

A NEW WORLD: EDUCATIONAL RESEARCH ON THE SENSORIMOTOR ROOTS OF MATHEMATICAL REASONING

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Recent developments in the theory and methods of cognitive science are enabling educational researchers to evaluate empirically the historical thesis that mathematical concepts are grounded in sensorimotor activity. My presentation will survey results from several recent design-research studies that have used eye-tracking techniques to capture the moment at which a student first sees the world in a new way. For the student, this spontaneous perceptual construction serves as a handy solution for coordinating the control of an interactive system. In turn, through cultural mediation this construction evolves into a new way of reasoning that becomes a mathematical concept. I will speculate on implications for educational technology.

I would like for you to know what I mean, when I talk about sensorimotor perception. And I would like for you to realize how new sensorimotor perceptual structures emerge through goal-oriented interaction with the environment. I further hope to convince you that learning new mathematical concepts begins with the formation of new sensorimotor perceptual structures. And I will show you how we can now track this process. The better we understand all this, the better we can design for learning.

A NEW WORLD: PERCEPTION REVISITED

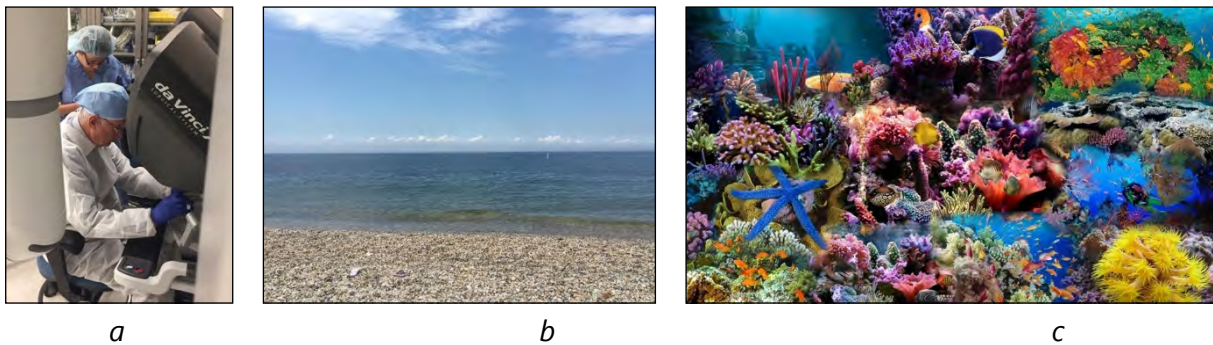


Fig. 1. *a* — robotics surgery; *b* — the Red Sea at Sinai; *c* — a coral reef

Figure 1*a* is a recent photograph of my father, Dr. Jack Abrahamson, Professor of Surgery. Jack will turn 90 later this year. Don't worry, he's not operating any more. The picture was taken last year, when he was touring the robotics operating theatre at the Medical School, University of California San Francisco. But I want to tell you a story about a non-medical experience Jack had. It was 40 year ago. So this is back in the 70s. The era of Leonid Brezhnev, I guess. I grew up in Israel. One summer we trav-

elled south to Sinai, to spend a vacation in a Bedouin village on the coast of the Red Sea (Fig. 1b). My brothers and I were underwater divers, so we had seen the coral reef before (Fig. 1c). But my father, although he grew up by the sea, in South Africa's Cape Province, and was an able seaman, had never used flippers or snorkel to go down under and see what lies below the water surface. Eventually, we talked him into trying. Jack put on a snorkel, a mask, and a pair of flippers and swam out to the reef. From far away, we watched his snorkel bob over and under the water. Finally, he came out and walked back up the beach. We'd never seen him before quite that way. Back then, Dad was a very serious person. Now his face was glowing, his jaw dropped, his eyes popped out. He uttered only three words: "A new world."

Now, to be sure, the reef had always been there, just below the water. The point was to see it — for *him* to see it, and, for that, one often needs the right conditions, including perhaps some special gear, and possibly some encouragement from those who create the conditions and facilitate the encounter. And yet those who create this encounter may, too, experience a new world — of witnessing another person learning.

So it's a new world in two senses. For the individual person — let's call them a student — the activity enables a new perceptual relation with the environment. You apprehend details and structures which were always there, even if *you're* experiencing them for the very first time. You might reflect on what you see and express it to yourself and other people. This new world is about a new phenomenal experience in the world as you found it. Yet for the observer — let's call them a researcher — the activity might enable seeing what the student is seeing in a way that has not been available before. Now, unfortunately, we could not see what Jack saw underwater or how he came to see it and make sense of it — we didn't have access to all that. To achieve that sight, one needs to "go underwater," that is, to use special instruments, like eye-tracking devices. This new world is about scientific breakthrough in modeling how new sensorimotor perception is formed when a person engages in goal-oriented situated activities, such as solving motor-control problems.

