studies investigate: "What is needed?," "What works?"; and "How does it work?," "What does this mean?" Also, we investigate how new technology may shape content.

An integral methodological component of design-research studies is the elicitation of student understanding and difficulty before, during, and after implementations of the designs. We interview students (Ginsberg, 1978), listening closely to their 'voice' (Confrey, 1991) and, in response, we create, modify, and introduce learning tools into implementations, often while the implementations are underway. We aim these tools as "Vygotskiian" supports—they constitute forms for students to articulate their understandings so as to hone, express, and struggle with their difficulties. That is, we embrace 'difficulty' as a positive cognitive and motivational factor stimulating individual problem solving and communication in the classroom forum.

Toward designing new learning tools, we survey literature on: (a) ontogenetic; (b) philogenetic; and (c) urban/rural ethnomethodological aspects of the target mathematical concepts; as well as (d) previous studies that evaluated, analyzed, and responded to student difficulty with these concepts; and (e) national and state standards and high-stake assessment studies. These resources inform an emergent domain analysis that situates the concepts vis-à-vis students': (1) familiar situational contexts; (2) cultural practices in which the concepts are (implicitly) embedded; (3) mathematical representations students are likely to recognize; (4) the vocabulary of the domain; and (5) K-12 roots and trajectories leading to and from the concepts.

Design Principles

Our educational design work is informed by reform-mathematics pedagogy and agenda (e.g., von Glasersfeld, 1990; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). We embed these constructivist and social-constructivist perspectives into the learning tools and classroom facilitation infrastructures we design so as to promote high-level mathematical discourse with an eye on engagement and equity. Toward these goals, we create project-based collaborative-construction activities, using both traditional and computer-based tools, so as to foster student personal and interpersonal construction of knowledge. Our design is further guided by the following perspectives and principles.

a. Constructionism. Students will best learn through voluntarily developing new skills to problem-solve the design, engineering, and construction of their own artifacts (Papert, 1991).

b. Connected mathematics, connected probability. Wilensky (1997) argues that standard mathematical curricula are ahistorical and 'acognitive'—they do not enable learners to experience the problems that stimulated the invention of current mathematical solution procedures (see also Lakatos, 1976). This results in *epistemological anxiety*—students know that their procedural knowledge is correct, yet they do not know why. Specifically for the domain of probability, Wilensky demonstrated how even mathematically-informed adults greatly gained from activities that enabled them to 'connect,' i.e., to ground the content in their intuition.

c. Learning axes and bridging tools. A *learning axis* (Abrahamson & Wilensky, 2005a), a theory-of-learning construct, is a 'space' extending between two necessary and complementary components of a mathematical concept. These conceptual components, each within the learner's comfort zone, are concurrently afforded by a single *bridging tool* (Abrahamson, 2004; Fuson &

Abrahamson, 2005), an "ambiguous" artifact designed to support students in linking up previous understandings, situational contexts, procedures, and vocabulary. Students learn by reconciling the *learning issues*—tensions between the bridging tool's competing conceptual components.

d. Stratified learning zone. A *stratified learning zone* (Abrahamson & Wilensky, 2005b) is an emergent and undesirable pattern in classroom participation in collaborative construction projects, in which a subset of high-achieving students have greater learning opportunities, and the more "menial" roles are assigned to other students. We seek to promote more equitable participation through implementing specialized facilitation infrastructures that sustain all students' engagement in the core problem-solving tasks without forsaking collaboration.

e. The resourceful classroom. Optimally, a teacher should be: (a) familiar with all media and tools; (b) fluent in the domain and design; and (c) flexible in navigating between resources in response to student initiative. Implications for professional development are that teachers should have opportunities to practice operating the various media and to anticipate student response.

f. Mixed-media learning environments: Abrahamson, Blikstein, Lamberty, and Wilensky (2005) describe projects in which a range of technologies and expressive tools are integrated into learning environments so as to enable multiple entry points. Students, who come to these flexible environments with different skills, inclinations, literacies, tastes, working habits, and passions, have increased opportunities for expression and for development of expertise. The artifacts students create in these environments reside after the implementation ends, acting as classroom referents in future discussion toward the social construction of further knowledge.

(For domain-*specific* design principles emerging form our research, see Abrahamson, 2005.)

