Fostering Mathematical Discovery: One Tutor's Strategies for Ushering the Construction of Proportional Schemas Via Mediated Embodied Interaction[1]

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How do instructors lead students to discover mathematical principles? Are extant frameworks for the analysis of instructors' professional practice still adequate, given recent advances in educational technology and the range of pedagogical approaches? We examined videographed data from the implementation of an embodied-interaction, discovery-based design for proportions. To analyze our data, we attempted to use professional perception cognitive–ethnographical frameworks (Goodwin, 1994; Stevens & Hall, 1998). However, several types of tutorial tactics "fell between the cracks" of these frameworks. We tabulate and exemplify these tactics to demonstrate how the frameworks may be expanded so as to apply to the current scope of pedagogy and media.

0. Background and Objectives

The Mathematical Imagery Trainer is a computer-based pedagogical system designed for students to ground core conceptual notions of curricular subject matter through embodied interaction that becomes progressively shaped and signified in disciplinary forms. Using remote control media, students engage a problem-solving activity involving the manipulation of virtual objects, with the objective of effecting a designated goal state of the system, such as making a screen green. Through discovery and practice, students develop interaction strategies by which to maintain the system's goal state while moving their hands, and thus they effectively develop a perceptuomotor schema that the designers view as the embodied conceptual core of the targeted subject matter. The instructor then introduces into the problem space new symbolic and referential resources, such as a Cartesian grid layered onto the interactive objects. Students recognize in these resources enactive or semiotic affordances for enhancing, explaining, or evaluating their interaction strategy. Yet in so doing, students implicitly distribute their strategy subgoals upon these resources' embedded affordances, so that their strategy becomes newly instrumented and effectively reconfigured and signified in ways that better resemble disciplinary practice. We have documented this two-stepped process and proposed the phrase "hooks and shifts" to describe the phases of bootstrapping mathematical notions via appropriating available cultural tools (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011).

The empirical context of improving and implementing the Mathematical Imagery Trainer continues to serve us as a laboratory both for improving the pedagogical artifacts and for developing theoretical models. In particular, we are exploring relations among grounded-cognition and sociocultural theory (Botzer & Yerushalmy, 2008; Trninic & Abrahamson, in

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press). Yet in addition to promoting pedagogical technology and theory, this line of work may inform teachers' discovery-oriented pedagogical practices.

The literature does not appear to provide teachers and teacher developers with principles for the facilitation of discovery-based instruction. For example, Ball, Thames, and Phelps (2008) elaborate on Shulman's (1986) seminal construct of pedagogical content knowledge to articulate dimensions of content knowledge for teachers. Whereas we find in our data great resonance with many of their dimensions of specialized content knowledge, still their paper leaves open the question of how to apply their model and, more specifically, how to facilitate activities such that students discover principles of the target content. Our current study attempts to fill this gap in the literature by articulating several types of pedagogical actions supporting reform-oriented mathematics instruction. In particular, we are trying to characterize what it is that instructors do that fosters students' hooking and shifting with mathematical artifacts.

We thus returned to our videotaped footage of the twenty-two 4th – 6th graders who participated in our tutorial clinical interactions, and we began to tag and classify the tutors' multimodal utterance in light of the emerging discursive context. It soon occurred to us that professional-perception frameworks (Goodwin, 1994; Stevens & Hall, 1998) hold much promise in developing a systematic characterization of the tutor's "tactics," as we began to call these instructional moves. However, certain qualities of our pedagogical design—its discovery-based, embodied, and semantically complex nature—appeared to require elaborations on these methodologies. This paper reports on a set of tutorial tactics that have not been previously ~~un~~articulated yet we view as instrumental in fostering student discovery of mathematical notions through engaging in future technology. By embracing some of these elaborations, professional-perception frameworks might bear greater methodological traction on guided, discovery-oriented, embodied-interaction activities. This proposed theoretical expansion may be beneficial to learning scientists interested in supporting and understanding mathematics learning, given the increasing ubiquity of embodied-interaction technology.

Section 1, below, builds context for this study by explaining the Mathematical Imagery Trainer as a design response to some enduring challenges of school mathematics. Section 2 focuses on qualities of our interactive design that appear to require certain expansions to the professional-perception frameworks. Section 3 presents empirical support for the utility of these expansions by way of a table that summarizes and exemplifies empirical evidence for a set of discovery-oriented tutorial tactics that are not easily captured in the existing frameworks. Finally, Section 4 offers summative comments, and Section 5 notes some limitations and implications.

1. Learning is Moving in New Ways: a Design for Proportion

In this section we introduce the Mathematical Imagery Trainer (hence "MIT"), an technological design conjectured to support deep learning of mathematical content. In particular, we will discuss an MIT for proportional relations (hence, "MIT–P").

The subject matter of ratio and proportion is didactically essential, because it underlies high-school content. However, many students in elementary school and beyond experience difficulty in understanding and using the core notions of proportionality. In particular, students incur difficulty in developing fluency with proportions that builds upon, yet is differentiated from, simpler non-multiplicative concepts, notations, and procedures (Lamon, 2007).