A NEW WORLD OF THEORY, TECHNOLOGY, AND METHODS

This paper is about the new world in both of these senses: a world of students seeing new structures, and a world of researchers who see this process as it is happening.

Upfront, here's the paper's take-home message. Three recent developments in the field of mathematics education research — in theory, technology, and methods — have been converging in a way that's enabling us to re-think early stages of conceptual learning, to create conditions that foster these stages effectively, according to some instructional objectives, and to understand how these early stages play out and how they might be monitored and steered. We're talking about theory of *embodied cognition*, technology for *embodied interaction*, and methods of *learning analytics*.

These are exciting days to be a cognitive scientist with an interest in the teaching and learning of mathematical concepts, because these three developments in the field have now matured and have come to a confluence.

Theory: embodied cognition

One development is in the philosophy of knowledge, epistemology, which is challenging the theories that researchers have been putting forth and evaluating with respect to how people learn. If once it was taken for granted that the brain functions as a computer-like central processing unit that is cut off from the sensorimotor modalities, now there is increasing interest in alternative proposals by which all cognitive activity is intrinsically perceptuomotor. These ideas, which have loosely been called embodiment theory, and include arguments that the mind is embodied, embedded, extended, and enactive, are renewing the field's interest in pre-AI models of the mind and of learning processes [Newen, Bruin, Gallagher, 2018; Shapiro, 2014]. Cognition, embodiment theorists argue, is not the manipulation of abstract symbols but, rather, cognition is perceptuomotor activity. Cognition is *modal*, in the sense that it is constituted in the various modalities, that is, by the very same type of neural activity that enables us to function in the world through sensation, perception, imagination, motor action, and by engaging cultural forms, artifacts, language, and practices that have evolved to support our survival and thriving. And so, how we think depends on how we know to move in the world, how we know to operate on various objects. Educators can therefore affect students' cognitive development by shaping their experiences with objects.

Some embodiment theorists claim there is no such thing as abstract entities. Rather, there are imaginary concrete forms. These can be complex and counterfactual, in the sense that they cannot be materialized. But mathematicians, like fantasy novelists and readers, are glad to suspend their disbelief and treat these objects as real or pseudo-real. Indeed, these objects may not be directly accessible to our senses as percepts in the world, but still they are phenomenologically real. Just as dreams are experienced as real, no matter how fanciful they are (cf. [Hutto, Myin, 2017]).

Embodiment theory ranges in its philosophical commitments and radicalism. Yet, however articulated, it is emboldening us to investigate and leverage the sensorimotor foundations of mathematical concepts [Abrahamson, Lindgren, 2014; Lindgren, Johnson–Glenberg, 2013].

Technology: human–computer interfaces for naturalistic embodied interaction

Another development is in technology, and in particular in Human–Computer Interaction (HCI). HCI is a major branch of engineering concerned with designing and evaluating technological systems for humans to interact efficiently with computational platforms. Recent advances in HCI include new platforms with human–computer natural-user interfaces that allow for intuitive, discovery-based interaction with information structures encoded in software [Antle, 2013; Dourish, 2001]. These advances in HCI are creating opportunities for educational designers to build learning environments in computational platforms that enable naturalistic inquiry through embodied interaction. Of particular interest are environments that implement activity genres for mathematics teaching and learning, in which students first build pre-symbolic, qualitative understandings of new notions through manual interaction with material

or virtual objects and only later adopt mathematical formulations of these physical movements [Abrahamson, 2014]. For instance, we can cast a mathematical concept in the form of an embodied-interaction regimen that privileges particular movement patterns, that is, particular enactment of situated skill. *This form of computational technology thus offers an epistemic interface between a mathematical concept and human sensorimotor perception, where both are cast in the movement modality, and the objective is for the human, through concerted inquiry, to figure out and match the computer's hidden movement form* [Howison et al., 2011].