Theoretical Resources of the Study

In their influential *prospect theory* papers, Tversky and Kahneman (e.g., 1974) demonstrated the fallibility of human probabilistic reasoning in decision making. For example, in one set of studies, their participants were asked to consider a battery of hypothetical situations and then evaluate the likelihood of statements pertaining to each situation. The experimental findings did not augur felicitous prospects for students studying probability and statistics. Yet, a tangential perspective on human reasoning posits that mental capacities pertinent to the study of probability may have evolved under conditions far removed from the ecologies of managers reading texts (Wilensky, 1991). From such a perspective, Gigerenzer (1998) advocates a reformulation of curricula so as to accommodate humans' *ecological intelligence*. For example, the formal notation of probability, e.g., ".7," pithily captures the anticipated ratio of favored events out of a set of random events, but a *natural frequency* representation, such as "70 out of 100," would better accommodate the way organisms encounter information. In like vein, this paper describes a study designed to identify breakdowns in students' informal probabilistic reasoning; examine the possibility that such breakdown is due to shortcomings in the formal expressive tools available to the students; and create tools that support learning trajectories. Specifically, we aim to support students in sustaining an understanding of—a *connection* (Wilensky, 1991) to—new procedural skills, as they learn probability (see also Konold, 1994; Metz, 1998).

Our research questions, coming into this study were: (1) What are 8th-grade students' entry understandings that are relevant to the subject of probability?; and (2) What learning supports—objects and classroom activities—could capitalize on students' entry understandings so as to help students develop understandings of more advanced situations? The paper discusses a set of interviews we conducted prior to a teaching–learning intervention in a middle-school classroom. We analyze student utterance and explain the learning supports that we designed in attempt to respond to our interpretation of this input in light of reasonable learning objectives.

Data Sources

The data are from an intervention that is part of a sequence of design-research studies conducted in urban middle-school mathematics classrooms, where we are investigating student learning of probability and statistics using learning supports of our design. Participants were representative of school demographics (25% White, 24% Black, 25% Hispanic, 24% Asian, 2% Native American; 26% ESL; and 63% eligible to free or reduced lunch).

Methods

We conducted pre-intervention interviews both to gauge G8 students' entrance knowledge into the experimental unit and as a datum line for measuring student gain. We selected 6 students (3 male, 3 female) sampled from three teacher-reported performance groups. All pre-interviews were conducted by the first author and lasted 14 minutes on average. In these *semi-clinical interviews* (Ginsburg, 1978), the student was given a problem to solve, and then the researcher and student discussed the student's reasoning. The problem was, "A fair coin is to be tossed three times; What is the probability that 2 heads and 1 tail in any order will come up?" (NCES, 2004; the solution is 3/8 or a .375 probability). The coin item was chosen both for its content and because only 3% of G12 USA students had solved it correctly (NCES, 2004). We wished to investigate the sources of student difficulty and probe for student understanding that could potentially be leveraged. The interviews were videotaped. Following, we conducted *microgenetic analyses* (Schoenfeld, Smith, & Arcavi, 1993) to identify and typify when and why each student moved from secure to tenuous grounds in attempting to solve the problem. Next, we compared students' problem-solving paths to reveal cross-student similarities and patterns. This analysis generated conjectured learning trajectories through the subject matter and informed a domain analysis towards a design rationale for developing tools, which this paper will overview.

Results and Discussion

Pre-interviewed students could not complete the solution of the compound-event item. Yet, they solved correctly single-outcome problems that each of them initiated in various forms. We will now demonstrate a data sample and then analyze and discuss all the data.

Data Sample—A Student Discusses Probability Problems

In working on the coins problem, a student, described by the teacher as below average, said:

> I think you should add more information about the math problem itself. Like, not just say, 'A fair coin is to be tossed three times.'

When asked what additional information she needs, she answered:

> Well, I don't know how to *say* this… that's my problem…[11 sec silence] Ok, I think that… Ok, the first line's perfect. But, I don't know how to tell you what I want in the second… That's what… It's getting me. Uhhm… [8 sec silence] I don't know what to say. I don't know, I don't know… I don't have the right words to come out.

She suggests a simpler problem, in which there is a box of 10 candies—6 caramel and 4 chocolate—and one is to determine the chance of getting a chocolate. She solves this problem ("4-out-of-10 chance"). When asked how this compares to the coin problem, she said:

> In the box of sweets, you know how many there is. You know that there are 6 of those and 4 of the rest….But here you don't know….I mean, you know there's 3 chances that you can get heads and 3 chances that you can get tails….But you're not as… you're not, like, as… as descriptive as you are on the other one….It *is* the same problem, because you're talking about the same concept….And, uhhm, but, it's not the same... well, you *could* say [that in the coin problem, analogously to the candy problem] there's a box, because it's like you're inferring what would happen.

