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How do mathematicians learn math?: resources and acts for constructing and understanding mathematics

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Abstract In this paper, we present an analytic framework for investigating expert mathematical learning as the process of building a *network of mathematical resources* by establishing relationships between different components and properties of mathematical ideas. We then use this framework to analyze the reasoning of ten mathematicians and mathematics graduate students that were asked to read and make sense of an unfamiliar, but accessible, mathematical proof in the domain of geometric topology. We find that experts are more likely to refer to definitions when questioning or explaining some aspect of the focal mathematical idea and more likely to refer to specific examples or instantiations when making sense of an unknown aspect of that idea. However, in general, they employ a variety of types of mathematical resources simultaneously. Often, these combinations are used to deconstruct the mathematical idea in order to isolate, identify, and explore its subcomponents. Some common patterns in the ways experts combined these resources are presented, and we consider implications for education.

Keywords Expert mathematicians · Topology · Proof · Reasoning · Knowledge resources

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1 Introduction

It is well established that understanding how experts in a given domain think and learn can provide important insights for educational practice (Bransford, Brown & Cocking, 1999). In mathematics, proof is an important point of intersection between experts and students. It is a central component of professional mathematical practice, and it is also considered an important topic for students at nearly every level of study (NCTM, 1989; Hanna & de Villiers, 2008). While there has been a great deal of research about the *construction* and *validation* of proofs by both experts and students, less is known about how proofs are used to learn and understand the mathematical ideas they concern (Mejia-Ramos & Inglis, 2009). But focusing specifically on how mathematicians use disciplinary materials such as proofs to learn unfamiliar mathematics can be especially useful with respect to education. It can provide insight into what sorts of experiences and behaviors help one learn new mathematical ideas effectively: both within the specific context of learning from proof, but also perhaps in terms of how one makes sense of unfamiliar mathematics more generally.

In this paper, we investigate *how expert mathematicians make sense of mathematical ideas that are unfamiliar to them* in the context of reading an unknown mathematical research paper that includes a novel proof. In doing so, we treat expertise as the acquisition and possession of specific knowledge or *mathematical resources*, and the ability to strategically bring those mathematical resources together to make sense of new and unfamiliar mathematics.

The contributions of this study are twofold: First, we introduce Resources and Acts for Constructing and Understanding Mathematics (RACUM): an analytic framework to represent, systematically and across participants, the connections between different mathematical resources that experts build when they make sense of unfamiliar mathematical ideas. RACUM helps to illuminate how expert mathematicians bring together different types of mathematical resources to engage in the "complex interaction between rigorous and intuitive thought" that is characteristic of their practice (Weber & Alcock, 2004, p. 209). Second, we use this framework to illustrate some of the ways that experts in our study used these different resources to make sense of an unfamiliar mathematical research paper, in an effort to better understand how they came to learn new mathematics and how we can use this knowledge to provide students with richer mathematical experiences.

2 Theoretical framework

Expertise in mathematics is not just characterized by the amount or organization of knowledge that an expert already has, but also by her ability to learn or construct new knowledge and productively integrate it with her existing understandings. As such, we begin this section by briefly reviewing literature on "adaptive" or "generic" expertise and on the role of proof as a powerful source of new mathematical knowledge. We then review three major strands of research that have contributed to our analysis of expert sense making: (1) that mathematical knowledge can be productively described as a connected network of resources that comprise or characterize a given mathematical idea, (2) that there are different types of such resources that serve different roles in one's mathematical sense making, and (3) that mathematical learning can be productively described as the building of a network across these resources. This theoretical background serves as the basis from which we build the RACUM framework.

2.1 Expert learning and proof

Learning new mathematics is an important part of expert practice for professional mathematicians. They must review papers, attend research colloquia, stay current on new techniques and theorems in their field, and otherwise keep track of advances even outside of their own research area. This ability to make sense of new mathematical material is similar to medical professionals' "generic" expertise, which differs from expertise within a specific area of specialization (Patel & Groen, 1991), or Hatano and Inagaki's (1986) description of "adaptive" expertise, whereby individuals are able to use existing knowledge in new situations. Better understanding such expertise is particularly relevant with respect to mathematics education because it involves an individual's reconstruction and comprehension of existing, but unfamiliar, mathematical knowledge.

Indeed, even at the expert level, the way that one interacts with and describes content can be very different depending on whether they are already familiar with that content. Roth and Bowen (2003) noted that when scientists are presented with an unfamiliar graph, they describe a number of "possible worlds" that the graph could represent, whereas scientists familiar with the graphs link them directly and "transparently." Wineburg (1997) documents how two historians, a specialist and a non-specialist, rely on very different aspects of the same historical documents in order to successfully synthesize and make sense of them. The non-specialist is described as having the ability to "develop new knowledge" using the documents, whereas the specialist is able to rely on his existing knowledge. This suggests that investigating how experts use their adaptive or generic expertise might have special relevance to educators because it can illuminate aspects of expert practice that may be masked by studying what experts already know well.

2.1.1 Learning from proof

Although proof is typically described as a practice in argumentation and validation, it can also serve as an important access point to new and unfamiliar mathematics. Bell (1976) suggested that proof can serve three different purposes: to *verify or justify the validity* of a theorem, to *illuminate* the reasons why that theorem is true, and to *systematize* aspects of the mathematical content of the theorem into axioms, concepts, results, and so on. De Villiers (1990) extended these categories to present five purposes: *verification*, *explanation*, *systematization*, *discovery*, and *communication*. Although all focus on the reasons a proof might be written in the first place, they also present some reasons that a mathematician (or student) may wish to read and make sense of existing proofs. Hersh (1993) argues that while proofs can be viewed as arguments for the validity of a claim, they can also be viewed as explaining a theorem and the mathematical content it concerns. Hanna and Barbeau (2008), extending Rav (1999), argue that proofs can also introduce new methods, tools, and strategies for solving problems.

The focus of this study is to identify the methods that experts use to learn mathematics from proof. Research investigating experts' construction and validation of proofs suggests that these methods will be varied. Weber and Alcock (2004) note that while a proof as accepted by the mathematical community is often in a syntactic (logic-based) form, experts often use semantic (meaning-based) reasoning methods when constructing proofs in order to identify and make sense of the mathematical properties and relationships they describe. This "complex interaction" (Weber & Alcock, 2004, p. 209) or "pairing" (Inglis, Mejia-Ramos & Simpson, 2007, p. 17) between empirical mathematics and the formal language used to describe it was vividly described by Lakatos (1976) with regard to the development and refinement of ideas within the mathematics community at large. Inglis et al. (2007) suggest that it is this very interplay between formal and informal reasoning that characterizes mathematical expertise in proving and perhaps also in comprehending a proof. We adopt this hypothesis and in this paper explore the nature of that interplay in the specific context of learning from proof.