Methods: eye tracking and multimodal learning analytics

And, finally, as educational researchers, how might we monitor this process? In particular, how do we capture a student's sensorimotor perception and any changes in this perception as the student engages in embodied-interaction learning activities? What might it even mean to know how a child is seeing the world? As I will explain, the instruments of eye tracking can help us gain some purchase on determining where a child is gazing and, through that, and in triangulation with the child's actions, speech, and gesture, to make educated guesses about how the child is seeing the world she is interacting with. Thus, a third development is in methods, in particular, instruments for measuring sensorimotor activity in a variety of modalities, then integrating these data and presenting them for analysis to search for behavioral patterns and trends in these patterns. This means that when students interact with the technological platforms we design for them to learn mathematical concepts, we can monitor our data, even in real time, for the emergence, regulation, and refinement of sensorimotor routines [Worsley et al., 2016].

So those are three lines of development in our field that are relevant to the work my lab does with our international collaborators — theory of embodiment, technology for embodied interaction, and multimodal learning analytics. Combined, these developments in theory, technology, and methods have created opportunities for the kind of research that led me to state that these are exciting days to be a cognitive scientist with interest in mathematics teaching and learning.

I would like to tell you about one line of research that our lab began exploring about ten years ago, which is building on this synergy of theory, technology, and methods. I will start by setting for you the context of this work, and then I will focus on a particular theoretical construct we call an “attentional anchor.” I will explain this construct, and I will present empirical work suggesting that this construct might play a role in future research more widely, on both the theory and the practice of mathematics teaching and learning.

The crux of the innovation is that we can use eye-tracking devices in order to monitor for changes in the way that mathematics students perceive the visual displays they are studying and manipulating. These changes in perception mark the formation of new sensorimotor schemes — new ways of perceiving the world so as to act upon it — and, in turn, these new perceptual structures constitute things we can measure, model, symbolize, and discuss, so that they become mathematical objects. Having real-time

access to how students are perceiving the environment, I will suggest, may change the way we teach, both in person and through artificially intelligent interfaces.

These are still early days in this line of research, and so some of the empirical data I will discuss are yet preliminary, and the applications are still being engineered. However, I would like to use the opportunity of this paper to share with you these developments and hopefully the excitement.

THE WORLD ANEW: LEARNING BY SEEING THINGS IN NEW WAYS



Fig. 2. Friedrich Fröbel “gifts”: a curricular regimen of pedagogical manipulatives

Many of us here are in the business of creating stuff for kids to learn math by. The idea goes back at least two centuries, to Friedrich Fröbel (see Fig. 2), who invented kindergarten, and it has been popular through the work of Maria Montessori, Caleb Gattegno, Vasily Davydov, Daniil Elkonin, Zoltan Diénès, and many others. Seymour Papert called these things “objects to think with.” And many of us have built their academic careers around investigating how people learn mathematical concepts through interacting with objects. I’m in this business, too, of creating, evaluating, and theorizing pedagogical regimens, including media and activities, for students to learn mathematical concepts.

Much of the literature in the research field of mathematics education, certainly in the collected proceedings from annual meetings of the International Group for the Psychology of Mathematics Education (PME) and its regional subsidiaries, such as this inaugural PME Yandex Russia meeting, is on how children learn through engaging with pedagogical materials and, therefore, how this learning should be facilitated and assessed. And a central idea in this body of research is that through engaging these materials, usually in an attempt to accomplish some particular assigned task, the students develop new perceptions of the environment that are vital to learning the concept in question. These new ways of orienting toward the environment are designed by mathematics educators so as to align with our civilization’s cultural heritage comprising productive ways of organizing our collective behaviors. And so these new ways



Fig. 3. The author and Neomi Sarah, at 4 years old, using hand formations to discuss the commutative property of multiplication. Ten fingers are 2×5 or 5×2

of *perceiving* the world emerge through, are mediated by, and are integrated in the use of new forms of *operating* on the world; forms that enable students to participate in the social enactment of cultural practices. Most parents do this intuitively, when they teach their children to count. Educators seek to emulate this naturalistic pedagogical acumen by formulating and theorizing effective principles for cultivating mathematical knowledge beyond counting (e.g., see Fig. 3).

This paper, too, is about children coming to perceive the world in new ways through operating on it. Where I am hoping to push the conversation forward is in suggesting that mathematics educational research is now at a point where we might revisit what we mean when we talk about students learning to perceive the world in new ways. In particular, I wish to suggest that we could pay more attention to the physical movements that students enact as they learn. I will argue that *by paying more attention to how children move, when they learn mathematical concepts, we could do a better job in theorizing the cultivation of perception*. For researchers, the new world is that, using eye-tracking instruments, we can see the moment a child comes to see the world in a new way. This insight could lead us to rethink the design of educational artifacts.

