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A STUDENT'S SYNTHESIS OF TACIT AND MATHEMATICAL KNOWLEDGE AS A RESEARCHER'S LENS ON BRIDGING LEARNING THEORY

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ABSTRACT. What instructional materials and practices will help students make sense of probability notions? Li (11 years) participated in an interview-based implementation of a design for the binomial. The design was centered around an innovative urn-like random generator, creating opportunities to reconcile two mental constructions of anticipated outcome distributions: (a) holistic perceptual judgments based in tacit knowledge of population-to-sample relations and implicitly couched in terms of the aggregate events with no attention to permutations on these combinations; and (b) classicist-probability analytic treatment of ratios between the subset of favorable to all elemental events with attention to the permutations. We argue that constructivist and sociocultural perspectives on mathematics learning can be reconciled by revealing interactions of intuitive and formal resources in individual development of deep conceptual understanding. Learning is the guided process of blending two constructions of problematized situations: the phenomenologically immediate and the semiotically mediated.

KEYWORDS. Design-based research, cognitive science, sociocultural theory, cultural semiotics, binomial, combinatorial analysis, sample space.

In this article, we experiment with a new publicizing format by extracting the intellectual background from the main text and consolidating and expanding it in notes that may prove useful for readers who wish to delve deeper into the issues underlying and motivating the research. These companion research notes may be accessed here; the notes will also be linked to suitable places within the text.

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1. INTRODUCTION

Background and Objectives

Educational researchers of probabilistic cognition generally agree that students should be given opportunities to work with random generators, such as coins, dice, spinners, as well as with computer-based simulations of experiments with these random generators. Using these mathematical objects, students are to explore two complementary genres of investigative probability activities broadly captured as 'theoretical' and 'empirical.' *Theoretical activities* include "classicist" combinatorial analysis, a rigorous procedure for determining outcome distributions anticipated to result from experimenting with the random generators. *Empirical activities* include "frequentist" experimentation, whether manual or computer simulated, by which actual outcomes are generated, aggregated, and represented (Jones, et al 2007).

By comparing inferences from their theoretical and empirical work, students are expected to evaluate and modify their analyses, develop vocabulary for key constructs (sample, distribution, random, convergence, etc.), and prepare to re-state these notions using standard inscriptional forms, such as probability expressions or the formula for binomial expansion. Yet, whereas researchers agree that curricula should target students' insight into this theoretical–empirical complementarity, they do not agree with regards to best practices for enabling this insight. For example, opinions range with respect to which media, activities, and activity sequencing may best support this inquiry.

Such questions about *instructional design* are informed by, yet feed back into, research questions related to *theory of learning*, such as: What perceptual experiences, cognitive and meta-cognitive capacity, and discursive and meta-discursive practices optimize students' chances of connecting the respective knowledge associated with theoretical and empirical activities? Moreover, any educational design is an expression of some *pedagogical perspective* (whether or not the designer acknowledges this), and ideally this perspective coheres with some assumed theory of learning.

However, scholars focusing on either theory of learning, instructional design, or pedagogical practice often articulate their theses in light of only one or two of these pillars of education – theory, design, and pedagogy – so that, their empirical results notwithstanding, the overarching coherence of their published work may be wanting (Kelly 2004, Sandoval & Bell 2004). In an attempt to triangulate issues of theory, design, and pedagogy, we shall address in this paper the following questions:

- What instructional materials and practices will help students make sense of probability notions as theoretical–empirical complementarities?
- In turn, what can we learn about the nature of deep mathematical learning from engaging students in such design?

Design solutions for basic probability vary widely, even within the general umbrella of design inspired by constructivist philosophy (see the notes [1] for varieties and purpose of designs). Accordingly, this paper does not attempt to arbiter among the designs, many of which have their own merits. Rather, the paper positions our recent design solution and in particular discusses the consequences of implementing this design, both for students' experiences and, through analyzing these experiences, for our own accumulating understanding of mathematical learning, within probability content and possibly beyond. Namely, through explaining the rationale of our activities and closely analyzing a case study of one student participant, we will create context for commenting on the nature of learning and its implications for design. So doing, we will be sketching an analogy between two dualities in need of bridging:

a) an intuitive-vs.-formal epistemological duality with respect to students' personal and cultural resources in learning mathematical content; and
b) a cognitive-vs.-sociocultural intellectual duality or 'dialectic' (diSessa 2008) between vying theories of learning.

Analogizing these dualities, we argue, bears vital keys both for the theory and practice of mathematics instruction. In order to examine these dualities, we begin by exploring the construct of 'cognitive conflict.' We will offer a reconciliationist interpretation of this construct, and this interpretation, in turn, will frame the introduction of the design employed in this study as well as the subsequent description and analysis of a case study.

Epistemological Resources in Need of Bridging

In the cited studies (see notes [1]), participants' initial anticipations of random distributions were by-and-large incorrect, yet the design successfully engaged the participants in activity-based reflection on these thwarted expectations as a means of ultimately promoting deep conceptual understanding. In some of these designs, the conflict students experience is between their own naïve intuition, which is empirically exposed as flatly incorrect, and the teacher's authoritative mathematics, which turns out to be superior for actual prediction. However, the role and value of cognitive conflicts in educational experiments as well as in classrooms has not gone uncontested (see this note [2] for further details). In particular, it has been recommended that learners should come to see "how mathematics will reconstruct their intuition" (Borovcnik & Bentz 1991, p. 75).

According to this alternative premise, when most students consistently respond incorrectly, it is worthwhile to examine whether they are responding not to the question as the researcher understands it but to their idiosyncratic interpretation of the question. Espousing this alternative premise, we posited that perhaps a different nature and timing of conflict would foster learning experiences better aligned with constructivist pedagogical philosophy. Specifically, we wondered whether contexts could be created in which students' cognitive conflict pertaining to basic notions of probability would be resolved not through rejection of incorrect intuitive suppositions in favor of validated mathematical procedures but as reconciliation or synthesis of the intuitive and normative. For established precedents to our view on dealing with intuitive pre-knowledge of learners, see this note [3].

Yet, can the intuitive and normative views of probabilistic situations be aligned? Classical studies have argued that humans do not reason rationally with respect to probabilistic situations (Konold 1989, Tversky & Kahneman 1974). It would thus appear that dissonance between the intuitive and the normative is inevitable. However, as Gigerenzer (1998) argues, such traditional experimental contexts do not enable participants to bring to bear cognitive schemas that the human species has evolved for operating effectively in *natural* situations pertaining to the mathematical study of probability. For more on the complementary pairs of immediate perception of phenomena vs. indirect, artificial construction of theoretical concepts, see this note [4].

- Our premise, in accord with constructivist and ecological views: students' intuitive inferences must, by *some* interpretation, be regarded as mathematically sound.

Accordingly, we need to determine how students are understanding our questions when they consistently err by normative mathematical standards. Moreover, we conjectured that students' alternative understandings of mathematical questions mark the pivotal issues that the design must target, because the students would not be able to make sense of the mathematical formulations of a problematized situation unless they could view the situations as mathematicians do.