2.2 Resources and processes for mathematical understanding

2.2.1 Mathematical knowledge as a network

Often, researchers interested in the flexibility and adaptive nature of mathematical understanding describe the structure of mathematical knowledge as a network of relations between different properties, objects, and procedures that come to bear on a given mathematical idea. Skemp (1976) suggested that a relational understanding of mathematics enables individuals to more flexibly solve mathematics problems by providing a number of pathways through which to connect ideas. Papert (1993, p. 105) argued that "...the deliberate part of learning consists of making connections between mental entities that already exist." This representation of knowledge is typically used in the context of mathematical problem solving: Individuals who are able to recognize similarities between different mathematical procedures, representations, figures, and so on are able to relate new mathematical problems to ones they already know.

It is less common for the network representation of mathematical knowledge to be used in the context of mathematical *proof*, though there are some indications that such a representation is useful. Wilensky (1993) and Cuoco, Goldenberg and Mark (1996) identify proof as a mechanism for connecting different aspects of one's mathematical knowledge. Burton's (1999) in-depth interviews of experts revealed that many described their own mathematical progress as a process of "making connections," and Papert (1971) has referred to the LOGO turtle as a "connection agent" that helps learners recognize different patterns of behavior that generate the same geometric result—leading to geometric proofs.

2.2.2 The role of different resources in learning

The resources that comprise a knowledge network are not necessarily of the same sort and do not necessarily serve the same purposes. Vinner (1991) has described a mathematical idea, or *concept image or concept frame*, as a collection of images, experiences, specific features, and other knowledge associated with the name of a mathematical concept. Schoenfeld (1985, p. 15) has identified these resources as "intuitions and informal knowledge, facts, algorithms, routine procedures, and knowledge about rules for working in a domain." Others have described how a mathematical idea or concept can be "deconstructed" (Tall, 2001) or "parsed" (Wilensky, 1991) into subcomponents—and that these deconstructions are not necessarily unique. Michener (1978) notes that mathematical understanding is not only a matter of possessing and connecting between varied knowledge resources ("items") but also requires an awareness of exactly *what purposes different types of mathematical resources serve*:

> In our view of understanding, a good part of the process is concerned with building and enriching a knowledge base. This includes creating associations of many kinds as well as items. It also involves differentiating between various kinds of items according to their function in acquiring knowledge, familiarity, and expertise. (p. 381)

Many studies have documented the specific roles for different types of mathematical resources, which can differ across levels of mathematical skill and among individuals. For example, experts are able to use mathematical definitions to modify their conception of a given idea, whereas students have more difficulty and are more likely to rely on examples, images, and experiences (Vinner, 1991; Gray, Pinto, Pitta & Tall, 1999). Alcock and Inglis (2008) have shown that examples can be used by even advanced graduate students to develop intuition regarding the semantic aspects of a proof, but also show this varies dramatically between individuals. Weber and Alcock (2004) showed that experts were better able than students to instantiate specific objects associated with isomorphic groups, which allowed them to translate associations between those objects into formal terms. Watson, Mason and colleagues have explored the specific role of examples (specifically, those generated by the learner) as resources for illustrating the generic properties of mathematical ideas (Watson & Mason, 2002) and learning new mathematical concepts. A large body of research has also investigated the role of visual resources in mathematical practice and problem solving (Stylianou & Silver, 2004). Few studies, however, focus on the strategic use of these different types of resources together, or in the context of experts' learning of new material.

2.2.3 Learning as building connections

Finally, if mathematical knowledge is represented as a network of resources, then learning new mathematics can be thought of as the creation and modification of that network. Sierpinska (1994) describes mathematical understanding as possessing "resources for understanding" and "acts of understanding," whereby resources are linked to one another in order to construct or re-construct a given mathematical idea. For example, a student who has memorized that $5 \times 2 = 10$ possesses this knowledge as a resource, but if that student is then asked to determine what 5×4 is and finds the solution by adding together two instances of 5×2 rather than simply recalling a memorized solution, she is performing an act of understanding.

Duffin and Simpson (2000) further categorize mathematical understanding by differentiating *building*, *having*, and *enacting* as different components of understanding. They describe *building* knowledge as "the formation of connections," *having* as "...the state of those connections at any particular time," and enacting as "the use of the connections available... to solve a problem" (p. 420). If we reconcile Duffin and Simpson's description of the components of understanding with the distinctions between *resources* and *connections* outlined by Sierpinksa, then having, building, and enacting knowledge serve as specific mechanisms by which connections between resources are established and used.

2.3 Research objectives

In this paper, we leverage this theoretical background—that knowledge is usefully characterized as a flexible collection of varied resources and that the process of learning involves the building of relationships between them—in the specific context of experts coming to understand an unfamiliar proof. We identify several classes of mathematical resources used by participants as *resources for understanding* and different ways of connecting between these resources as *acts for understanding*. Finally, we use this representation to evaluate the methods experts use to make sense of the mathematics presented in an unfamiliar proof. By tracking the development of a network of different resources over time, we hope to gain insight into what Michener (1978) refers to as the "functional role of items" (p. 11)—that is, an epistemological understanding of how different types of mathematical resources contribute to one's learning in different ways.

In the following sections, we describe our study, review the mathematical research paper presented to participants, and introduce our analytic framework in detail. We then use this framework to explore three specific questions:

– What are the resources experts use to make sense of unfamiliar mathematics?
– What are the roles of different types of resources for making sense of unfamiliar mathematics?
– How are those resources combined to build an understanding of unfamiliar mathematics?

3 Methods and analytic framework

3.1 Participants

Participant	Professoral rank	Areas of research
Greg	Graduate Student	Quantum Theory
Mike	Professor	Number Theory, Representation Theory
Mark	Professor	Group Theory, Coding Theory, Combinatorics
Joe	Graduate Student	Functional Analysis
Ted	Asst. Professor	Statistics and Probability
Ana	Assoc. Professor	Ergodic Theory, Dynamical Systems
Saul	Professor Emeritus	Complex Analysis
Eric	Professor	Probability and Fourier Analysis
Myron	Assoc. Professor	Partial Differential Equations
Aaron	Professor	Differential, Algebraic and Enumerative Geometry, String Theory

Ten mathematicians—eight professors (assistant, associate, full and emeritus) and two advanced PhD students who had completed their qualifying exams participated in the study. They came from a variety of 4-year universities, the majority of which are classified Carnegie Doctoral/Research Universities-Extensive. Participants were identified through a combination of faculty recommendations and university directory listings and contacted via email to see if they would agree to be interviewed. In the email, we told participants that we were interested in how experts reason about mathematics and that they would be provided with an unfamiliar proof and asked to discuss the ideas presented within. However, only one was female, which makes it difficult to speak to potential gender differences or generalizability beyond this specific participant sample. Of the ten participants, seven had obtained tenure, and five were full professors.