TOGGLING THE WORLDS: PERCEPTION OF AMBIGUOUS FIGURES

Movement is difficult to talk about, because — well..., it keeps moving! So in order to say something about the phenomenology of movement, let us step back and begin by speaking about the phenomenology of something much simpler — static images.

Ambiguous figures are popular, because, similar to optical illusions more generally, they offer an intriguing perceptual experience (see Fig. 4). As we shift our foveal visual orientation onto different regions of these images, our perceptual construction of the image toggles between two alternative and often mutually exclusive potential meanings of the image [Tsal, Kolbert, 1985]. In turn, *reflecting* on this experience, we may realize that visual sensory perception is active (not passive), constructed (not inherent),



Fig. 4. Three famous illustrations bearing ambiguous figures.

From left: Louis Albert Necker's cube; Edgar Rubin's vase; and Joseph Jastrow's duck/rabbit

relational (not monistic), subjective (not objective), and mostly tacit (not conscious). Consequently, given these important insights onto sensory perception, these images are often used in introductory courses on sensation and perception.

Yet I would like to point out that with all these famous images, we are not asked to do anything physical — just to observe and interpret. Can we change how a person sees an object by asking them to do different things with that object? To accomplish this, it may help to select an object that we commonly use actively as a tool. As in the case of the classical ambiguous images, above, I am about to create experiential circumstances that could affect how you frame your perception of a sensory display. But, unlike these images, I will attempt to manipulate your sensory perception of the object through changing your motor orientation toward the object. Ready?

AFFORDANCES, THE PHENOMENOLOGICAL QUALITY OF THINGS

Consider a pencil (see Fig. 5).



Fig. 5. A pencil — what it is is what you do with it

Now imagine that you are about to use it in each of the following ways: to

- write
- erase
- sharpen
- pop a balloon
- drum
- scratch your back

- hand it to a child
- twirl it around your finger
- balance it in the middle
- balance it on the tip
- save a ladybird from a puddle
- break it in two

As you considered putting the pencil to these various proposed uses, you may have noticed your body orienting in different ways, each appropriate to the object's specific *ad hoc* utility relative to its proposed function. Your sensory organs, such as your eyes, may have shifted to specific regions of the object, such as its middle (to balance it there), even as your motor organs, such as your hands, arms, and upper torso, tensed and shifted ever so slightly, in preparation to enact a particular form of contact, such as a grasp, each type of contact attuned to particular properties of the object. These types of sensorimotor impressions, which are both nuanced and ephemeral, bind us to objects in the environment by way of proto-action perception, which Gibson [1977] called *affordances*. Note that an affordance is not “in” the object irrespective of the observing organism, nor is it “in” the organism irrespective of the object. Rather, an affordance is inherently relational [Heft, 1989]. When you consider writing with a pencil, you both see the pencil in a way that is specific to its writing function and you organize your motor capacity to enact writing movements.

You have just participated voluntarily in an experimental activity designed to achieve two technical objectives: (a) to sever your physical orientation toward an object from your sensory perception of the object; and (b) to keep changing the purpose and consequent morphology of your unconsummated imaginary engagement with the object. I invited you to engage in this humble introspective exercise, because I could thus occasion for you an opportunity to experience the manifold of constituent somatic, kinaesthetic, and proprioceptive micro-sensations of your preparatory motor disposition toward a perceptual construction of the environment. Normally, in the stream of doing things with objects, these feeble constituent qualia of sensorimotor activity — how you are seeing an object, and how your body is preparing to engage it — are tacitly and irreducibly enmeshed below the radar of consciousness. As Mechsner writes,

affordances are not only perceived as properties of the affording object. By way of an educated phenomenological sensitivity we may also experience the specific way our body is related to the objects of our interest [2003, p. 240].

In this sense, what an object *is*, at least in our ongoing unreflective phenomenology — which, arguably, characterizes the vast majority of our humdrum hominid experience — is how we are *using* it. In fact, the object will not be salient or accessible to our consciousness as a thing, unless our flow of immersive being-in-the-world breaks down [Koschmann, Kuuti, Hickman, 1998]. But I needed for you to unpack your natural

being-in-the-world into its respective sensory and motor factors, because I wish to discuss the constitutive role of sensory perception in forming our capacity to engage in the motor enactment of movement forms by which we accomplish the control of task-oriented manipulation. As such, I needed for you to temporarily experience perception and action as disjoint.