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From a Domain-Analysis Perspective on Student Voice Toward a Design Rationale

Ontology of probability. Student utterance revealed "ontological fuzziness" (see also Piaget & Inhelder, 1975; Wilensky, 1991) regarding three key elements of the domain of probability: the stochastic device (e.g., the coin[s]), the combinatorial space of all possible outcomes, including favored and unfavored events (e.g., HHH *HHT HTH THH* HTT THT TTH TTT), and specific outcomes of the sampling action (e.g., THT). Of these three elements of stochasm, only the stochastic device is an a priori substantive tangible object of consistent appearance. The other two elements—the space of all outcomes and the specific outcomes—either cannot be directly seen (the combinatorial space) or they are constantly changing (specific outcomes). One source of student confusion could be that they had only worked on single-outcome problems, in which there is congruence between items in the sampling "population," e.g., each and all of the green and blue marbles in a box, and the space of all possible outcomes of the stochastic action, e.g., all single marbles that can be drawn from that box. Also, note that constructing a combinatorial space wherein symbols replace icons demands advanced representational skills. Informed by this analysis, we concluded that students need opportunities to 'concretize' (Wilensky, 1991) both the combinatorial space and specific outcomes.

Modeling compound-event situations on single-outcome situations. Students' relative comfort with single-outcome probability problems suggested they may be able to use the *single*-outcome model recursively in problem-solving *compound*-outcome problems. For instance, to concretize the combinatorial space in the 3-coins problem, one could first determine all eight possible outcomes and write each outcome on a slip of paper. Then, one could put these eight slips in a box and select one at random. Thus, the "candy-box model" can apply to compound-event situations. Following, we describe an analogous stochastic object that uses marbles.

Design Solutions

In this section we present some of the stochastic objects and activities we designed in response to student difficulty, as evidenced in the pre-interviews and analyzed above.

Figure 1. Design, development, and use of the marble scooper. From left: an image of a computer model for the 3-D "print"; the final product; scooping from a box of marbles; a teacher shows the classroom a scooped sample of 9 marbles; a "9-block" for combinatorial analysis.

The marble scooper and the 9-block. The *marble scooper* (see Figure 1, above) is a device for sampling a fixed number of marbles out of a vessel containing many marbles, e.g., an equal number of marbles of two colors. We have built a scooper that samples exactly nine marbles. A sample (Figure 1, second from the right) may have, e.g., 5 green marbles and 4 blue marbles.

The scooper is a unique stochastic object—unlike in coins or dice, a particular sample is not an inherent physical aspect of the device but is constituted only through an interaction between the device and the "population" of marbles. Decoupling the stochastic *object* from its *outcomes*, the scooper may help in conceptualizing the combinatorial space of the 9-marble compound event as a spatial "variable" with color "values." Also, the intrinsic 2-D spatial form of the scooper, an array, suggests a topology for managing the construction of the space—an outcome is not just, e.g., "5green/4blue," but a specific arrangement of these. To help students concretize and build the space of all different possible 9-marble combinations, we designed the *9-block* (Fig. 1, on the right), a 3-by-3 square grid in which each small square can be either green or blue.

Figure 2. 9-block activities. From left: a sample 9-block; assembling 9-blocks, classified by the number of green squares in each; the "combinations tower"—the combinatorial space of the 9-block; a computer experiments that produced the distribution of the 9-block; a statistics activity.

Related activities. We ask students to determine the chance of getting exactly 5 green marbles. (There are $2^9 = 512$ unique items in this space, and a 5green/4blue combination is expected to occur 126/512 = ~.25 of the time). Students literally build the combinatorial space (the *combinations tower*), working in crayon-and-pencil and computer environments (Figure 2, above). Operating with these 512 9-blocks is functionally analogous to operating in a *single*-outcome space ("candy"), yet conceptually this recursively models a *compound*-event situation on a single-outcome situation. Next, students work with computer-based simulations that generate distributions of compound events (Figure 2, second from right). The visual resemblance of this distribution to the combinatorial structure (compare to Figure 2, center) stimulates inquiry into the law of large numbers. The 9-block also features in statistics activities (Figure 2, on the right), where it constitutes a sample taken from a large blue-and-green "population" of squares.

Situating the Study in the Larger Project (*ProbLab*) and Future Work

Both combinatorial analysis and computer-based experimentation contribute to student understanding of probability (Abrahamson & Wilensky, 2005b), and student conceptual learning may be understood as a synergy or reconciliation of these complementary activities (Abrahamson & Wilensky, 2005a). That is, the cognition of 'probability' can be seen as a theory–process dialectic: students learn to "proceduralize" combinatorial analysis and, in turn, to ground in the products of this analysis an anticipation of empirical distribution. Further work is needed to develop and evaluate a unit that promotes such learning-as-reconciliation in a resourceful classroom by using the bridging tools we have described. Such a unit (see *ProbLab*, Abrahamson & Wilensky, 2002) will be guided by activity-design principles including juxtapositions of: (a) different stochastic devices; (b) different embodiments of the same devices; and (c) different data-analysis perspectives on experiment outcomes (Abrahamson, 2005).

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