3.2 Protocol

Students and professors who wished to participate were given semi-structured clinical interviews using a think-aloud protocol (Ericsson & Simon, 1984; Chi, 1997; Clement, 2000). Each was provided with the same mathematics research paper (Stanford, 1998) not directly related to any of the interviewees' specific fields of research. This paper was selected for its relative accessibility in terms of complexity and vocabulary. They were asked to read the paper and try to understand it such that they would be able to teach it to a colleague. They were also probed to describe what they understood of the mathematical ideas presented as they read, if this did not come up naturally in the course of the interview. Interview data were videotaped, transcribed, and coded by the first author and an independent research assistant. The coding system is discussed below.

We would like to make clear that our study uses *transcripts of experts' talk* as a primary source of data. As such, when we refer to *acts* and *resources*, we are referring to acts and resources as identified in the transcripts and understand that these may not always be complete accounts of expert's mental acts and resources. Because of this, it might be that experts were more likely to elaborate on their understandings in detail for the purpose of the interview in an effort to explain the mathematics to the interviewer. However, experts were aware that the interviewer was familiar with the mathematics contained within the proof, and during the interview debriefing, many experts affirmed that they felt their "think aloud" speech was comparable with the processes of sense making that they might engage in on their own.

3.3 Coding methods

We developed our coding scheme using iterative grounded theory (Glaser & Strauss, 1977), though heavily informed by the literature on mathematical understanding and resources described above. The first author and an independent research assistant coded three full-length transcripts (>25% of the total corpus of data). Interrater agreement was 70%, adjustment for randomness using Cohen's $\kappa \approx 0.64$ (95% confidence interval $\approx (0.56, 0.73)$), a relatively strong agreement given the number of categories and levels of codes that exist. After the two coders met to discuss and resolve coding conflicts, the coding scheme was adjusted and interrater reliability rose to 97%, Cohen's $\kappa \approx 0.97$ (95% confidence interval $\approx (0.93, 1.0)$).

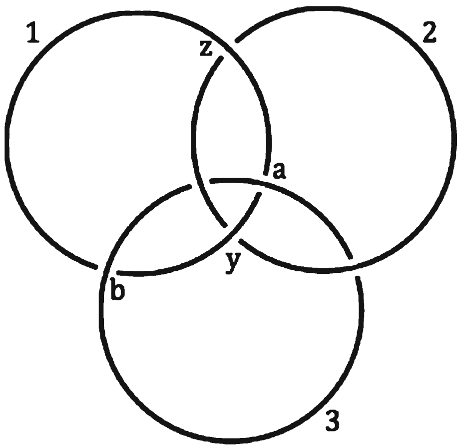
4 The mathematical research paper

The research paper that we provided to participants (Stanford, 1998) is in the domain of geometric topology and concerns *links*, which can be thought of informally as arrangements of circles of rope (called *components*) that are entwined with one another. The paper explores the conditions under which those links can be made *trivial*, meaning that all components can be pulled apart from one another without breaking them. If, as a result of the entwining of components, a piece of one component passes over (or under) a different one in a two-dimensional projection of the link, the point at which they cross is called a *crossing*, and changing which component passes over or under the other is called *changing crossings*. A link is *Brunnian* if any single component can be removed from the arrangement to make the link trivial. A link is *k-trivial* if it has k sets of crossings such that for each crossing set i, if all of the crossings in i are changed, the link will be trivial even if the crossings in one or more of the other k sets are changed as well. Five theorems in total are explored in the paper.

The theorem and proof used for this study establishes that *Brunnian links* of n components are *(n−1)-trivial*. Consider, for example, the Borromean Rings, a well-known example of a Brunnian link illustrated in Fig. 1 (this example is also featured in the research paper itself). We label the components of the link 1, 2, and 3. The link is Brunnian because if either component 1, 2, or 3 is removed, the remaining components will no longer be entangled with one another. The link is also 2-trivial because we can choose two sets of crossings: $S_1 = \{a, b\}$ and $S_2 = \{y, z\}$, for which if all crossings in S_1 are changed, the link will fall apart whether or not we change the crossings in S_2 and vice versa. However, if the unlabeled crossings are changed, the link would no longer be trivial. Note that other sets of crossings can be chosen: For example, we could select the two sets to consist of set S_1 and a set comprised of the unlabeled crossings instead.

The proof that accompanies this theorem argues that for any n component Brunnian link L, we can pick a nonempty proper subset of indices $T \subset \{1, 2, ...n\}$ of the link and create crossing sets S_i comprised of all "under" crossings for each component $i \in T$ (like we did in the above example of the Borromean Rings). If we change all of the crossings in each S_i, we obtain a link $L(T)$ that leaves one or more components entirely "underneath" all of the selected components. The components that were not selected are arranged in the same manner that they were, except with some components removed. But since L was Brunnian, that means that the bottom

Fig. 1 The Borromean Rings



subset of the link, now with some components removed, is trivial. Likewise, since the mirror image of a Brunnian link is Brunnian and the "upper" portion of $L(T)$ is a mirror image of the original link with the bottom components removed, it is also trivial, and hence, the entire link is.

5 Coding framework

Although primarily developed through iterative analysis of participant interviews, our coding framework is also heavily influenced by much of the literature reviewed in the theoretical background. We extend Sierpinska's (1994) resources/acts distinction in order to better understand the processes by which mathematical understandings are constructed. To do so, we identify mathematical knowledge *resources* referenced by participants during their interviews (many of these loosely correspond to the resource types discussed in the theoretical framework—such as definitions, examples, "deconstructions" of a mathematical object, and so forth). We also identify *acts* of understanding as questioning, resolving, or explaining by experts. At least one resource, and often more than one, is included in every act of understanding.

Although these categories were derived for this study specifically and thus could be an artifact of the structure and content of the task provided to participants, we contend that they are applicable generally to a wide range of domains of mathematics. To illustrate this, as well as to make the coding scheme more accessible to readers unfamiliar with the mathematical content used for the study, we include hypothetical examples pertaining to even numbers for each code category. In this section, each code description includes (a) a definition of the code category, (b) a hypothetical example to illustrate how each category can be applied to descriptions of even numbers and the rationale for its inclusion in that category, and (c) a real example obtained from interview data and the rationale for its inclusion in that category.