Now, of course, we all know that the image in Fig. 5 is a pencil. That is, we might, on the one hand, take the categorial stance that a pencil is a pencil is a pencil; regardless of what we intend to do with this pencil, the simple fact stands that this still is no more yet no less than just that — a pencil. As such, there would be little to any room for ecological psychology in the scholarly work of sorting and defining the phenomenal manifold. By this oppositional view, there is much epistemic utility in acknowledging the objective identity of objects, which transcends all contextual and intentional circumstances, impervious to the hazards of the observer's knowledge, skill, objective, sentiment, dispositions, or wherewithal. Notwithstanding, as mathematics-education researchers, who care to understand how students learn through manipulating objects they encounter in instructional activities, it is, on the other hand, important for us to query the source and constitution of a child's contextual orientation toward these objects.

Intellectual concern for the implicit meanings students bear for artifacts they are manipulating as well as for the emergence of explicit mathematical meanings from these activities is typical to scholarship on individuals' guided mathematical sense-making. This concern is discussed from a variety of perspectives in our field's literature on the epistemology, ontology, and ontogenesis of mathematical entities, such as through the framework of instrumental genesis [Vérillon, Rabardel, 1995], radical constructivism [Steffe, Kieren, 1994] sociocultural theory [Saxe, Gearhart, Seltzer, 1999; Sfard, 2002; Stetsenko, 2002], or various semiotic approaches [Bartolini Bussi, Mariotti, 2008; Font, Godino, Gallardo, 2013; Radford, 2014].

I draw much inspiration from these contributions to the literature, and yet my interest is on what I believe is a giant gap, in most of this work, with respect to modelling how humans engage objects and wherefrom concepts therefore emerge. What is missing is movement [Sheets–Johnstone, 2015]. In theorizing students' learning through manipulating objects, researchers for the most focus on the outcomes of manipulation — *what* the students do with the things, such as sorting, joining, or counting them. What is less theorized is *how* the students manipulate the objects, that is, the movement forms that students enact in performing the assigned tasks. I wish to add to the field's conversation on learning-through-manipulating a focus on students' experience of moving — the moving itself that gets things done in the learning space (e.g., see [Sinclair, 2018]). Our earlier exercise with the pencil (Fig. 5) was designed to sensitize the reader to the tacit phenomenological stuff that, I maintain, much of learning is made of. As I now explain, the passage from unreflective phenomenology of movement to reflective consciousness of objects hinges on the construction of perceptual structures. Thus, the very perceptual structures that enable the enactment of movement are cognitive grounds for what will become a mathematical notion.

ENACTIVISM BY DESIGN — FROM PHENOMENOLOGY TO CONCEPTS

If humans' pervasive phenomenology is unreflective immersive doing in the world, where do mathematical concepts come from? And then, once we are satisfied with some working hypothesis about the origins of mathematical concepts, how might we put this theory into practice, in the form of activities for students to learn mathematical concepts? As I now explain, my laboratory's design-based research efforts have been inspired by *enactivism* [Varela, Thompson, Rosch, 1991], a theoretically informed and empirically validated epistemological perspective from the philosophy of cognitive science. Still, applying a high-level theory to educational practice often requires a mid-level pragmatic framework [Ruthven et al., 2009]. We have been applying enactivism to our pedagogical agenda by formulating *embodied design*, a pedagogical framework for mathematics education. Embodied design articulates heuristic principles for building and implementing activity genres that draw on students' naturalistic perceptual and motor capacities [Abrahamson, 2009; 2014; 2015; 2017]. In particular, the *action-based genre* of embodied design delineates steps for engineering learning environments that foster conceptual learning at the sensorimotor–sociocultural interface. In their seminal book, *The Embodied Mind*, Varela et al. [1991, p. 173] explain their philosophy of cognitive science, enactivism, as follows:

In a nutshell, the enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided.

At the Embodied Design Research Laboratory in the Graduate School of Education at the University of California Berkeley, we have been evaluating the potential relevance of the enactivist position as a guiding framework for building mathematics learning activities and examining students' experiences using these resources. As designers, we use the enactivist credo to reverse-engineer our learning environment.