We approached this design problem by analyzing the target content from an embodied-cognition perspective that implicates human reasoning as grounded in spatio–dynamical imagery

(Barsalou, 2010). An appeal of this hypothesis for researchers of mathematics education is its resonance with fundamental tenets of genetic epistemology, essentially the thesis that mathematical concepts emerge through reflection on the enactment of perceptuomotor schemas (Piaget, 1968). The grounded-cognition hypothesis further resonates with the thesis that mathematics is a multimodal, multi-media, and multi-semiotic praxis (Bamberger, 1999; Bautista & Roth, 2012; Lemke, 1998, 2003; Nemirovsky, 2003; Núñez, Edwards, & Matos, 1999; Radford, 2002; Rotman, 2000; Skemp, 1983). We therefore conjectured that some mathematical concepts are difficult to learn because everyday experiences do not occasion opportunities to embody and rehearse perceptuomotor schemes underlying those concepts. Specifically, we conjectured that students' canonically incorrect solutions for rational-number problems—"additive" solutions (e.g., "2:3 = 4:5" or "2/3 = 4/5" - cf. Behr, Harel, Post, & Lesh, 1993)—indicate a lack of kinesthetic–visual action images to ground proportion-related concepts (Fischer, Moeller, Bientzle, Cress, & Nuerk, 2011; Goldin, 1987; Pirie & Kieren, 1994).

Accordingly, we engineered an embodied-interaction computer-supported inquiry activity for students to discover, rehearse, and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation. At the center of our instructional design is the Mathematical Imagery Trainer for Proportion (hence "MIT–P"; see Figure 1, below), which is an embodied-interaction system designed to foster the development of perceptuomotor schemas grounding notions of proportion. Participants used both hands to remote-control a pair of virtual objects on a computer display monitor, one object per each hand, in attempts to "make the screen green." When students first engage in this activity, the screen will be red (see Figure 1a). Unknown to them, the screen would be green only when these two objects' respective heights above the bottom of the screen relate by a particular ratio that is set on the tutor's computer console. Students move their hands about until they happen to find a combination of cursor locations that corresponds with a green screen. If the ratio is set at 1:2, the screen is green only when the right-hand cursor is twice as high above the bottom of the screen as compared to the left-hand cursor (see Figure 1b).

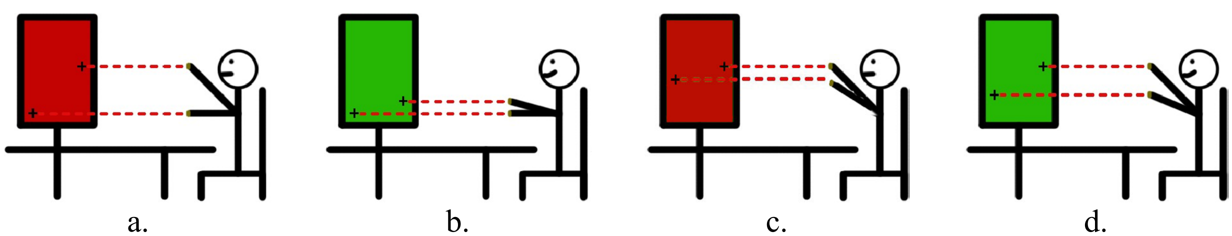

Figure 1. The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 ratio, so that the right hand needs to be twice as high along the monitor as the left hand. A schematic interaction sequence: (a) while exploring, the student positions hands incorrectly (red feedback); (b) stumbles on a correct position (green); (c) raises her hands *maintaining constant distance between* them (red); and (d) corrects position (green). Compare 1b and 1d and note the different distances between the cursors.

Once children discover a "green spot," we ask them to find another green spot. Invariably, they respond by moving their hands up or down, maintaining a *constant* spatial interval between the crosshairs, which causes the screen to turn red (see Figure 1c), whereupon they correct back to a green screen by adjusting the distance (see Figure 1d). Eventually, they establish a relation between the hands' elevation and interval, stating, for example, that, "The higher you go, the bigger the distance" (compare Figures 1b and 1d).