5.1 Resources

Parent (a) Parents are aspects of some mathematical idea that are inferred from participants' own experiences or understandings but that are not explicitly mentioned in the research paper. (b) In the case of even numbers, one might assume that operations such as addition and multiplication can be performed with even numbers because they are a type of integer, or they may relate even numbers to the practice of matching in order to create two equal groups of objects. (c) In the context of our study specifically, general notions of knot theory and descriptions of knots and links in terms of physical objects such as string or rope were the most commonly occurring parents. In the example below, Mark is using two parents: a colloquial definition of *knot* and his preexisting understanding of *trivial* as the most simple mathematical case of some phenomenon.

> MARK: Well in terms... if I think of a knot, I think of a piece of rope that's knotted. And I assume trivial means there really isn't a knot there.

Fragment (a) Fragments are divisions of a mathematical idea or object into smaller pieces that can be manipulated or otherwise explored. Although fragments can comprise a part of the definition of a mathematical object, this is not necessary. (b) In the case of even numbers, an individual might notice that an even number of objects can be split into two equal groups, or several groups of 2. These groups can then be manipulated or considered as objects in their own right. (c) Popular fragments in our study included the individual components that comprise a link, the crossings where different components pass over or under one another, and subcollections of components or crossings. Below, Ana describes the triviality of a link in terms of each component ("circle") being unlinked from each other or disjointed.

> Ana: ...the rest is trivial whatever that means, and I'm assuming that means the *disjointed circles*.

Example (a) Examples are specific instantiations of the idea being considered that are immediately available to an individual, either via recall or because it is provided in the proof materials. (b) For example, an individual might immediately think of the number 2 as a simple example of an even number. (c) In our study, two specific examples were provided within the text of the proof: the Borromean Rings and the Whitehead Link. In addition, many participants referred to the Hopf link, the simplest nontrivial link configuration comprised of two interlocked components. Below, Mike recalls that the Borromean Rings, a popular structure in knot theory, is one specific example for which if one component is removed, the others fall apart.

> Mike: Yea, *the Borromean Rings* should be.. I know enough.. they have that property.. that when you unlink a component, you get a trivial knot.

Prototype (a) Prototypes are instantiations of the idea being considered that are assumed to be representative of more than a single instance of the idea. (b) A prototype of an even number might be the number 10, if it is used to represent a more general class of numbers ending in 0 which the learner knows are always even. Although prototypes might be similar or equivalent to a formal definition, they are not used as such—but instead as a particularly powerful example for which manipulations and conclusions may be extended to a larger class of instances. (c) Prototypes were the least common type of resource mentioned in our interviews but were used extensively by one participant who was familiar with a generalizable strategy for constructing Brunnian links with n components, Joe. Below, Joe explains that he assumes that any manipulations he performs or properties he observes in his "canonical" Brunnian link construction are true of any other Brunnian link because they are isotopic.

> Joe: So, my definition was sort of, construct a canonical example and say this is, *any Brunnian link is isotopic to this Brunnian link*, so.. it's like a representative of equivalence classes of brunnian links, so...

Construction (a) Constructions are specific instantiations of the idea being considered that are developed ad hoc, usually by combining fragments, examples, and/or prototypes in some way. (b) An individual may construct a specific example of an even number by combining, for example, several groups of 2 ($2 + 2 + 2 + ...$), or by adding a number they know to be even with a prototype of an even numbers ($8 + 10$). (c) Several experts constructed examples by taking an existing example of a Brunnian link and adding a component, or by creating new crossings in the existing link. Below, Mark constructs an example of a Brunnian link by interlocking two of his fingers and then a third. Although the actual structure he creates is not Brunnian (this would be difficult to do using fingers), he describes the construction as though it were by then removing one interlocked fingers and illustrating that the other fingers can be pulled away.

> Mark: Okay. And so if I got something like that *[forms circle with one finger] and [interlocks with another finger, and a third]* something interchanging here, if I remove one of the links the other two come apart, then that's what they're talking about.

Definition (a) Definitions are complete descriptions of the behavior, structure, or properties of the focal mathematical idea, which accounts for all instances of the idea. (b) Definitions of even numbers might include statements such as "an even number is any integer multiplied by 2," or $\{2k|k \in \mathbb{Z}\}$. (c) Several definitions existed in the text of the proof used in our study and were used by participants, such as the definitions of a link, a Brunnian link, or a trivial link. In addition, when experts described the properties or behavior of a mathematical idea in a general way that they explicitly noted would apply to the universal set of objects, this was coded as a definition. Below, Joe is providing a definition of Brunnian by noting that for any number of components, if any one component is removed ("thrown away"), the link becomes trivial.

> Joe: ...he's saying *if we have n-components of Brunnian... whenever [turning page] you look at... whenever you throw away one of the components you have something trivial.*

5.2 Acts

In our coding scheme, each instance of a mathematical resource resides inside an *act* of mathematical understanding. Three such acts are identified: questions, resolutions, and explanations. If an expert states that she does not understand some aspect of the proof, that statement is coded as a question. If the expert is not immediately familiar with some aspect of the proof but is able to use features described within to arrive at some conclusion, that statement is coded as a resolution. Finally, if an expert asserts that she is already familiar with phenomena described in the proof or simply describes that existing understanding, that statement is coded as an explanation. In this section, we present each act along with an example from our data.

Question A turn of talk is coded as a question if the participant articulates a lack of knowledge about some aspect of the proof and does not make an explicit attempt to make sense of that unknown aspect. In the quote below, Joe makes clear in a question that he does not understand what *n-triviality* means.

> Joe: Okay. [takes pencil] Okay, so they're saying something about... n-triviality, *I've never heard of that...*
> Interviewer: Do you have any idea of what that might mean?
> Joe: *Not a clue.*

Resolution A turn of talk is coded as a resolution if the participant is unfamiliar with some aspect of the proof but is able to use other components of the proof to arrive at some conclusion about its meaning. Below, Ana uses several resources to make sense of the definition of *changing crossings*: She first selects a specific example of a Brunnian Link (the Borromean Rings provided in the text of the paper) and manipulates specific fragments (crossings) of that example in a way that she suspects is "changing crossings." She then analyzes the resulting structure using the definition of Brunnian in order to determine whether her interpretation of "changing crossings" is correct.

> Ana: So he's saying here are these crossings, these are in one set and these two are in another set. But then what does it mean to change them? [pause] So suppose I picked um one corresponding to A, what am I supposed to do what does that mean to change them? *I wonder if it means to go from an up crossing to a down crossing, so let's try. [pause] oh yeah, see I do think I'm right*, because that circle is disengaged by changing these two crossings okay. So changing crossings means going from up crossing to down crossing.