Thus, we posited that it is desirable to design instructional contexts that elicit the student's and teacher's differing views so as to create instructional opportunities for targeting this conflict directly (Borovcnik & Bentz 1991). To enable productive discourse around the conflict, we wished to embed the conflict in activities involving an actual object – the conflict might then become the dyad's struggle to articulate what this object actually is.

- We pursue this conjecture: At the core of *cognitive conflict* is a situation in which *two alternative perceptual constructions of an object* obtain, where *one* is more *intuitive yet mathematically delimited* and *the other* is *mathematically valid yet unintuitive.*

This conjecture builds on the anticipation that semiotic forms, in which students are required to re-present their perceptions of naturalistic situations, may be either more or less compatible with students' mathematically sound intuitive inferences: More on the relations between mediate and immediate, natural and artifactual knowledge may be seen from this note [4]. Consequently, the educational interaction becomes one of negotiating ways of seeing the world; learning transpires as an internalization of the dialectic between intuitive and normative views.

In order to foster situations that enable learning as a reconciliation rather than a substitution of knowledge, we searched for a context in which students' *initial* intuitive judgments, which are often expressed in naïve forms, would be *affirmed*, rather than disproved, as mathematically normative. Our working design solution was to create a problem context wherein students would:

a) observe a random generator and cast judgment with respect to the anticipated outcome distribution that would result from experimenting with it;
b) build the expanded sample space of this experiment; and then
c) recognize consonance between their initial inference and structural properties of this space, such as apprehending proportional relations between the subset of favorable events and all possible events, or relative to other event subsets.

Participating in such a design, we projected, students' experience would change dramatically from being proven wrong to learning to argue why they are right. Specifically, the design was to foster *semiotic leaps* – experiences of insight, in which learners appropriate mathematical formulations of phenomena as rhetorical means of warranting their tacit judgments for these same phenomena, in line with discursive practices of argumentation and proof valued by mathematicians (Abrahamson 2009a, 2009b).

Thus, students would experience symbolical mathematical artifacts not as superior alternatives to their intuitive notions but as disciplinary enhancements of these notions (see note on related research [5]). The focus of the current empirical study is on understanding the nature and promise of students' struggle to coordinate intuitive and disciplinary views of phenomena pertaining to the targeted concepts, binomial distribution and sample space.

Theories in Need of Bridging

Our case study, which demonstrates a learning trajectory typically observed in this project, illuminates a pedagogical dilemma germane to any design program informed by the following theoretical dialectic – the reader may find more details in note [6]. Namely, our educational experiments are informed by two competing background theories:

- The radical constructivists view learning as subjective construction of mathematical concepts (von Glasersfeld 1987). Within a similar orientation, proponents of Realistic Mathematics Education seek to promote students' *guided reinvention* of mathematics. The challenge of this "bottom-up" approach is that it is very difficult to forge a viable and effective curriculum upon a guided-reinvention pedagogy.
- The socioculturalists perceive mathematical learning as a process of acculturation into mathematical practice through engaging artifacts within organized social activity (going back to Vygotsky 1978/1930). The implication of an acculturation model, however, is that students may not be able to make any sense of the mathematical solution procedures they learn to perform.

Thus, on the one hand, we cannot wait for students to "reinvent the wheel," but on the other hand we do not wish to impose solution procedures despotically. Therefore, whether new mathematical tools are introduced surreptitiously or brusquely, still the pedagogical challenge is to make these tools sensible.

- How, can we help students perceive cultural mathematical tools as extending their intuitive grasp?
- How can cognitive and sociocultural approaches be reconciled to illuminate this learning process?

Bridging Epistemological Resources – Bridging Theory

We have thus discussed two gulfs:

(a) an epistemological gulf between students' intuitive ways of seeing "raw" phenomena as opposed to seeing mathematical analytical constructions of these same phenomena; and

(b) a theoretical gulf between constructivists and socioculturalists with direct implications for instructional practice.

We will argue that design-based researchers' constructivist-vs.-sociocultural theoretical conflict, reflected in students' intuition-vs.-artifacts emergent conflict, can both be resolved. The following questions lead our investigations:

- How are design-based researchers to operate who are convinced by the rationale of both the constructivist and socioculturalist theory?
- Are design-based researchers uniquely equipped to reconcile these vying theoretical approaches, specifically because we study students' mathematical cognition by engaging them in activities based on artifacts we ourselves design?
- If so, what narratives of human learning might we author that abide with both theories?

To resolve both conflicts, we must recognize mathematical notions as epistemologically equivalent to scientific notions and accept the implications of this view for the project of facilitating students' guided re-invention of mathematical concepts. Namely, both mathematical and scientific notions are grounded in phenomenal experiences – either of actual situations that are perceived directly or of situations that are perceived indirectly through mediating (semiotic) artifacts such as diagrams. Yet, just as the primary phenomenology of magnets is epistemologically distinct from the science of magnetism purporting to explain it, so the phenomenology of randomness is epistemologically distinct from classicist probability commonly used to predict it (see also Liu & Thompson 2002).

This epistemological gulf between tacit and cultural formulations of mathematical and scientific phenomena suggests that students, similar to historical mathematicians and scientists, have to notice within situations under inquiry unforeseen properties or patterns and recognize their pertinence for warranting intuitive inferences regarding these same situations [7].

Below, we begin by presenting the specific design solution we have been implementing as a means both of investigating theories of learning and, potentially, promoting the instruction of probability. Specifically, we will comment on an apparently pivotal learning process occurring even prior to and apparently supportive of the coordination of theoretical and empirical probability. After introducing this design, we will discuss in depth a case study of a middle-school student engaged in a problem-solving activity based on this design.

2. A DESIGN FOR THE BINOMIAL

The design employed in this study included materials geared to support student activities pertaining both to empirical and theoretical probability. The empirical-probability instruments were a physical random generator as well as computer-based simulations of experiments with this generator, and the theoretical-probability activities involved cards and crayons for creating the sample space of the experiment.

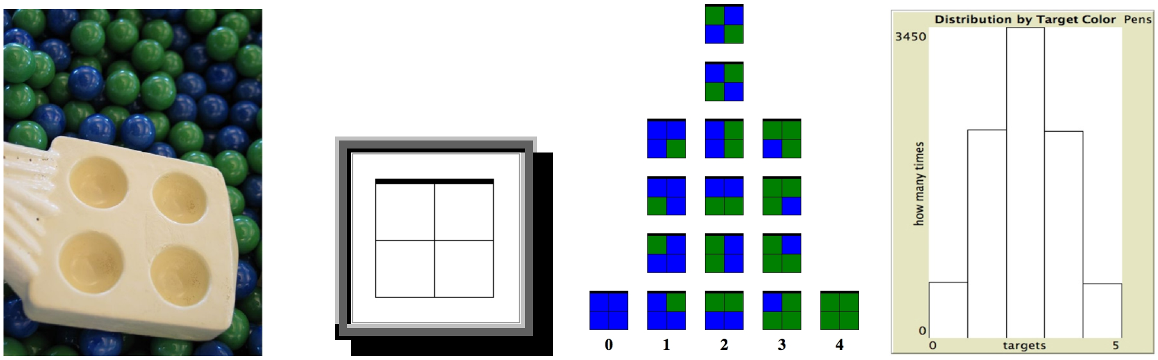
A single mathematical object, the *4-block* – a 2-by-2 grid of four squares, each of which can be either green or blue – crossed all the materials. The empirical instantiation of the 4-block was in the form of a *marbles scooper* (see Figure 1a), a utensil for drawing samples of (four) ordered marbles out of an urn-like tub containing hundreds of green and blue marbles of equal numbers (see Movie 1).