The interview protocol then calls for the researcher to introduce a grid onto the screen (see Figure 2a-c, below), and students typically respond by re-articulating their qualitative strategies quantitatively. Recall that students begin with statements such as "The higher you go, the bigger the distance." Once the grid appears, students typically determine an additive interaction scheme that mathematizes their strategy, for example, "For every 1 that I go up on the left, I go up 2 on the right." Next, numerals are introduced upon the grid (see Figure 2d, below), and students determine an explicit multiplicative scheme, for example, "The right one is always double the left one." This scheme enables them to predict and evaluate any green spots before enacting them. The interview continues with other ratios, such as 1:3, 2:3, 3:5, etc.[3]

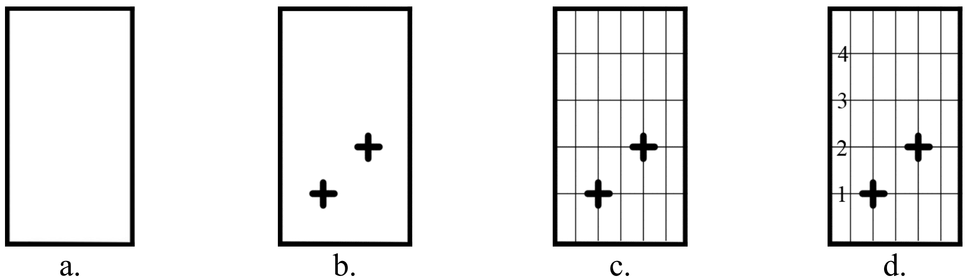

Figure 2. MIT-P display configuration schematics, beginning with (a) a blank screen, and then featuring a set of symbolic objects incrementally overlain onto the display: (b) crosshairs; (c) a grid; and (d) numerals along the *y*-axis of the grid. Not featured here is a ratio table.

Figure 3. Child and tutor co-remote-control two cursors on a computer monitor. The child is attempting to accomplish the designated task objective of making the screen green. The instructor is structuring the inquiry process by pacing the left hand as an independent variable.

Throughout the interview, the tutor poses tasks, clarifies instructions, and intervenes in the child's inquiry process with questions, hints, demonstrations, inference prompts,

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[3] The protocol then continues with a final hands-on activity that we do not treat in this paper. Therin, the control mechanism is changed from manual to numeral: we introduce a ratio table that students need to fill in, and then the computer "plays out" the number inputs by moving the crosshairs automatically from one number pair to the next and giving the appropriate color feedback. We enable students to go back and forth between these interaction modes.

collaborative enactment, etc. For example, in Figure 3 (see above) a tutor and child co-operate the two remote-control hand-held devices. It is precisely these types of rich interactions that this paper attempts to characterize and classify, because we believe they are essential to effective instruction, at least in the context of embodied-interaction designs. Auspiciously, the inherent dynamical physicality of MIT inquiry-based activities bears the methodological advantage of making visible—for the instructor, for the analyst—nuanced aspects of cognition and instruction that often cannot be seen but only inferred.

Having described the activity design and outlined students' typical responses, we will step back to highlight several dimensions along which the MIT activity differs from naturalistic and vocational learning. These differences, we believe, explain the apparent shortcomings of professional-perception frameworks to capture aspects of MIT-based tutorial interaction. The empirical sections of this paper will be then be offered as evidence supporting this thesis.[4]

2. Challenges of Embodied-Interaction Discovery Activities for Science of Learning: Improving Methodological Tools vis-à-vis Experimental Practice

The forms of activities at the center of our inquiry are of the general type "embodied-interaction," and the structure, substance, framing, and facilitation of these activities are inspired by discovery-oriented pedagogical philosophy. As such, these activities, and hence the empirical data gathered at our implementation sites, differ from what one is likely to see in settings where vocational instruction takes place. We believe that these differences should bear on how tutorial interactions such as these are researched. In particular, we submit, experimental learning activities may require a stretching of extant analytical frameworks so as to capture how a student is learning and how an instructor is guiding this learning. In practice, this should compel researchers to adjust how they attend to, parse, code, model, and describe aspects of interaction. Below, we articulate three dimensions along which embodied-interaction discovery-based instruction appears to be fairly unique in ways that bore on how we analyzed our empirical data.

First, research on the development of professional vision has predominantly treated explicit instruction that might be characterized as "show and tell" (but see Mariotti, 2009; Radford, 2010). Yet as educators who believe in the cognitive, affective, and deutero advantage of discovery learning over explicit instruction (Bateson, 1972; Freudenthal, 1991), we are interested in understanding how instructors usher learners toward insight—a "show and *don't* tell" pedagogy, if you will. Therefore, in order to investigate guided inquiry, we submit that interactions could be studied as bi-directional and dialogic, rather than unilateral and expository.

[4] MIT tasks are defined in terms of a specified goal state of the interactive system, which the student is to effect—a target phenomenal invariance or dynamical conservation that the student is to generate. The MIT-P task is to sustain a green screen while bimanually manipulating two virtual objects on the computer screen. As a learning activity, this task is dramatically different from traditional schoolwork, because the solution method is unknown to the child. Moreover, this task is different from what mathematicians often do, because there is no theorem to prove (but see Thurston, 1994, on mathematicians' "aimless" yet highly productive exploration). Rather, this task is closest to forms of inquiry that scientists engage, for example a botanist or entomologist who first encounter a specimen of an unknown species and are trying to understand its properties, or a chemist who has built a new compound and is attempting to determine its reactions to various agitations. But then again, scientists who discover an undocumented phenomenon, material, or star, etc., do not know a priori what specific behaviors they might witness (cf. "green"), so that their interactions with the phenomenon are not initially oriented toward generating any specified goal state. As such, the MIT task is positioned between closed- and open-ended inquiry-based learning activities.