Explanation A turn of talk is coded as an explanation if the participant is already familiar with some aspect of the proof and describes their understanding of that aspect without additional investigation. Below, Greg gives an explanation of *n-triviality* (Note that this explanation is not correct. It is the crossings between components that are changed, not the deletion of components themselves that matters for *n*-triviality):

> GREG: So it means I can remove... as long as I pick... I've labeled some set of these links, rings, knots... as long as I've labeled some set of them and I remove any of them, we get triviality.

5.3 Identifying patterns of understanding

Finally, the above codes were combined. Each turn of talk by the participants was coded first as a way of understanding—a question, resolution, or explanation—and then was coded for each resource used within that way of understanding. Below we illustrate how these two codes are used together to with a specific interview passage:

> Mark: Here he says we obtain $L(T)$ by changing, so he's talking about changing crossings, but he doesn't say what he's changing them to and that confuses me.

This turn of talk is coded as a QUESTION because the participant does not proceed to engage in any activities that help him to figure out what "changing crossings" might mean. Within that question, the participant explicitly identifies two substructures or FRAGMENTS of a mathematical link, the *sublink* $L(T)$ that is constructed in the paper by changing all sets of crossings with indices in T and *crossings*.

6 Results

Our main questions for this study, as outlined in Section 2.3, were:

- What are the resources experts use to make sense of unfamiliar mathematics?
- What are the roles of different types of resources for making sense of unfamiliar mathematics?
- How are those resources combined to build an understanding of unfamiliar mathematics?

We organize our results as follows: In Section 6.1, we review general trends in mathematical resource use organized by participant and act of understanding. Our main findings are that when QUESTIONING or EXPLAINING, experts used DEFINITIONS and FRAGMENTS most frequently, and when RESOLVING experts used INSTANTIATIONS (which include EXAMPLES, PROTOTYPES, or CONSTRUCTIONS) and PARENTS most frequently. In other words, it seems that experts were more likely to employ empirical resources when actively making sense of some aspect of the mathematics they did not understand. However, all experts used a variety of resources and did not appear to adopt a clear style or preference for specific types of resources.

Next, in Section 6.2, we describe and illustrate a number of specific patterns that were recurrent in our interviews across participants. We find that when QUESTIONING and EXPLAINING, experts combine empirical (INSTANTIATIONS and PARENTS) and formal (DEFINITIONS and/or FRAGMENTS) resources for the purposes of identifying or exploring *relationships* among different mathematical substructures and properties. However, when RESOLVING, experts often used these resources to deconstruct the focal mathematical object in order to *identify or isolate its specific subcomponents* in a more piecemeal manner. We conclude with a brief discussion of the implications that this study has for our understanding of learning from proof and for mathematics education more generally. We note that although our sample is not large enough to make any claims about the generality of our findings, these results lend some insight into the complex nature of expert learning and provide a preliminary attempt at describing that process in a way that systematically illuminates similarities and differences both within and between participants.

6.1 Frequency of resource use

Across participants in our study, there was considerable variety in the number and type of resources used for different acts of understanding. However, some consistencies did emerge. One of the most simple and illuminating results of our analysis was that across all ten participants, passages coded as RESOLUTIONS involved the highest average number of resources, and with the exception of one participant (Aaron), passages coded as QUESTIONS involved the lowest (Table 1). All depended on a variety of resource types: most frequently fragments and definitions, but also parents and instantiations (Table 2). This is not necessarily surprising: We might expect that as one tries to make sense of something that is unknown, one may rely on many different sources of information. What is interesting is that RESOLUTIONS were most likely to involve FRAGMENTS, INSTANTIATIONS, and PARENTS but involved few or sometimes no DEFINITIONS (Tables 3 and 4).

These aggregate results also reveal that while experts seemed to have different preferences regarding what resources to use for what purpose, most were more likely to use empirical resources (PARENTS and INSTANTIATIONS) for RESOLUTIONS (Table 4), and all of them used many different types of resources throughout the interview (Table 2). In other words, although FRAGMENTS and DEFINITIONS were the primary mathematical resources presented in the paper provided, empirical resources and outside knowledge played a considerable role in these experts' reading of that paper. This is consistent with the hypothesis that experts do not necessarily rely on one method or adopt a consistent preference for what resources they rely on to comprehend proof but instead leverage a wide collection of resources when needed.

Table 1 Average number of resources per QUESTION, EXPLANATION, and RESOLUTION

	QUESTION	RESOLUTION	EXPLANATION	ALL
Greg	2.00	**5.00**	3.17	3.25
Mike	1.50	**2.75**	2.67	2.50
Mark	1.38	**4.50**	2.30	2.15
Joe	1.93	**3.13**	2.28	2.51
Ted	1.75	**3.89**	2.36	3.00
Ana	1.90	**3.06**	2.67	2.63
Saul	1.27	**2.80**	2.75	1.95
Eric	1.45	**2.25**	1.50	1.62
Myron	2.00	**2.89**	2.11	2.36
Aaron	2.33	**2.63**	2.20	2.43
Average	1.75	**3.29**	2.40	2.44

The highest average number of resources for an act type is highlighted in bold for each participant

As we shall illustrate in the next section, experts often used several different types of resources in combination with one another as they worked to understand the new mathematics presented in the paper.

Finally, for most participants, FRAGMENTS were mentioned more often than DEFINITIONS (Table 2) and especially during RESOLUTIONS (Table 1), which suggests that the act of deconstructing and making meaning of specific subcomponents of the mathematical idea is an especially important part of experts' first reading of a mathematical paper. Certainly one would expect specific aspects of the topological links discussed in this paper—such as crossings and components—to be mentioned because they are an important part of the proof presented. It appears, however, that often experts spent a lot of effort defining and understanding these subcomponents in their own right *without explicit reference to the definitions provided in the proof*. This will become more clear in the next section.