In this particular design we explicitly did not wish for students to initially take many scoops, because we did not want them to engage in experimentation that might have led them astray from mathematically correct predictions. Thus, we aimed to keep the scooping to the bare minimum necessary to enable students to understand the mechanism of this random generator.

The 4-block featured again in the form of physical cards – initially blank 2-by-2 matrices (see Figure 1b), which participants colored in using green and blue crayons so as to create the sample space of the experiment. The expanded sample space of this experiment includes sixteen unique cards, and a target structure of the activity was the *combinations tower* (Figure 1c) – a particular assembly of the sample space, classified from left to right by the number of green cells in the 4-blocks (zero through four) and arranged in vertical columns. Computer simulations of the experiment varied according to how they represented the experimental outcomes.

Whereas some simulations featured conventional histograms with stark columns that by default do not keep a record of the specific configurations of green and blue cells (see Figure 1d and Applet 1), other simulations featured icons of actual sampled 4-blocks that stack up, as the experiment runs, in five "stalagmites" (Abrahamson 2006) (see Figure 2 and Applet 2). Whereas the marbles-scooping experiment is, strictly speaking, a hypergeometric *approximation* (because it is without replacement), the computer simulations are truly binomial.

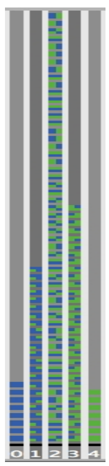
Note the hybrid nature of the combinations tower (Figure 1c): it assembles the sample space in a columnar structure that is visually similar to the histogram featuring the expected distribution of actual experiment outcomes (Figure 1d). The rationale of this hybridity was to create a bridging context between these theoretical and empirical semiotic artifacts as a means to facilitate students' juxtaposition and coordination of these complementary structures into a grounded understanding of the binomial.



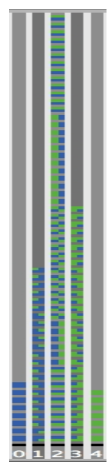
a. The marbles scooper for the actual experiment.
b. A 4-block card for the combinatorial analysis.
c. Combinations tower – an assembly of sample space.
d. Actual distribution produced by a computer simulation.

Figure 1. Theoretical and empirical embodiments.

a. Press "SETUP", then "GO" — the view shows the samples in the order they were generated



b. Press "SORT OUTCOMES" to sort the samples within each column by combinatorial type.



c. Press "PAINT OUTCOMES BY GROUP" to discriminate the sixteen groups by arbitrary colors.

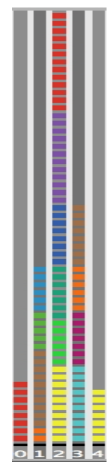


Figure 2. Three different views of an outcome distribution generated in the NetLogo virtual experiment "4-Block Stalagmite," corresponding to three modes of interaction with the simulation.

Specifically, the design for the first two phases of the instructional sequence was for:

(a) students to predict the expected distribution in hypothetical experiments with the marbles box (Figure 1a), i.e., to appreciate that the most likely scoop has exactly 2 green and 2 blue marbles, since neither color density is advantaged in the box, and that the all-green or all-blue events are equally rare; and

(b) next, students were to build the combinations tower (Figures 1b&c) and perceive it as means of anchoring their unformulated sense of distribution and consequently accept the 1 : 4 : 6 : 4 : 1 ratio as a quantitative enhancement of their initial qualitative judgment.

Thus, by working first with the random generator itself and then constructing its sample space, students were to triangulate their intuitive, presymbolic prediction for the empirical distribution with mathematical interpretation of the expanded sample space.

The computer modules were aimed to validate this triangulation by demonstrating that the empirical outcome distribution indeed converges on the 1 : 4 : 6 : 4 : 1 structures. In addition, a function of these simulations – that they parse the histogram columns into their inner sub-subsets of elemental events (see Figure 2) – was designed to support students in comparing between the theoretical sample space and the empirical outcome distribution and, specifically, in seeing the empirical distribution as scaling-up the sample space in a transformation which we have named a *stochastic–multiplicative stretch* (Abrahamson & Cendak 2006).

That is, each of the 16 equiprobable elemental events in the sample space occurs roughly the same number of times, so that in an experiment with, say 1,600 trials, we expect each of these elemental events to occur roughly 100 times, resulting in an actual experimental distribution of roughly 1×100, 4×100, 6×100, 4×100, and 1×100 samples in the five columns, respectively. The unique structure of the virtual experiments as well as the capacity to rearrange its elements automatically facilitate a visual mapping from the 16 elemental events in the sample space to the 16 chunks of empirical outcomes on the computer screen, thus supporting the development of bridged theoretical–empirical conceptual models we aimed to foster.

Yet, as we shall see, even before they arrive at working with the computer-based simulations, students already have opportunities to face what appear to be fundamental issues inherent to the topic of the binomial – negotiating intuitive and mathematical views of what constitutes the elemental object in the experiment and hence in the sample space. In fact, whereas we anticipated much difficulty around issues of randomness that lie in coordinating the sample space (theoretical) with actual experimental outcome distributions (empirical), we found that students are comfortable with the ideas of chance inherent to this theoretical–empirical mapping, at least as this mapping is constructed in our particular design (Abrahamson & Cendak 2006).

Therefore, the case study reported herein focuses instead on the earlier challenge of coordinating intuitive and mathematical views of the experiment – views corresponding to the marbles box and combinations tower, respectively. We will see that Li, a middle-school student, initially viewed the experiment as constituted of five possible outcomes – no-green, one green, two green, three green, and four green – whereas the interviewer wished to analyze the outcomes further into the expanded sample space that contains sixteen objects, not only five, with 1, 4, 6, 4, and 1 elemental events classified in five event sets. The crux of this report is a description of how the dyad first came to agreement, and this discursive moment of bridging intuitive and mathematical perceptual views is then re-contextualized as epitomizing prospects of bridging constructivist and sociocultural theoretical views.