Namely, the instructor has to understand what the students are looking at and how they are looking at it just as much as, if not more than, the child has to understand what and how the instructor is observing (cf. Newman, Griffin, & Cole, 1989; Sherin, Russ, Sherin, & Colestock, 2008; White, 2008). As such, a study of students' emerging professional vision as mathematicians and a study of mathematics tutors' professional practice are in effect two mutually informing sides of the same ethnography.

Second, research within the paradigm of instructional ethnography often models conceptual ontogenesis in terms of learners developing from not-knowing to knowing, such as in disciplining the novice eye to interpret brain images (Alač & Hutchins, 2004). Yet we are also interested in how people learn by coordinating among a *set* of conceptually meaningful views on a situation, that is, by shifting from knowing-one-way to knowing-another-way (cf. Godino, Font, Wilhelmi, & Lurduy, 2011, p. 257). Instructors ushering this particular type of shift, it seems to us, do not orient as much as *re*orient learners' view toward the visual displays—a pedagogical objective that plausibly demands more nuanced forms of interaction and, accordingly, more nuanced models of interaction.

Third, PP ethnographic research predominantly treats cases of passive perception, in the sense that learners are guided to make sense of preexisting perceptual stimuli in the domain of scrutiny. In embodied-interaction designs, however, learners themselves generate the data they analyze—their posture and motion are inextricable aspects of the emerging information. As such, the learners' physical actions are not only pragmatically subservient to the exploration, they are epistemically central to field of scrutiny—they *are* the data (cf. Kirsh & Maglio, 1994). We thus adhere to a conceptualization of perception and action as cognitively inextricable aspects of perceptuomotor schemas. Accordingly, we attend to what learners are doing as much as to what they are saying. Furthermore, we monitor how the learners are guided to develop these ways of doing, just as we monitor how they are guided to develop ways of seeing, especially given that our design fosters actions that only later become signified mathematically. As such, the description and documentation of actions are brought to the fore of our analyses, not only as supplementary gestural information that may disambiguate verbal utterance but as integral components of learning. Following some brief comments, below, on the methodology of this study, we will present its results.

The corpus of data consists of eighteen videographed sessions, in which Grade 4-6 students from a K-8 suburban school in the San Francisco East Bay participated voluntarily in either individual or paired task-based semi-structured clinical interviews (duration: mean 70 min; SD 20 min). The students had not been exposed formally to the concepts of ratio and proportion, and in any case pre-intervention assessments demonstrated that they did not have proportion-related schemas as available means of organizing number pairs into meaningful sequences.

We analyzed the data both as individuals and collaboratively (Schoenfeld, Smith, & Arcavi, 1991) with the objective of seeing, agreeing upon, naming, and sorting the tutor's tactics. We set off with only a broad working definition of tutorial tactics as "on-the-fly discursive actions for productively mobilizing student inquiry during actual tutorial interactions." We treated our emerging finding of tactics using principles from grounded theory (Strauss & Corbin, 1990), by which categories emerged and were checked across the corpus of data. Yet in addition to this "bottom up" approach, our categories became progressively informed "top down" by professional-perception frameworks, as we realized the purchase of these frameworks on our data. For example, we created categories corresponding with "highlighting" and "coding" that are used in those frameworks respectively to designate the showing and naming of features in the domain of scrutiny relevant to the practice. We were also informed by interaction parameters we had previously identified as governing these same students' inclination to adopt the mathematical instruments and bootstrap advanced strategies (Abrahamson et al., 2011).

Coding a tutorial tactic is a nettled business, because one wishes to code the tutor's discursive actions, and yet the ad hoc meaning of each utterance is necessarily informed by the interaction leading up to it. We decided to code according to the tutor's presumed intention, given both the history of the interview up to that moment and the next goal in the protocol. However, we did not code these utterances according to their apparent effect on the student's behavior, because the student would not necessarily respond to the action per the tutor's intention or not respond at all. In these evaluations, we adopted a phenomenological, naturalist, hermeneutic–dialectic approach, in which participatory analysts first act as the measuring tools, then interpret and negotiate their judgments (Guba & Lincoln, 1982, 1989, 1998; Harris, 1976).[5]

Through the analysis process, it soon became evident that a great proportion of what the tutor was doing could be interpreted as domain-general discursive practice, such as dialogue maxims (Grice, 1989) and means of regulating conversation (Schegloff, 1996). Furthermore, the tutor's actions could largely be interpreted as means of "entering the child's mind" (Ginsburg, 1997), that is, as clinical techniques for probing the child's ongoing thoughts. This is to be expected in a task-based semi-structured clinical interview that was designed so as to evaluate the utility of a set of instructional materials and accompanying facilitation protocol (diSessa, 2007; Goldin, 2000). Yet as we became adept at seeing these many aspects of the tutor–student interaction, we could better discern against their background a set of tactics that appeared to be unique to discovery-oriented instructional methodology for hands-on inquiry learning. Importantly for our thesis, these tactics did not appear to lend themselves too well for coding via professional-perception categories.