6.2 Patterns of resource use

In this section, we delve into further analysis of exactly how and why experts used different types of mathematical resources together to make sense of an unfamiliar mathematical idea. We do not intend this to be an exhaustive catalogue, but rather a beginning look at the interplay between specific resources as they are used together by experts to learn new material. Therefore, we will focus specifically on points of contact between what might be considered more and less formal resources—in other words, between *empirical resources* such as specific INSTANTIATIONS of a

Table 2 Average number of FRAGMENTS, DEFINITIONS, PARENTS, and INSTANTIATIONS per act

	FRAGMENTS	DEFINITIONS	PARENTS	INSTANTIATIONS
Greg	0.72	**1.56**	–	1.11
Mike	**0.97**	0.89	0.11	0.78
Mark	**1.48**	0.73	0.08	0.40
Joe	**1.38**	0.80	0.04	0.23
Ted	**1.05**	0.97	0.26	0.32
Ana	**1.38**	0.94	0.10	0.12
Saul	**0.88**	0.84	0.26	0.28
Eric	0.37	**1.00**	0.39	0.25
Myron	**1.05**	0.77	0.16	0.35
Aaron	**0.95**	0.78	0.14	0.51
Average	**1.02**	0.93	0.15	0.44

The highest average number of resources per act is highlighted in bold for each participant

Table 3 Average number of FRAGMENTS and DEFINITIONS per act

	Fragments				Definitions			
	Q	R	E	All	Q	R	E	All
Greg	–	1.00	**1.17**	1.08	1.00	**2.00**	1.67	1.56
Mike	0.25	**1.50**	1.17	0.97	**1.25**	0.58	0.83	0.89
Mark	0.63	**3.00**	0.80	0.50	**1.20**	0.50	0.50	0.73
Joe	0.86	**2.00**	1.28	1.38	**1.07**	0.74	0.59	0.80
Ted	0.33	**2.00**	0.82	1.05	**1.08**	1.00	0.82	0.97
Ana	1.00	**1.69**	1.44	1.38	0.80	0.81	**1.22**	0.94
Saul	0.45	**1.20**	1.00	0.88	0.73	0.80	**1.00**	0.84
Eric	0.27	**0.50**	0.33	0.19	1.00	1.00	1.00	1.00
Myron	0.71	1.22	1.22	1.05	0.86	**0.89**	0.56	0.77
Aaron	0.67	**1.26**	0.93	0.95	0.67	**0.95**	0.73	0.78
Average	0.57	**1.54**	1.02	1.06	0.94	**0.96**	0.90	0.93

The act type with the highest average frequency of each resource type is highlighted in bold for each participant

mathematical object or the use of a nonmathematical PARENT and more *formal resources* such as FRAGMENTS and DEFINITIONS, which symbolically and abstractly define the properties and relationships that characterize that object. As we describe in the previous section, for most participants, RESOLUTIONS were most likely to include the most resources together at once and were more likely to include PARENTS or INSTANTIATIONS. Below, we shall first review how resources were used together in RESOLUTIONS, and then move on to briefly discuss how they were used together in QUESTIONS and EXPLANATIONS.

6.2.1 Resource use in resolutions

As we expected, experts did use specific (and hence less formal) instances of the mathematical objects being discussed in the paper—namely topological links—to make sense of certain terms, relationships, or properties as they appeared in the proof. However, they used different types of instantiations in different ways. Typically, experts used EXAMPLES provided within the research paper as objects to manipulate and deconstruct in order to isolate and identify specific structures associated with a given term. Once these specific structures were identified, they used them to create or manipulate CONSTRUCTIONS and PROTOTYPES to understand the relationships between mathematical substructures and the resulting logic contained within the proof. Finally, experts used colloquial PARENTS during the early stages

Table 4 Average number of INSTANTIATIONS and PARENTS

	Instantiations				Parents			
	Q	R	E	All	Q	R	E	All
Greg	0.33	**2.00**	0.63	0.63	–	–	–	–
Mike	–	0.33	**0.50**	0.32	0.17	0.08	–	0.08
Mark	–	**1.00**	0.20	0.25	0.10	–	**0.13**	0.08
Joe	–	0.30	**0.38**	0.28	0.03	**0.08**	–	0.04
Ted	0.08	**0.56**	0.55	0.38	0.18	**0.33**	0.25	0.26
Ana	0.10	**0.25**	–	0.14	–	**0.29**	–	0.10
Saul	–	**0.60**	0.25	0.20	**0.50**	0.20	0.09	0.26
Eric	0.09	**0.50**	0.17	0.19	–	**0.25**	0.09	0.11
Myron	0.29	**0.56**	0.22	0.36	0.11	**0.22**	0.14	0.16
Aaron	**1.00**	0.26	0.27	0.32	**0.27**	0.16	–	0.14
Average	0.19	**0.64**	0.32	0.31	0.11	**0.18**	0.09	0.13

The act type with the highest average frequency of each resource type is highlighted in bold for each participant of the interview to identify both the components of and relationships between mathematical objects.

In Table 5, we feature three instances where experts used the examples, definitions, and fragments provided by the proof in order to isolate and identify a specific substructure of links. In all three cases, they are using these examples to resolve the question *What is X?* by applying the existing definitions and fragments to the provided examples in order to isolate unknown features and associate those features with the terms they may represent. Ana identifies what it means to *change crossings* by manipulating components of the provided example, observing the result, and concluding that since the result was as predicted by the definition of Brunnian, her manipulation was, in fact, the manipulation associated with the term *changing crossings*. Mark uses a provided example and a definition to construct his own *crossing set* and compare it with the crossing set provided by the paper, to determine whether his understanding of a crossing set matches that provided. And Eric refers to both of the provided examples to identify which components are common to both, in order to identify the substructures ("loops linked together in a different way") that will serve an important role in the proof.

In contrast, when testing or attempting to understand in more general ways how substructures of the mathematical object were related, experts often used constructions rather than examples. These constructions were dynamically generated as experts identified which components of the mathematical idea were related, and in which ways. In other words, while EXAMPLES were used to answer the question *What is X?*, CONSTRUCTIONS (and, in the case of one participant, a PROTOTYPE) were used to answer the question *How does X relate to or affect Y?*. These constructions were generated by experts by combining specific fragments together into a novel structure, or by adding fragments to the examples provided by the proof. At times they did not exhibit the relationships and properties that they were constructed to explore,

Table 5 EXAMPLES, DEFINITIONS, and FRAGMENTS were combined to identify specific terms

ANA: Identifying *changing crossings* using a FRAGMENTS, a DEFINITION, and an EXAMPLE

So he's saying here are these crossings, these are in one set and these two are in another set. But then what does it mean to change them? So suppose I picked um one corresponding to A, what am I supposed to do what does that mean to change them? I wonder if it means to go from an up crossing to a down crossing, so let's try. Okay so if I turn this into a down crossing, and this also, that circle... oh yeah, see I do think I'm right, *because that circle is disengaged by changing these two crossings okay. So changing crossings means going from up crossing to down crossing...*

MARK: Identifying *crossing set* using FRAGMENTS, a DEFINITION, and an EXAMPLE

...so you're defining a crossing and giving it the same subscript if it involves the same two components I think. Is that right? So this one involved 2 and 3 here and here. Alright so... S2 involves 1 and 2, and I assume that's supposed to be S3, is that right? Or no that's not right. This is S3, S3, S1... ok this is S1 alright. *So if you take two of them and list all the places that they cross, that's what this is. It divides it up into a partition.*

ERIC: Identifying *link* using a FRAGMENT and two EXAMPLES

I'm trying to see how these are part of the same object. In other words, these have a common context, these are examples of links, or... *see in one case you've got a circle and a figure 8 and they're linked around in a certain way is if they were a piece of string, and here you've got three loops which are linked together in a different way*. So this has got to be two examples of this general thing he's talking about.

and sometimes they were incomplete and only a subset of the fragments of the mathematical object being explored. In Table 6, Ana builds a construction by adding a new component to an example already provided in the proof in order to determine why the strand of a component in one of the line's sets will always pass over a strand in that set's complement. Joe explores what is required for a link to be *n*-trivial by constructing an ad hoc set of components that cross over one another several times. Aaron creates a drawing of only lines crossing—ignoring that components are closed loops—in order to better understand how a link can be independently undone in *n* ways.