3. METHODS

Design-Based Research

The data used for this study were collected as part of a larger project, *Seeing Chance* that we are conducting in the design-based research approach, which is a *conjecture* driven paradigm: researchers investigate the plausibility of some teaching or learning *mechanism*. The approach goes back to Brown (1992); Cobb, et al (2003) or Confrey (2005) are more recent sources; for more details on this research paradigm, see this note [8]. For example, we conjecture that grounding students' analytic understanding of stochastic distribution in their intuitive population-to-sample expectation of frequency may enable deep conceptual learning of the binomial, and that such grounding may be achieved by engaging students in activities supported by materials and facilitation techniques designed specifically to achieve this pedagogical goal.

Upon embarking on a new project, design-based researchers instantiate their conjecture in the form of a first-take set of instructional materials. Then, through a sequence of study cycles, the researchers iteratively modify and calibrate the design in response to consistent patterns in participant students' observed behaviors. Reciprocally, by reflecting on the nature of these calibrations, the researchers define and refine their theoretical models, often developing new constructs that may obtain beyond the narrow realm of the specific content and improve further research of this nature ("humble theories," Cobb, et al 2003).

Fostering Cognitive Conflict over Competing Perceptual Constructions

Li was one of 28 Grade 4–6 students (9–12 yo) each of whom we engaged in a 1-hour one-to-one semi-structured clinical interview. For our reasons for this method and its relative merits compared to other options, see this note [9]. All sessions were audio/video recorded for subsequent analysis. An interview protocol structured the following activity sequence.

Participants:

(a) predicted the actual outcome distribution in a hypothetical stochastic experiment conducted with the marbles box and scooper;

(b) were guided to build the sample space of the experiment and arrange it in the form of the combinations tower; and

(c) operated the computer-based simulated experiments.

We were in particular interested to see what, if any, connections students would discern among these three activities. Throughout the interview, we used prepared questions as well as extemporized follow-up questions to elicit students' observations and reasoning. The interviewer steered the student toward experiencing a succession of cognitive conflicts pertaining to conceptual issues key to the domain. In line with the design-based research framework, *learning axes and bridging tools* (Abrahamson & Wilensky 2007), we sought to identify learning issues core to the domain and so we focused the object-based discursive interactions on these issues.

Typical of our designed cognitive conflicts is that students attempt to interpret models of phenomena under inquiry as they would the source phenomena. So doing, the students ignore that analytic models may re-present phenomena on the basis of unintuitive properties. Viewing these models naïvely may thus engender a reading that contradicts the initial inference, even if the models in fact support the intuitive inference.

Specifically, the conflicts resulted from competing perceptual interpretations of objects that we intentionally designed to be ambiguous. For example, one of the sixteen specimens in the sample space, embodied in the form of a 4-block with a green square in its top right-hand corner, might under certain contextual circumstances be interpreted by a participant as an *elemental event* (one of sixteen equiprobable elemental events), yet a participant who does not understand the relevance of permutations to the combinatorial analysis might refer to the same object as signifying *any* of the four elemental events with exactly one green square and three blue squares in any order (thus referring to a single object as an *aggregate event*).

We anticipated that students would need to acknowledge these alternative ways of looking at the 4-block in order to use the sample space as a warrant of their informal inference for the experiment. Namely, we came to realize, students' intuitive sense of distribution is tacitly couched in only five *objects* (loosely: no-green, 1-green, 2-green, 3-green, and 4-green), whereas the formal expression of distribution explicitly considers all 16 objects grouped in five *categories*:

[BBBB]
[BBBG, BBGB, BGBB, GBBB]
[BBGG, BGBG, BGGB, GBBG, GBGB, GGBB]
[GGGB, GGBG, GBGG, BGGG]
[GGGG]

Figure 3. Grouping of all combinatorial objects into five classes.

Our interpretation that students would tacitly couch their intuitive sense of distribution in only five objects, with disregard to the internal order of the four singleton events in each, builds both on earlier studies (Abrahamson, et al 2006) and on Xu & Vashti (2008), who ran age-appropriate analogous experiments with 8-month-old infants.

The classical experiments by Tversky & Kahneman (1974), too, can be interpreted as demonstrating humans' natural inclination to ignore the order of singleton events in compound-event outcomes. Thus, when a study participant claims that a coin flipped four times is more likely to land on HTHT than on HHHH, the participant may understand the item as a comparison between "2 Heads and 2 Tails *in any order*" and "4 Heads." Indeed, "2 Heads and 2 Tails in any order" is a more likely compound event than "4 Heads" – it is six times as likely.

Bringing all the above to a concise summary, our focused research objective was to examine for any challenges students face as they learn to coordinate:

(a) their 5-object, aggregate-event-based, population-to-sample, qualitatively correct, tacit sense of distribution, which they experience in the context of a problematized source random-generator phenomenon; and
(b) a 16-object, elemental-event-based, quantitatively correct, explicit expression of distribution, which reformulates properties of the same source phenomenon.

A Cohort of Researchers' Journey towards Understanding Students' Insight

Whereas in previous studies we have taken broader analytic scopes of these data, the current paper focuses on a single case study so as to discuss in detail relations among situated mathematical problems, intuitive perceptual constructions, and guided engagement with instructional media, with implications for an integrated cognitive and sociocultural theory of mathematical learning.

Li, our case-study participant, is a 6th grade (11 years old) student evaluated by his mathematics teachers to be in the top third of his class with respect to his achievement and participation (for case studies of students of other levels and ages, see Abrahamson 2009a, Abrahamson, et al 2008). We selected Li, because his behavior was typical of his age/achievement group, yet he was particularly articulate in expressing his reasoning. A video excerpt of 3.5 min. featuring the culmination of Li's learning process serves as the focus of our discussion in the next section (see Movie 2 and the transcription of it.).

Over several years, our research team collaborated in intensive microgenetic qualitative analysis (Schoenfeld, et al 1991) of the Li video excerpt in an attempt to build a coherent explanation of his behavior in light of his entire interview as well as our whole corpus of data that included other participant–students' interviews. We approached the dyad's utterances, gestures, and inscriptions as multimodal, multimedia, multi-semiotic-system goal-oriented expressive acts, in which objects' emergent meanings are negotiated, often tacitly, by strategically making salient their perceptual features and relations (see notes [10] on related work).

The results section, below, attempts to communicate not only our final interpretations of these data but also a key change in our understanding of the data, which occurred along the data-analysis process. Explaining our own journey – and not only its results – may be of interest, because it underscores how difficult yet how crucial it is to listen closely to students (Confrey 1995) and in particular to appreciate their legitimate alternative constructions of situations and questions (Borovcnik & Bentz 1991).

The validity of our analyses should be judged against the raw data afforded in the form of Movie 2, and the generalizability of our claims should be evaluated in light of our larger set of findings, which we repeat here below yet elaborate elsewhere (Abrahamson 2009b).