3. Results and Discussions

The objective of this study is to increase our understanding of instructional practices supporting student reinvention of mathematical concepts. Our data are a set of tutorial clinical interviews from a design-based research project that has been examining the potential utility of embodied-interaction technology for mathematics education. In the course of investigating the tutorial tactics observed in those data, we became obliged to expand on the professional-perception frameworks that were partially guiding our taxonomy of tutorial tactics. These frameworks, which have proven to be powerful lenses on mainstream instruction, did not appear to work quite well as lenses on non-mainstream instruction with future technology. Below, we present tutorial tactics identified across the data corpus, then reflect on the scarcity of such analyses in professional development. In Table 1, below, the first categories capture domain-general aspects of task-based interaction, and the later categories treat aspects of embodied-interaction inquiry.

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[5] The tutor—the first author—is a professor of mathematics education. He has conducted hundreds of semi-structured task-oriented interviews for his research and has administered thousands of professional tutoring sessions in the private sector. The premise of this study is that one effective way of determining useful pedagogical practice is to examine the practice of an expert and attempt to model this practice along salient dimensions that appear pertinent to this effectiveness (see, e.g., Lampert, 1990, for a notable precedent of an instructor-researcher as a reflective practitioner). As things stand, the novelty of these designs implies a scarcity of opportunities to study their facilitation, so that we would have been hard-pressed to find compatible research sites.

Table 1. Tutorial Interaction Tactics Supporting Mathematical Learning via Hands-On Problem-Solving Activity

Tactic	Description	Example
CONTEXT GENERAL		
A. Para-Content Framing	Administrative, logistical, and procedural actions to organize, monitor, and optimize interlocutor's comfort and engagement.	Adjusting S's grip of electronic device.
B. Clinical-Interview Techniques	Research methodology that overlaps with many tutorial strategies, e.g., eliciting the S's vocabulary through direct questions or indirect discursive ploys, echoing S speech, probing S reasoning, etc. (see Ginsburg, 1997)	T: "So how could we call your other theory, then?"
C. Discursive Practices	Amid ambiguous expression, interlocutors take discursive measures to interrogate and establish meanings. The interlocutor with superior contextual positioning suggests effective signs and elicits confirmation.	T: "What do you mean by 'the same way'?"
TUTORIAL		
1. Establish Joint Problem Space	Establishes the perceptual field, significant elements, mode of physical interaction, available resources, and task. Typically occurring at the beginning of the intervention, T might later re-evoke this information for clarification and/or introduce new tasks. These introductory communications are explicit and direct as compared to implicit and hinted cues during inquiry.	Setting up the MIT-P activity by introducing and naming situation elements (tr, camera, monitor, laptop), interaction mode (lifting/lowering tr), and task (making the screen green).
2. Elicit Orientation of View, Strategy, Reasoning, Vocabulary, and Knowledge	Performs actions intended to glean information about S's perception, planning, acting, and prior learning *that are not self evident*. T evaluates this information in terms of its sources, rationale, consistency, and coherence as these relate to the design's targeted views and strategies. Whereas these solicitations may stimulate reflection (see 4., below), they are initiated primarily so as to inform T of that which cannot be seen or heard.	<u>Ex 1</u>. T: "How are you counting? Do that same thing for me, please." <u>Ex 2</u>. T: "What do you think Dan was doing?" <u>Ex 3</u>. T: "Have you learned multiplication?"
3. Orient S's Perceptuomotor Interaction Toward Phenomenon's Critical	Uses discursive means to steer learner toward sharing similar ways of perceiving, planning, and acting in relation to accomplishing tasks in the domain of scrutiny. Based on S's manifest/elicited orientation of view, T evaluates that S is prone toward an impasse	