Finally, experts used PARENTS early on in their reading of the paper both to identify what specific terms in the proof referred to, as well as to establish the potential relationships between different components of the mathematical objects discussed. However, when parents were used, they were less likely to be used in the specific, deconstructive-and-reconstructive manner that examples, constructions, and prototypes were. In other words, while experts referred to several of the specific fragments that comprised instantiations (either to identify what those fragments were, or to identify how they were related), experts used parents in a more unitary, analogical manner to more broadly define the nature of the mathematical properties they illustrate. In Table 7, Mark uses his colloquial knowledge of knotting and mathematical triviality to determine that a trival link may be one without a knot, Ted uses the same analogy to determine what might constitute a crossing, and Saul uses his colloquial understanding of the term twisting to determine that such a manipulation may not involve disconnections.

Table 6 CONSTRUCTIONS built from FRAGMENTS and EXAMPLES were used to test relationships

ANA: A CONSTRUCTION from FRAGMENTS and an EXAMPLE to test set relationships.

Clearly it has to go under and over, but why does it have to hit someone in the complement. So here we have one two and three, um, and so what if I picked T it's just gonna be 2. Here's 2, so... um... any strand from this, *so here's a strand* [Ana adds a component to form the Borromean Rings]... okay look it's... so it's still true when there are three of them but *what if I had a fourth one like this* [Ana draws an additional component crossing over and under the components of the Borromean Rings], but of course I don't know if that's a Brunnian link anymore. Okay. So if I were really reading this for myself, here's what I would do is I would sort of put a question mark and say does Brunnian imply this, and I'd have to check this.

JOE: A CONSTRUCTION from FRAGMENTS to test a DEFINITION.

So, I think... I know, I think what he's saying is you have all these crossings... I have to draw it, you have a bunch of crossings going on, [drawing], and he's saying you make sets by grouping these together... I don't know yet, I'm gonna say no right now. I'm gonna say... I'm gonna say... I'll just maybe do three groups here [draws] and I think he's saying... what can you do... [reads] once you change the crossings, you flip all the crossings in any one of those sets... [referring to drawing] this isn't one of those examples, but if it were... then changing the crossings in one of those sets would make the whole thing unfold, regardless of what you did to the others.

AARON: A CONSTRUCTION from a collection of FRAGMENTS to clarify a DEFINITION.

so you've got... and then there are also these, so i don't really know what's happening, i dont know if he will explain, but you've got [draws a hashmark-like thing] lots of crossings, and i'd imagine that you have something here to undo and something here to undo and something here to undo and you can do it n different ways, and um the undoing is independent of what's happening at the other places.

Table 7 Parents were used "wholesale" to define specific terms and possible relationships

Mark: Using a parent to establish the definitions for *link* and *trivial*.
Well in terms... if I think of a knot, I think of a piece of rope that's knotted. And I assume trivial means there really isn't a knot there. (Ok) So I would assume a trivial link ... I wonder if it's just a knot that's closed at the two ends and there's not a knot there. That's what I would guess.

Ted: Using a parent to identify the *crossing* fragment.
Well I'm guessing it means if you have two ropes then they're going like this [forms an x with his fingers].

Saul: Using a parent to establish the definition of *twisting*.
Well, I know... in, in normal everyday language, what twisting could be, I'm not sure in this conception of it topologically, I assume it means some sort of a transformation, which um, uh, changes the the certain kind of distortions but doesn't allow other kinds, maybe it won't allow disconnections.

6.2.2 Resource use in questions and explanations

While empirical resources such as instantiations and parents were often used during resolutions, they were used less during questions and explanations. When instantiations were used in questions, they were used to emphasize the unknown relationships between fragments and definitions, or to describe the manipulations for which effects were still unknown. More often than in solutions and explanations, these instantiations were ad hoc constructions, and they focused on specific subcomponents of the mathematical idea rather than the entire object of study. In Table 8, Joe asks a question about the relationship between two fragments in the proof,

Table 8 Questions identify unknown relationships among definitions and fragments

Joe Questioning the relationship between the fragments $L(T)$ and S_n.
So, the link... okay, so what confused me about this is the link L, this L of T depends on the choice of these sets of crossings S_1 through S_n, so I'm thinking... and which also depends on the diagram, so I'm... it doesn't seem like a well-defined concept at the start, I don't know yet.

Myron Identifying fragments needed to relate the definitions *Brunnian* and *n-trivial*.
At this point if I were serious and in this field, I would think how would I actually prove this, um, so I've got an n-component link, so each of these guys has to intersect, has to have crossings, with at least one other, there could be... I suppose each guy, each... hmm. Each one has to have at least two crossings I suppose, and a certain number of crossings, I don't know how you would show this.

Aaron Using a construction to identify an unknown relationship between *crossing set* and *component* fragments.
[Draws a central component, with incomplete linked components] If I have one link arbitrarily, and I want to find $n-1$ other links, so I want to find $n-1$ other sets by saying pick all the crossings involving one of the other links, another of the other links, this is going to be $n-1$ disjoint sets, because there's not a crossing that involves 3 links. So the fixed link and 1 link, the fixed link and 2 links, up to the fixed link and $n-1$ links. Now, I would imagine from the definition these have to be nonempty sets of crossings, but maybe not. Or maybe that's something which he's gonna tell us, that they're all nonempty. Namely that there's some link that intersects every other link? I don't know.

Myron enumerates the fragments required in order to establish the relationship between two DEFINITIONS, and Aaron constructs a generic partial example of a link to identify which relationships between FRAGMENTS he needs to better understand. This further establishes that in our study (and perhaps more generally when learning about new mathematics from disciplinary materials), experts were more interested in assigning meaning to specific components and identifying substructures of the mathematical ideas expressed, rather than immediately seeking to understand how all of those components relate to the stated theorems and definitions in a unified way. Indeed, it seems that QUESTIONS served to establish exactly which syntactic aspects of a mathematical idea must be tied together in order to create a proof, while INSTANTIATIONS and PARENTS are used to populate that structure during RESOLUTIONS by clarifying DEFINITIONS, identifying FRAGMENTS, and testing potential relationships.