4. ANALYSIS OF THE LI CASE STUDY

Background: The Baffling Issue of Students' Metonymic Treatment of Elemental Events

When asked to determine the most likely scoop, all but a single student guessed that it would be a two-green-and-two-blue (2g2b) scoop. When asked how they had performed these population-to-sample inferences, students were not too articulate – they said that they "just saw" the situation that way. All their utterances were constructed as aggregate descriptions of the outcomes – they rarely referred to the alternative arrangements of green and blue marbles *within* a given scoop. Below, by way of introducing our analyses of the focal video data from the Li interview, we begin by describing a general finding (Abrahamson & Cendak 2006) that greatly challenged us yet ultimately led to our current hypotheses, which we then elaborate.

After a participant had completed the construction and assembly of the sample space, such that all 16 elemental events were arranged in the form of the combinations tower, the interviewer would lift two cards – a card from the 3-green (3g1b) column and the single 4-green (4g) card – and ask the participant to judge whether these two events are equally likely to be drawn from the marbles box or whether one of them is more likely than the other (see Figure 4).

By and large, students would claim that the event represented by the specific 3g1b card is more likely to occur as compared to the 4g event card. This, despite our taking measures to ensure that the interviewer and student were referring to the same physical object. For example, the interviewer would repeat that he is referring to the particular 3g1b card, not to the group, and the student would concur that s/he understood the distinction. Similar results were obtained for other between-column card pairs. When asked this question regarding a pair of *within*-column cards, however, students would state that these events were equiprobable because the cards were literally "the same."

The dyad would then discuss the between-column issue, until the interviewer was satisfied that the student stably differentiated between a compound event *qua* elemental event (one of 16) and a compound event *qua* aggregate event (one of 5). These negotiations lasted between 1 to 12 minutes. Curiously, the older and higher-achieving students took the longest to stabilize, perhaps because they were more reflective and self-exacting.

In Abrahamson, et al (2008) we put forth the construct *inadvertent metonymy* as a description (but not yet an explanation) of students' response on this item. By 'metonymy' we meant that the student referred, for example, to a single 3g1b card as though it were carrying the cumulative probability of the entire aggregate set of four elemental events in the 3g1b column. By 'inadvertent' we qualified students' semiotic act by acknowledging that the metonymy was unintentional. With the construct of 'inadvertent metonymy,' we aimed to:

a) highlight that the dyad agreed on the referent but disagreed on its sense – we thus shifted our analysis from attending exclusively to semantic and syntactic features of the discursive act to a broader view encompassing semiotic and pragmatic dimensions of the situation and deconstructing these pivotal interpersonal dimensions so as to deepen our understanding of students' individual cognition (see also Borovcnik & Bentz 1991);
b) suggest that the dyad's disagreement had been tacit – it was exposed only through interaction – and, reflexively, that the researchers understood this communication breakdown only through post-facto intensive collaborative analysis;
c) valorize both dyad-member views as viable, given the individuals' respective mathematical knowledge at that time; and, ultimately
d) hone subsequent data analyses by focusing on the interlocutors' idiosyncratic *sense* for agreed referents and asking why the students were inclined to see individual elemental events as aggregate events and what such legitimate yet non-normative construction implies for the prospects of these students appropriating the normative view without forsaking their intuitive sense of the situation.

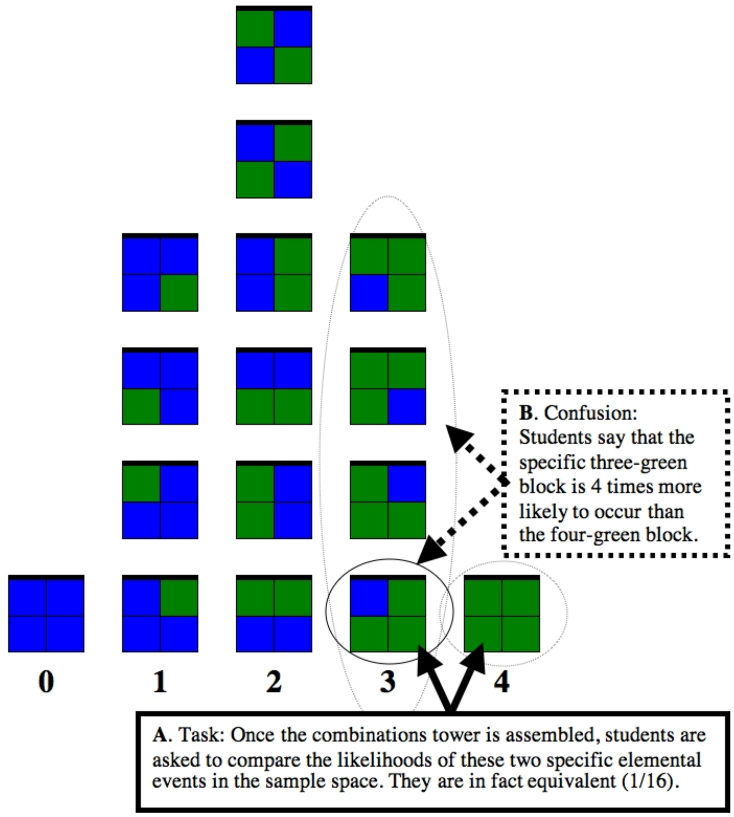


Figure 4. A student's ostensible misunderstanding of the targeted content is explained as a legitimate idiosyncratic construction of the interviewer's question: The student 'sees' elemental events as aggregate events, because s/he objectifies the sample space as built of five "things" and thus views the other elements as redundant duplicates.

In retrospect, though, we have recognized that 'inadvertent metonymy' was a figment of the researchers' epistemology – an ontological innovation that helped us make a *first* pass at the baffling data by situating the students' rhetorical act within our own semiotic frame, yet a first pass that did not furnish a viable explanatory model for the students' reasoning from the *students'* perspective. That is, the metonymy was never inadvertent – not once did students acknowledge their own aggregate construction of a specific 4-block as a metonymy, even when the students finally differentiated and named the two competing frames (e.g., "number-" vs. "place-wise").

That is, the students had never, in the first place, meant for the 4-block to stand in for anything at all other than its iconically similar scoop. Thus, in referring to a particular 4-block as an aggregate event, the participants were veritably seeing the card in that way, notwithstanding its "glaring" spatial configuration, in the eyes of the interviewer. And yet, from a methodological perspective, posing and considering the hypothetical construct of 'inadvertent metonymy' was a crucial analytic phase in our long-term analysis. Indeed, in the remainder of this section we will share more recent insights that this interim construct subsequently engendered.

We begin by reiterating that the mathematically informed researcher brings to bear well-articulated frames that the student has yet to develop. Namely, the researcher clearly differentiates between elemental and aggregate events. Plausibly, in some particular discursive contexts the researcher might be inclined to refer to a particular elemental event metonymically so as to mean the entire event. Yet when the property of likelihood is at stake, permutations must be enumerated as singular elements, such that the metonymical reference to the card must give way to regarding the card as a unique entity within the sample space. The researcher would then probably refrain from referring to a card metonymically, because s/he would recognize the referential confusion such signification might engender.