Features	with respect to discovery of properties critically pertinent for productive engagement with the problem. So T ushers S toward fertile data-gathering locations, orientations, or aspects of the situation.	
3.1 Highlight	Takes multimodal discursive measures to make salient aspects of domain.	
3.1.1 Feature	Draws S's attention toward an object or aspect of the domain, in a way that modifies S's perceptual construction of the domain, such as by suggesting directionality and introducing fictive motion.	T: "Ok so what do you think of the numbers *going down* this way?"
3.1.2 Restructure	Introduces conceptual model from another domain, so as to steer S to mentally reframe a set of data, such as by "rewiring" correlations along two dimensions of interaction.	S analogizes color gradients to a rainbow, T suggests traffic light instead.
3.1.3 Objectify	Responds to S's description of an action or relation in the joint domain of scrutiny by using a cognate noun (possibly a homonym) to staple a new referent as a semiotic resource for further discussion. Peirce named this "hypostatic abstraction," a form of diagrammatic reasoning (Bakker & Hoffmann, 2005).	S: "I think they have to be diagonal from each other." T: "What about up high—would it be the same diagonal?"
3.2 Code	Initiates, negotiates, and establishes consensual means of reference:	
3.2.1 Code Aspects	...to specified aspects of the domain (elements, properties, allusions, etc.);	T: "Same kind of 'game' but different... 'rule,' I guess."
3.2.2 Code Actions	...to possible actions in relation to these aspects (e.g., actions with or upon objects, manipulation strategies, principles, etc.).	T: "Ah....as if you're holding a ladder?"
3.2.3 Re-Code*	Substitutes S's situated term with a more general mathematical term.	S uses "distance," T quotes him back as through he had said "difference."
3.3 Customize Interaction Parameters	Introduces a new case, so that S attempt by induction to apply previous method. T selects this case strategically, e.g., to foster cognitive conflict.	
3.3.1 Select Qualitative Case	Refers to an interaction dimension and uses qualitative descriptors to refer more specifically to a degree/extent along this dimension.	T asks S to try "higher" on the screen.

3.3.2 Select Quantitative Case	Refers to an interaction dimension and uses quantitative descriptors to refer more specifically to a particular value along this dimension.	T: "So if your LH is at half, where should your RH be?"
3.3.3 Customize Margin of Error	Modifies S's task demands by tightening or loosening the interaction margin of error in response to manifest evidence of S's apparent (physical) capacity.	T adjusts the "tolerance" value on the console to help S enact a green screen.
3.4 Coach Interaction	Distributes, orchestrates, guides, (co-) performs, (co-) simulates, and monitors physical interaction operations. [overlaps/intersects with other categories]	
3.4.1 Demonstrate	Enacts an optimal perceptuomotor interaction, possibly accompanied by explanation that highlights features and subgoals of this performance as well as their relation to data and principles, so that S will then imitate.	T shows that it is possible to sustain constant "green" screen while moving hands.
3.4.2 Guide	Takes turns in performing solo interaction, while other person guides the performance physically and/or verbally.	T holds both S's forearms and guides them up and down.
3.4.3 Distribute	Coordinates and paces manipulation. In appropriate contexts, T might establish her own actions as the ad hoc independent variable, thus structuring the data gathering yet enabling S to discover each dependent-variable datum.	T holds LH tr, S holds RH tr. T paces S along a sequence of paired-hand positions that may prompt discovery.
4. Scaffold Reflection and Elaboration	Invites S to seek coherence within prior data and inference: highlights S's utterances; prompts recollection of pertinent data; and provides epistemic forms to organize data, identify conflicts, and formulate inference.	
4.1 Encourage Evaluation	Asks S to attempt to confirm their theory with empirical data. Suitable when Ss do not appreciate a problem or conflict in their own inference.	T: "Do you want to try and show me what you mean?"
4.2 Ask for Summary	Asks S to relate previous activity and conversation succinctly. Enables S to select, chunk, structure, represent, and generalize; and enables T to assess S.	T: "So… if you wanted to summarize what you've seen so far?.."
4.3 Support Summary	Performs actions that help S recall, organize, and condense the data into a briefer report, which foregrounds patterns.	T holds LH tr to co-reproduce with S a sequence 1-2, 2-4, 3-6...

4.4 (Re-)Organize data	Makes implicit patterns in the data more salient via using multimodal and material semiotic resources for the production of implied or actual inscriptions or via rearranging objects in the physical problem space. Re-encodes data in forms that help S recruit associated meanings and production rules.	Ex 1. T tabulates S's reported data on a board to help S notice latent numerical patterns. Ex 2. T asks S to re-order ratio utterance RH:LH as LH:RH.
4.5 Recount	Reminds S what s/he had said and done earlier by re-evoking/reenacting a previous episode, including actions, discussion, inferences. Yet so doing, T explicitly or implicitly introduces supplementary structure.	Returning S from the 3:2 challenging numerical case to the previous, easier 1:2 case, T recaps findings in this case.
4.6 Problematize	Prompts S reasoning by restating findings or inferences as juxtaposed one to the other. Underscores logical tensions.	T: "And so, is that like 'doubling' or not?"
4.7 Generalify	Frames S's utterance/action as rule-based, 'a case of' bearing validity beyond the particular case. This creates ontological shift in the status of utterance/action.	
4.7.1 Echo	Repeats S's factual statement or a fragment thereof in a manner that connotes an opportunity for inference, for generalization. T thus highlights but does not code aspects of data S had detected.	S: "....about...I donno... three squares higher." T: "*Three* squares higher."
4.7.2 Launch	Builds upon S's own words a sequence, pattern, or incomplete proposition that cues or implies a need for closure.	T: "2 ahead, then 3 ahead, so what will it be now?" S: "I think 4."
5. Valorize	Uses explicit speech acts and/or affective inflection to communicate positive or negative judgment pertaining prospective utility of S action or notion; informs S of the quality of their ideas or performance; cues to focus/drop particular efforts.	
5.1 Positive Valorizing	Praises, encourages, exclaims admiration, etc. Positively valorizing may encourage a challenged or frustrated S to perseverate amid difficulty.	T: "So, you're getting pretty good at this!"
5.2 Negative Valorizing	Marks S action as imperfect, usually by hems and haws, etc. Negative valorizing is liable to disempower S. Uses sparingly, sensitively, mostly to draw attention.	T: "Let's make three... Woops... Is that working?"