In this way, QUESTIONS enabled experts to identify the mathematical resources—specifically, the FRAGMENTS and DEFINITIONS—that they knew were connected, but for which they did not understand the nature of that connection. It makes sense, then, that questions most closely mirrored the patterns of dissemination in the paper itself, and that FRAGMENTS and DEFINITIONS played a more central role in their own right than they did when involved in SOLUTIONS, where they were often referred to from within the context of specific INSTANTIATIONS.

Finally, like questions, EXPLANATIONS dealt with the relationships that existed between the more formal FRAGMENTS and DEFINITIONS presented in the proof. Unlike questions, however, as experts explained known terms and relationships, they still referred to EXAMPLES or CONSTRUCTIONS to illustrate their explanation. These informal resources here, in contrast to their use during resolutions, were used illustratively and "wholesale"—that is, experts referred to those examples as an instance of some more formal idea that they were explaining, but did not elaborate on the specifics of how those examples fit. Even when constructions were used rather than preexisting examples, experts used them to illustrate a final result, relationship, or behavior of

Table 9 Explanations

MARK A CONSTRUCTION of *component* FRAGMENTS to explain the DEFINITION *B*runnian.
Okay. And so if I got something like [interlocks two fingers] that and something interchanging here [locks another finger], if I remove one of the links the other two come apart, then that's what they're talking about. Okay, so back here... so that's what he meant in his very first sentence, ok.

MYRON The *Borromean Rings* EXAMPLE to explain the DEFINITION *Brunnian* and *component* FRAGMENTS.
That means just that it comes apart. So the Borromean rings are just three independent rings, and if you take any one out then the other two are unlinked and it's a counterexample to this putative conjecture, a conjecture that one might make that the linking number, the pairwise linking number of rings is all you need to tell if things are linked.

JOE A CONSTRUCTION from FRAGMENTS to explain the DEFINITION of *Brunnian*.
So, if you just... cut one, and take it away from the link. [Okay.] Yea. So you can't do it with... uh... well, (pause) you can't sort of do it nontrivially with two, like if you want to link two you have to go like that [interlocks fingers], but if you want to do three, you can do it where no two of them are linked, but you can throw a third one in there not linking it to any two, but linking the two together.

the mathematical structure, rather than to identify what its subcomponents were or how they interacted. In Table 9, Mark and Joe both use constructions comprised of components to explain the behavior that characterized a Brunnian link, and Myron indicates that the Borromean Rings are an instance of a Brunnian link.

7 Discussion

Often, expertise in mathematics—especially in the realm of proof and formal argument—is described as an interaction between a diverse set of reasoning strategies. While we are beginning to understand more about how this interaction occurs in proof construction (Weber & Alcock, 2004; Inglis et al., 2007), this study shows that it also occurs in the context of *learning new mathematics* from proof. In our study, experts navigated this interaction in very specific and intentional ways—to isolate and identify exactly what mathematical substructures are associated with the symbols that are presented and manipulated in the proof (rather than only to verify or better understand the relationships established therein). They used definitions, symbolic structures, and specific examples to deconstruct mathematical ideas into their fragment components. To test the definitions and theorems presented in the proof, they then often recombined these fragments in novel ways. Indeed, the interactions between different reasoning strategies observed in this study might best be described not as a movement between, but rather the simultaneous connection and coordination among mathematical resources such as definitions and mathematical objects on one hand and specific instances of those objects and contextual knowledge on the other.

It makes sense that experts spend a great deal of time and effort understanding the underlying mathematical objects involved in a proof, especially one that is unfamiliar. This has also been observed in proof construction: Weber and Alcock (2004) note that although undergraduate students attempted (logic-based) proof productions, they were unable to do so in part because they were unable to “instantiate the groups in a meaningful way”—in other words, they were unable to recognize exactly what pieces and properties of groups constitute the symbols upon which syntactic proofs can be built. They contrast this with a doctoral student who, upon instantiating a single group, was able then to identify a property that was critical to developing a successful proof.

We believe that our findings have some important implications for education: especially for the need to provide students experiences with and opportunities for the *deconstruction* and *coordination* of mathematical resources. By *deconstruction*, we mean the explicit attention to the different fragments that make up a mathematical object of study (or, conversely, how a mathematical object of study can be decomposed into different fragments in different ways) and how those fragments can be explored. By *coordination*, we mean the ways that those deconstructions, along with existing mathematical resources such as examples, definitions, fragments, and even everyday understandings, can then be brought together to not only underpin, but serve a generative role in the testing and generalization of mathematical relationships.

For example, experts in our study used terminology in the paper, provided examples, and their own ideas and manipulations to identify specific components of the mathematical objects used in the proof. After identifying these components, experts often created their own examples by recombining these different component types in different ways. This simultaneously provided the experts with a specific, manipulable instantiation of the idea as well as a means to hypothesize and test more general relationships. This sort of activity has also been shown to help young students move toward sophisticated proof-like behavior. Consider, for example, the case of third graders who by exploring different methods for decomposing an even number into different component groups (many groups of two, or two even groups), were able to develop a sophisticated explanation of why adding two odd numbers produces an even number (Ball, Hoyles, Jahnke & Movshovitz-Hadar, 2002). While manipulatives and established symbol systems (such as base-ten blocks or the place value system) do give students opportunities to practice structuring and deconstructing mathematical objects in different ways, our findings suggest that an important activity may be to challenge them to find new units into which those objects can be deconstructed, and using those units to construct new examples of mathematical objects with special properties of behaviors (such as an even or prime number). This may also help students to bridge the gap between specific examples and methods for generalizing those examples while still providing a semantic basis for reasoning about mathematical relationships.

Our goal with this paper was to make two major contributions to the study of mathematical knowledge construction: an analytic framework that helps to reconcile research into expert mathematical reasoning, and the investigation of mathematicians reasoning about novel mathematical material. We have described in depth the RACUM (Resources and Acts for Constructing and Understanding Mathematics) framework and have shown how such a framework can be used to illustrate the ways in which mathematicians coordinate different sorts of mathematical resources in order to make sense of unfamiliar mathematics. We found that rather using examples as illustrative tools, experts used them in combination with other available resources in order to identify the units from which the object of study can be flexibly constructed and tested. These deconstructed components were then employed by experts to understand the individual relationships and substructures that defined the mathematical object of interest and the behaviors that underlied its formal definition. While certainly this is not the only practice of importance in learning and understanding mathematics, it is a rich and complex one that can give educators some insight into successful individual construction of knowledge, and deserves detailed and systematic study.

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