The students, however, initially lack relevant conceptual structures for selecting between aggregate- and elemental-event views of a card. Indeed, the prospects of a learner to build mathematically valid structures by coming to see objects in new ways is entirely contingent on the learner's personal goals for the activity that frames these perceptual constructions. At this point in the interview, these students do not yet know that attention to permutations is crucially conducive of combinatorial analysis; in fact, they do not at all know they are engaging in what we call combinatorial analysis! Thus, the students had no goal-oriented premise to appreciate the distinction we attempted to highlight between aggregate and elemental views.

As noted, our discovery of this 'inadvertent metonymy' shifted the data-analysis toward attempting to understand our participants' default aggregate-event-based perception of each of the 16 singular cards in the sample space. We therefore asked, "What activities give rise to students' aggregate orientation toward individual 4-blocks in the sample space?" As the narrative of the Li case study will reveal, we have implicated the available media – specifically the interaction between students' contextual intuitions grounded in the marbles box and the materials made available for subsequent combinatorial analysis – as explanatory of the researcher's confusion over students' aggregate views of individual 4-block cards in the sample space.

In fact, we will present students' persistent aggregate-event construction of elemental events as a *desirable* transition toward normative understanding of the sample space. We will furthermore submit that the available media were challenging for Li's conceptual development yet instrumentally so – the media bootstrapped insight. In particular, we will interpret Li's difficulty as *rupture* caused by the very *semiotic means of objectification* (Radford 2003) made available for him to reify his tacit psychological objects.

Thus, the Li case study will problematize cultural tools as double-edged swords, from a constructivist perspective. That is, the tools are to accommodate students' intuitions, yet in order to enhance those intuitions, these tools – perhaps *necessarily* – incorporate "frictive" elements that may initially jar with the intuitive constructions. We further posit that some personal cognitive conflicts may emanate from interpersonal discursive conflicts over vying mental constructions of instructional artifacts.

Ontological Imperialism of Available Semiotic Means of Objectification

Setting the Scene

At the beginning of the interview, Li was asked to predict "what we will get" when we scoop from the marbles box. Interpreting the question as referring to the *probable*, Li gazed at the box and responded that 2g2b would be the most common draw. Subsequently, Li was handed the stack of empty 4-block cards and asked to show "what we could get." Interpreting the question as referring to the *possible*, Li set out to show all the different events. He built a total of five events (see Figure 5, the bottom row) and stated that he had completed the task. Namely, Li was content that the five cards exhausted the sample space as he perceived it.

For example, Li saw the card that has a single green square in the top right-hand corner (see in Figure 5, directly above the "1" numeral) as signifying "1g3b" (in any order) and made no allusions to the fact that there are four unique permutations that all have exactly one green cell and three blue cells (see, in Figure 5, three additional faded-out cards above the bottom card).

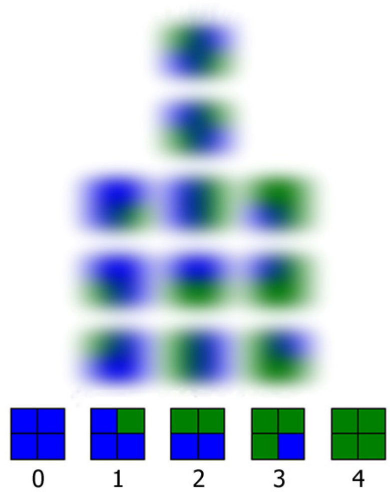


Figure 5. Metaphorical representation of Li's mental construction of the combinations tower. Li views the five 4-blocks in the bottom row as exhausting the sample space, because he sees each one of the 4-blocks as an aggregate event, not as an elemental event, in contrast to how the interviewer sees it. Therefore, Li views the remaining eleven elemental events above the bottom row as irrelevant to the task of showing all possible outcomes – he states explicitly that any one of these eleven 4-blocks could stand in for the 4-block at the bottom of its respective column.

A Covert Communication Breakdown

Whereas Li saw this particular card (see Figure 5) as showing the aggregate event 1g3b, the interviewer perceived this same card as showing one of four elemental events that collectively comprise the aggregate event 1g3b. Thus, the card is an ambiguous semiotic object whose idiosyncratic sense depends on an individual's knowledge frames and contingent goals. The dyad, oblivious to these vying frames, tacitly differed over the meaning of the card. Consequentially, Li and the interviewer experienced a communication breakdown, because they did not realize they were seeing their joint referent differently.

This miscommunication, we propose, was caused by what Bamberger & diSessa (2003) have called *ontological imperialism*. Li's communicative goal is to objectify in material form his sense of distribution in the marbles-box experiment – he had not seen order in the scooper and now does not see order in the cards. Yet, the media put at Li's disposal as expressive means of representing the sample space – the empty 4-blocks – pre-impose a particular mathematical view on the experiment that explicitly foregrounds the spatial configuration of the event, in the same way as the scooper was designed to foreground the configuration:

One cannot, in principle, color in an order-less 4-block, because each of the four cells in the 4-block is assigned a unique color. Li, of course, was unaware that by virtue of using the available media, he had represented his intuitive judgment such that a skilled user of the medium would read into these representations meanings – new layers of signification – that Li had not intended or even harbored. But the interviewer, in turn, did not yet enjoy the hindsight of two years of data analysis and thus was perplexed by Li's insistence that the five cards he had created exhausted the sample space.

On the Prospects of Appropriating Mathematical Vision

Yet how might an expert guide a novice to attend mathematically to instructional materials and especially when the mathematical view is at odds with the intuitive view? Could it be that novices adopt the mathematical view toward details of the materials only after the materials appear meaningful as a whole? That is, could it be that holistic perceptual judgments of mathematical representations precede understandings of their analytic premises? Is the logic of a semiotic artifact contingent on its global discursive function?

The spatial configuration of colors in the 4-block is a priori "visible" only to practitioners in this disciplinary domain (see Goodwin 1994, on 'professional vision'), for whom order counts in the context of conducting combinatorial analysis. For Li to attend to order, the dyad would first need to acknowledge that the student and he are differing in their *orientations of view* (Stevens &

Hall 1998) toward the cards; they would then need to name these orientations; and finally negotiate whether one of the orientations is more advantageous toward achieving the common goal.

At this point in the interview, though, Li was oblivious to the goal of combinatorial analysis, and so the process of building the five cards appeared to him as no more than an opportunity to reiterate his earlier statement about the marbles box, now using semiotic templates that enable him to distribute his statement onto a permanently displayed set of scoop images.

If he were given suitable semiotic means for marking upon the five cards their respective expected frequencies, we conjecture, Li may likely have done so. One could imagine a variety of such means, e.g., using a red marker to indicate upon the five cards the "intensity" of their respective felt likelihoods. Conversely, one could give Li five egg cups, in which he would place actual marbles that would move around amorphously.

Yet only combinatorial analysis offers a semiotic means in line with the cultural practice of mathematical argumentation – the combinations tower's five columns inscribe the felt frequencies as the events' vertical projections and, so doing, render the qualitative notion quantitatively explicit. Notwithstanding, the central question, for developing theory of learning as well as curricula, is precisely how to help Li see the columns as inscribing his felt frequencies.