6. Pre-Orient: Frame New Semiotic Resource as S's Prospective Means of Better Enacting, Explaining, or Evaluating Interaction Strategy	Affects uncertainty as to: (a) details, quality, or validity of S interaction strategy; or (b) referent of S speech/gesture. T states that a new symbolic element is about to be introduced into the working area, then specifies its purpose as bearing rhetorical utility in clarifying ambiguity and/or furnishing more specificity. T thus implicitly problematizes S's work by suggesting it was inadequate. T both creates discursive need for repair and offers means for repair.	T: [numerals on] "So what we're gonna do now... Can you see the little numbers that appeared on the left?.... I wonder if those numbers in any way can help you explain to us the rule."

Note: T = the tutor; S = the student; Ex = example; LH = left hand; RH = right hand; tr = hand-held electronic tracker device
*The analysis has revealed some rare cases of "re-code" that seem to be inadvertent tutor actions, that is, they apparently do not result from deliberate tactics.

The motivation for this study was a need to understand instructional practices for supporting student engagement in embodied-interaction learning activities. The rationale of the study was to return to videographed data of interviews that led students to discover mathematical notions, and to identify therein tutorial tactics that supported this discovery. We created Table 1, a taxonomy of tutorial tactics, by negotiating between a bottom-up coding of observed tactics and a top-down general orientation from the professional-perception frameworks. In light of differences between our design and conceptual instruction usually treated by those frameworks, we expected to reveal previously unarticulated tutorial tactics. In so doing, we further expected to evaluate the frameworks could accommodate our design. Both expectations were confirmed. As is evident in Table 1, the new tactics are qualifications on, and supplements to the frameworks.

First, the tutor oriented the student perceptuo*motor*—not just perceptual—orientation toward the activity; he customized the student's engagement parameters; and he coached via demonstration and hands-on guidance. Second, the tutor did not explain to the student any logical or mathematical principles but instead structured and steered the students' reflection and inference. Third, by problematizing the student's interaction strategy as either similar or dissimilar to the student's previous strategy for the same problem, the tutor encouraged the student to juxtapose two action plans: these plans were functionally equivalent, in the sense that they elicited the same feedback from the technological device, and the tutor pushed the student to explain this functional equivalence logico–mathematically. We now elaborate on this third point via an example.

The following two interaction strategies for the Mathematical Imagery Trainer are mathematically commensurate ways of advancing from a given number pair that relates by a 1:2 ratio to another: (a) for every 1 unit you raise your left hand, raise your right hand by 2 units; (b) whenever you re-place your left hand to a new point along the grid, re-place your right hand double as high as the left. The first strategy embodies an additive model of ratio, in which the conjunction of the *a* and *b* elements in the *a:b* ratio can expand via linked iteration into a set of proportional ratios, such as 1:2, 2:4, 3:6, etc. The second strategy embodies a multiplicative model of ratio that creates a set of proportional ratios via preserving the functional relation ("double") between the *a* and *b* elements of the *a:b* ratio. Elsewhere, we are examining cases of students who attempted to determine connections between these strategies (see Abrahamson, Negrete, & Gutiérrez, 2012).

4. Conclusion

Theories of professional perception (Goodwin, 1994; Stevens & Hall, 1998) offer powerful constructs for modeling instructional interaction. Yet, we assert, for these theories to be durable and effective, they should be updated and expanded vis-à-vis current perspectives on learning (embodied cognition), technological advances driving innovative learning environments (embodied interaction), and pedagogical frameworks (guided inquiry).

As evidence for the above assertion, we have presented findings from the analysis of one constructivist tutor's instructional tactics. During our analysis, we attempted to work with categories and constructs from professional-perception frameworks. However, these frameworks did not readily fit our empirical data, so that we were compelled both to qualify some of the categories and to build new categories.

As embodied-interaction technology becomes increasingly accessible, we might do well to pay renewed theoretical attention to the constitution of a domain of scrutiny, and not only to how an instructor guides students' seeing of a domain. In particular, when students need to physically generate the domain even as they are investigating it, an instructor may need to coach this activity by hinting, highlighting, and coding aspects of *physical*, not only perceptual orientation. That is, future design such as the Mathematical Imagery Trainer may increasingly engage children's perceptuo*motor* engagement, a naturalistic mode of learning. If disciplined-perception frameworks were to assimilate dimensions of instructional interaction suggested by our analysis of embodied-interaction discovery-based tutorials, they could stand to inform pedagogy of indirect instruction. Via indirect learning, students may better develop 21st Century skills, which expand traditional vocational conceptualizations of what it means to be a professional.