In sum, at this point Li did not yet know that by attending to the order of the four singleton events in the scoop he would be able to create a set of objects, the sample space, that could *collectively* signify his presymbolic notion regarding the expected outcome distribution in the marbles-box experiment. The mathematical pertinence of the analytic procedure he was asked to engage was thus temporarily suspended – the procedure could be *instrumentalized* (Vérillon & Rabardel 1995) only once Li knew what the analysis was an instrument *for*.

The goal of the activity of coloring the cards was therefore enfolded into their "hidden" property of order, and this goal was waiting to emerge *retroactively* only once the sample space were completed and Li saw it as objectifying his intuitive judgment. In other words, for Li to understand the utility of combinatorial analysis, he would first need to blend and anchor his unarticulated sense of frequency, which emanates from the marbles-box context, into the 16 cards, thus synthesizing the intuitive and formal.

Yet, to do so, Li would need to sustain this unarticulated sense of frequency throughout performing the combinatorial-analysis procedure so that it is still phenomenologically available at the critical moment of completing the combinations tower. However, ontological imperialism predicts consequences more dire than communication breakdowns between a student and a teacher.

Namely, the student may reflexively come to see the world through the lens of the imposed representation and in so doing may lose unarticulated notions vital for connecting these mediated representations to the original immediate experiences (Bamberger & diSessa 2003). As we shall see, Li was indeed affected by the emerging meanings of the mathematical objects he himself had created. We now continue to describe the interview, at the point where Li enters a transitional phase of generative confusion.

From Heteroprobability to Equiprobability and Back Again: Negotiating toward Insight

Gazing at the five cards he had just created (see Figure 5, the bottom row), Li retracted his initial informal inference that 2g2b would be the most likely event and, instead, stated that these five events are equally likely to occur. Why did Li behave thus? There are reports in the literature on the *equiprobable bias* – learners' tendency to judge as equiprobable events that are in fact heteroprobable (Falk & Lann 2008, LeCoutre 1992). It might appear judicious, then, to label Li's behavior as exhibiting this same bias.

However, a unique feature of our interview protocol – that participants first guessed correctly the expected frequencies in a random generator and only then built its sample space – suggests that the bias results not from initial intuitions for a random generator but from engaging the analytic activity of building and/or investigating its sample space, yet using unfamiliar expressive objects. That is, the error is due to semiotic activity, not due to situated judgment.

Returning to our argument, above, recall that Li was not provided semiotic means of inscribing into the five 4-block events he created their respective intuited frequencies. And yet, Li is not aware that he has not fully inscribed his intuitions – both the objects (events) and their properties (frequencies). Therefore, Li tacitly trusts that the cards are carrying all the information he had ostensibly inscribed into them – the five objects as well as their relative-frequency properties. But, of course, the five event–cards do not in fact bear encoded information regarding their respective frequency properties, and so this information may be lost.

That is, crucial aspects of a presymbolic notion are liable to be attenuated once the notion is filtered through a person's constrained fluency with an unfamiliar representational system, consequently breeding conflict between the intuitive and the inscribed. Without intervention, this conflict may persist, creating un-connected knowing of mathematical procedures.

The interviewer's objective then became to enable Li to regain the initial sense of heteroprobable distribution, by which 2g2b was the most likely event, and coordinate this sense with a normative reading of the expanded sample space that highlights the plurality of the 2g2b event set in comparison to the other sets. The interviewer thus asked Li to create the permutations on the five combinations. Li builds the remaining eleven cards (see in Figure 5, the faded-out cards above the bottom row) yet protests that these new permutations are not pertinent to the analysis, arguing that the initial question had never been about permutations.

Li's contention, paraphrased, is, "If you want me to predict the frequency of *combinations*, why should I care about all these *permutations*?" For Li, it is as if the interviewer had asked him to measure a coin's diameter as a means of determining its fairness – the property being measured appeared to him completely irrelevant to the property in question.

Immediately, though, Li states that the middle column implies a 6/16 chance for the 2g2b event. It appears as though Li recognized in the differential heights of the columns a means of expressing their differential property of likelihood, which he had intuited earlier, in the context of the marbles box. That is, even though he has been agnostic of the permutations, the "more-ness" of the 2g2b column *as a whole* as compared to the 3g1b column *as a whole* implies a corresponding inequality between the aggregates 2g2b and 3g1b with respect to the dimension of likelihood, so that 2g2b feels more likely than 3g1b (on the "More A – More B" intuitive heuristic, see Stavy & Tirosh 1996).

Li thus performs a semiotic leap from his presymbolic gestalt image of distribution to an artifact that models the same notion mathematically. Yet… as his gaze wanders to other parts of the tower, specifically to the 4g element, Li reasserts his equiprobability bias, by which the five aggregate events are equally likely [11]. Li's hesitance and fickleness are reminiscent of Sfard (2007) who discusses the typical "intimations" and "implementations" of students' emerging understandings for mathematical artifacts (see Animation 1 for a schematic representation of Li's shifting attention).

The interviewer then prompted Li to re-consider the marbles box, thus presumably creating an opportunity for Li to re-infer 2g2b as the most likely event. Next, the interviewer referred back to the combinations tower. In what followed, the interviewer used a rhetorical mode so as to demonstrate for Li an apparent tension between Li's inferences from his direct apprehension of the situation (the marbles box) and the model of this situation (combinations tower). Specifically, the interviewer wished to underscore for Li that his immediate intuition (regarding the five objects in the marbles box) was heteroprobable whereas his mediated inference (for the objects in the combinations tower) was equiprobable.

The interviewer's rationale was to restate for Li his own equiprobability bias for the combinations tower while highlighting variability in the heights of its five columns. To so do, the interviewer produced a contradiction between the contents of modalities of his own communication, speech and gesture. Namely, while the interviewer verbally restated Li's view that all aggregate events in the combinations tower are equally likely to occur, simultaneously he gesturally drew Li's attention to the five columns. Li responded to this rhetorical contradiction by reaffirming that 2g2b would occur 6/16 of the time.

Thus, Li appears to appropriate the sample space as a means of objectifying his intuitive inference for the experiment that it analyzes. Crucially, it is *not* the case that Li initially understood the principle of permutation expansion and subsequently drew inferences from the completed sample space. On the contrary – the mathematically correct inferences were based not on the combinations tower but on the source situation, the marbles box. Moreover, all the way through to his insight, Li repudiated the rationale of attending to any 4-block in the combinations tower beyond the five in the bottom row.

In fact, Li was eventually willing to consider the five *sets* of elemental events as relevant to the task only because he recognized that the columns' respective heights resonated with his intuitive sense of heteroprobability and symmetry for the outcome distribution in the marbles-box experiment, irrespective of the permutation material that made up these columns. Only then, perhaps still reluctantly, did Li infer the implication of anchoring frequencies in the columns' heights. Namely, if the height *properties* of the five columns correspond to the frequency properties of the five intuitive events, then the *identities* of these five columns' contents should map onto the five events, too. Thus, Li spoke of some events (in the marbles box) as occurring more often than other events because "*these* [2g2b elemental events] have more than these [in another column]."

Such comparison immediately instantiates the entire column – not just the single icon at its base – as signifying the event under scrutiny. Switching from five objects to five columns, Li "smeared" upwards the respective identities of the five bottom blocks, invoking the *sets* of elemental events, not just the five bottom cards, as objectifying his five intuitive events. So doing, Li turned from tacitly construing each of the cards as aggregate events to explicitly construing them each as elemental events. Li's perception was thus disciplined. At that moment, the notion of an aggregate event could be first articulated as, e.g., "events that have 2g2b *in any order*."

Retroactively, Li accepted the *process* of combinatorial analysis as instrumental in predicting distributions, but only because its *product*, the combinations tower, captured his intuitions for expected relative frequencies.

5. CONCLUSIONS

Cognition

We have identified a critical tension between intuitive and formal constructions of situations involving a random generator of compound events: intuitive inferences are based on viewing the situation as enfolding through a set of aggregate events, whereas mathematical analyses of these situations are based on viewing the same situation as also enfolding through an expanded set of elemental events.

For example, people who see a HTTH result of flipping a coin four times, may tacitly construct this outcome as “2H and 2T in any order.” Thus, when they are asked to compare the likelihoods of HTTH and HHHH and they select the former as more likely, we might believe they have inferred that “P(HTTH) > P(HHHH),” a mathematically incorrect statement, whereas what they mean is that “P(2H2T) > P(4H),” a mathematically correct statement (see also Borovcnik & Bentz 1991). Our perspective raises challenges for Tversky & Kahneman (e.g., 1974), by whom a “P(HTTH) > P(HHHH)” claim is flatly illogical, because we are suggesting that the logic of people’s decision-making must be evaluated in light of their subjective construction of the situation, and that such construction is a function of education not logic.

Furthermore, note that the probability activities we discussed in this paper were “theoretical,” not “empirical.” Whereas we concur with our colleagues that critical learning of probability occurs in coordinating its theoretical and empirical aspects – and indeed our studies included significant work with computer-based simulations – we have demonstrated herein that the coordination of tacit (aggregate) and mathematical (elemental) views of probabilistic phenomena may play a major role in student learning of binomial distribution, even before any samples have been drawn.

Design

That said, we are certainly not suggesting that intuitive views of the world suffice for mathematical literacy. On the contrary, we have grappled with the design problem of how to foster students' mathematical learning, building on their intuitive views. Specifically, we demonstrated one way of potentially enabling learners to sustain their aggregate-event-based intuition even as they learn to appreciate the rationale and utility of elemental-event-based analysis.

We found that students who make aggregate-event-based qualitative judgments of random generators' outcome distributions can nevertheless adopt an elemental-event-based quantitative approach to mathematical models of these distributions. To do so, students need guidance in perceiving the sample space as enabling them to express their intuitive judgments. Then, and only in retrospect, the students may accept the combinatorial-analysis *process* as meaningful.

Pedagogy

Cognitive conflict need not entail a student abandoning intuitive knowledge in favor of mathematical knowledge. Rather, we have demonstrated a case of cognitive conflict between phenomenologically immediate and semiotically mediated ways of seeing problematized situations. Namely, students may have trouble accepting mathematical models, because these models parse the world in ways that conflict with their tacit parsing (see also Bamberger & diSessa 2003).

Yet students may be guided to appropriate the mathematical model through a process we call a "semiotic leap" (Abrahamson 2009b). That is, the students recognize that the inscribed mathematical model, as a whole, captures aspects of their presymbolic image better than their naïve model does (cf. Sfard 2007). This semiotic leap triggers inquiry processes described by C. S. Peirce as "abductive."

We see this process of negotiating tacit and cultural constructions of "mathematical situations" as elaborating on Donald Schön's thesis that learning is the process of synthesizing the intuitive and formal. In our case study, this synthesis was enabled as an assimilation of a situation's mathematical parsing onto its intuitive parsing. This process is supported through careful design of mathematical objects that we call "bridging tools" – variants on classical mathematical representations that invite perceptual judgments akin to those performed upon the source situation they model.

Theory of Learning

A consistent concern of this paper has been the relation between two theories of learning, constructivism and socioculturalism. At their extreme, pedagogical approaches inspired by each of these theories of learning bear very different implications for instructional design. Design-based research studies, we have demonstrated, are pragmatic arenas for examining the prospects of reconciling theories of learning, because the caveat of creating coherent activities requires a coherent theory of learning. Inspired by principles of the constructivist perspective, we assumed that students' assertions are subjectively meaningful, and we thus strove to reveal these meanings. Inspired by principles of the sociocultural perspective, we provided students with ready-made equipment and guided them into the associated practice of mathematical analysis.

Our study reaffirmed the constructivist principle that students' perceptual construction is contingent on their understanding. At the same time, we demonstrated the challenges students face in making sense of cultural tools. We found that teachers can play a vital role in helping students construct new mathematical ideas upon their existing knowledge, even if the existing and new notions appear conflicting. In particular, we now view teaching as the craft of helping students objectify inferences based on tacit knowledge in the form of new mathematical artifacts. In turn, we view educational designers' charge as devising and researching effective semiotic artifacts affording this guided synthesis.

Equipped with these new insights, we may dissolve apparent tension between constructivist and sociocultural theory – a theoretical dissolution that, in turn, may breed design solutions for guiding students' safe passages along disciplinary trajectories from intuition to inscription. Thus, bridging tools may support the consilience of theoretical and empirical perspectives both in mathematical learning and in research onto this learning.

Limitations and Future Work

Whereas we have focused on a single case study here, our insights into this case built on a total of seventy-five interviews that used the same protocol as well as related research into other multiplicative constructs. However, our comments on instructional design are largely post facto, because we are only beginning to understand our own craft of design (Abrahamson 2009a, Abrahamson & Wilensky 2007).

A future challenge will be to engineer bridging tools that foster semiotic leaps toward targeted mathematical constructs, such that the viability of the emergent design framework can be held against a priori predictions. Furthermore, we are interested in the durability of students' new insights as well as in the completion of their passages toward symbolical inscriptions.

Specifically for the learning of probability, we wish to further explore whether and how our design enables students to understand cases of compound elemental events with $p \neq 0.5$ (Abrahamson 2009a) and how these activities may scale up to classroom group work.

AUTHOR'S NOTE

This paper builds on Abrahamson (2008), so this author wishes to re-convey his gratitude to the ICME 11 TSG 13 reviewers and editors for their insightful comments that have advanced my thinking. For discussion of our earlier work as well as for further references supporting our heuristical design framework, see Abrahamson (2009b).

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ANNEX

For the annex of all extra material to this paper, follow this link.

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