Our findings bear relevance also to scholarly discussions around pedagogical methodology for fostering effective science inquiry practice (e.g., "structuring and problematizing," see Edelson & Reiser, 2006; Reiser, 2004). Notably, some instructional designers of scripted inquiry-based learning environments are informed by the belief that scientific methodology can be scaffolded through stipulating that children log and share their observations, articulate hypotheses, conduct controlled experiments, reflect on their findings, etc. (Slotta & Linn, 2011). Whereas we share the objective that children develop powerful inquiry practices, we wish to see more learning environments that foster greater continuity between natural inquiry and disciplinary practice (Gopnik, Meltzoff, & Kuhl, 1999; Karmiloff-Smith, 1988; Lakatos & Feyerabend, 1999). Granted, the particular embodied-interaction problems that we have implemented so far in the Mathematical Imagery Trainer perhaps do not demand of students to manage as much information as do inquiry tasks in biology, and, more generally, it is not unproblematic to compare design frameworks across STEM disciplines. However, the tutorial tactics fostering inquiry-based discovery identified in this study may be useful in the design of computer-based curricula as well other science domains. So doing, though, we foresee the greatest challenges in determining *what* elements on the screen children are attending to and, moreover, *how* they are attending to those elements and what *meanings* these elements evoke.

5. Limitations and Implications

We have evaluated the utility of professional-perception frameworks to shed light on non-standard expert–novice instructional sessions. Based on our findings, we have asserted that these frameworks require elaboration along several dimensions, which we have specified. Our assertions have been contingent on the assumption that our empirical data are representative of the phenomena in question. However, that may not be the case.

One obvious limitation of our study is a sampling problem. Namely, we have looked at the practice of but one tutor, whose professional practice might be idiosyncratic and thus be of little external validity. That is, we should exercise caution in making claims that our findings of this tutor's tactics generalize to, and might even inform the practice of the greater population of tutors.

A second problem is that this tutor operated in a particularly constrained setting. Namely, the tutorial session implemented a task-based semi-structured clinical-interview protocol serving a design-based research study, so that perhaps the tutor was after data as much as after learning, and these objectives may not always have converged.

Although these limitations all point to a need to gather further data, the particular settings might actually bear some opportunity for internal validation. Namely, a possible methodology for triangulating our findings using the same corpus of data would be to compare the practice of expert and novice tutors who participated in the study. That is, data from the current project have offered us some preliminary evaluation of our findings, because in addition to the PI, several researchers-in-training participated in conducting the interviews. Consequently, we have had opportunities to witness several occasions of novice tutors exercising tactics that appeared less effective. These observations, in turn, were very instrumental in "opening our eyes" to what the more experienced interviewer was in fact doing. Namely, that which appeared "natural" or seamless in the experienced interviewer's practice thus became reified as a deliberate tactic. Whereas these observations are anecdotal, a brief example in Appendix A may serve two ends. Namely, we hope to demonstrate: (a) the viability of characterizing localized tactics as a means of studying professional practice; and (b) the potential of this line of research to inform the training of tutors and researcher–interviewers.

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Appendix A: Novice Tutorial Tactics

Charlie (pseudonym), a novice tutor, was facilitating his first session ever, as part of his training as a graduate student. The elementary-school study participant had been manipulating the two cursors on the screen and had determined a demonstrably effective strategy for moving her hands while keeping the screen green. Charlie had thus reached the point along the protocol where he was to overlay a virtual grid upon the screen. He said to the student that he was about to bring up something on the screen and that she should see *whether that changes anything*. He then lit up the grid. The child picked up the tracker devices that had been lying on the desk. Just as before, she located a "green spot" and then lifted her hands further up, maintaining a green screen in accord with her strategy. No, she reported, nothing has changed.

Of the total of a near two-dozen students who participated in this study, this student was the only one who responded thus. Other students tended to appropriate the grid as a means of better enacting, explaining, or evaluating their strategy. In retrospective analysis, we realized that how the tutor frames the introduction of a new artifact partially predicts whether or not the student engages it as a useful instrument (Gutiérrez, Trninic, Lee, & Abrahamson, 2011). Thereafter, Charlie learned to frame the introduction of new symbolic artifacts as potentially promoting the interaction.

Incidences such as this, which we have been archiving for training purposes, are essential in the preparation of interviewers, because they occasion opportunities for supervisors to flesh out implicit dimensions of their own practice. As such, for a PI charged with training graduate students as much as with conducting research, videotaped documentations of such incidences are vital for building a laboratory's capacity. Yet for the particular methodological needs of the current study, these incidences accentuate the rationale of our approach. Namely, professional tutors exercise a repertory of specific tactics that affect the nature (if not quality) of students' engagement in learning activities.