We begin by asking ourselves, what are the cognitive structures we would like our students to develop? Answering this question depends on our learning objective. If, for example, we would like for the students to develop the concept of proportionality, then we consider what might be a dynamic instantiation of this concept; that is, we are looking for some movement composition that mathematicians would recognize as a clear schematic exemplar of the target concept. Pratt and Noss [2010] use the term *phenomenalization* to capture this creative process of making “concrete” realizations of “abstract” concepts. For example, we might determine a movement composition for proportionality, in which two objects rise side by side at different speeds (see Fig. 6). Once we have designed this movement form, which we call a *conceptual choreography*, we then ask how a student might enact this movement form. This particular movement form of two rising objects could be enacted bimanually, with each hand raising one object. As such, we have determined a sensorimotor pattern that students should develop through participating in the activity we are about to create. From here, we deduce what would be the actions composing this sensorimotor pattern, here it would

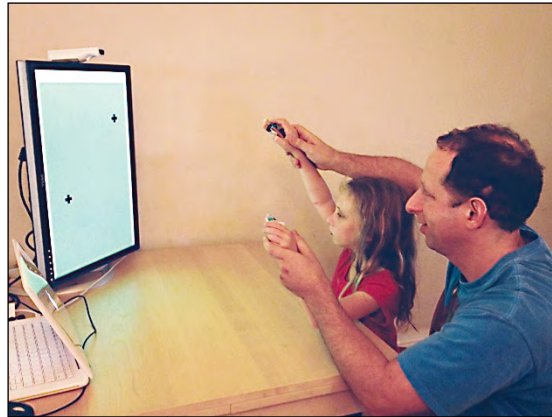


Fig. 6. The author and Neomi Sarah, at 8 years old, using remotes to co-enact a conceptual choreography for proportionality, in which two cursors rise at a 1 : 2 ratio

be a left-hand action and a right-hand action, where each hand is rising. Finally, we ask how these actions might be perceptually guided so as to perform the conceptual choreography. That is, what could be a particular perceptual orientation toward the sensory display of the two rising hands that could facilitate the coordination of this bimanual motor action in proportionate form? As we will explain, one way of controlling two hands moving at different speeds, which can otherwise be a challenging feat, is to focus on the spatial interval between the hands. This gap has to increase as the hands rise and decrease as they come back down.

BUILDING A NEW WORLD: EVALUATING EMBODIED DESIGN

I have demonstrated how a philosophy of cognitive science (enactivism) can be implemented in the form of a pedagogical framework (embodied design). In particular, I have exemplified how *we cast developmental stages of enactivist ontogeny as structural and procedural elements in the action-based genre of embodied design for mathematics learning*. As such, *our engineered learning environments, which are carefully designed HCI systems, are crafted to simulate the ecological conditions that would elicit from students the development of new sensorimotor perceptual structures*. We had determined these particular structures as pivotal for students to experience the phenomenalization of a mathematical concept that is targeted by our design. In turn, *students engaged in our activities develop these perceptual structures spontaneously, as their pragmatic means of solving an emergent motor-control problem they encounter in the course of attempting to perform an assigned task involving the manipulation of objects*. For example, they construct a new Gestalt — the spatial interval between two virtual objects on a screen — as their “steering wheel” for coordinating the bimanual work of moving two objects in parallel at different speeds. We use the phrase *Mathematics Imagery Trainer* to name this type of learning environment that we build for students’ sensorimotor perceptual construction of proto-conceptual structures. These sensorimotor perceptual structures evolve from proto-conceptual to conceptual once students adopt mathematical frames of reference, as we explain below. *By token of eliciting naturalistic behaviour*

within a highly crafted environment and nurturing these behaviors into normative disciplinary expression, the Mathematics Imagery Trainer fosters conceptual learning at the sensorimotor–sociocultural interface.

The Trainer featured in Fig. 6 was the first of our attempts to leverage, evaluate, and investigate the enactivist–constructivist thesis – viz. that cognitive structures emerge from recurring action-oriented sensorimotor patterns – as a *modus operandi* for crafting educational design. Results from clinical testing of the Trainer were first presented at PME-NA 32 [Reinholz et al., 2010]. Students who participated individually or in pairs in our task-based semi-structured clinical interviews were able to enact movement forms that satisfied the task requirement of keeping the screen green while raising or lowering the virtual objects. Their multimodal explanations suggested that they were developing a succession of increasingly sophisticated strategies for solving the bimanual motor-control problem that they encountered in the course of attempting to perform the task. Through iterated attempts, the study participants became conscious of new dimensions of operating the objects, and they explored for optimal values along these dimensions.

Figure 7 illustrates paradigmatically the sequence of interaction events commonly observed across students, as they figure out that their hands should move not at the same speed but at different speeds.

When we then introduced symbolic artefacts onto the screen – first a grid, and later numerals along the vertices (see Fig. 8) – students endorsed these features into their sensorimotor scheme, and yet in so doing the scheme changed. We concluded that

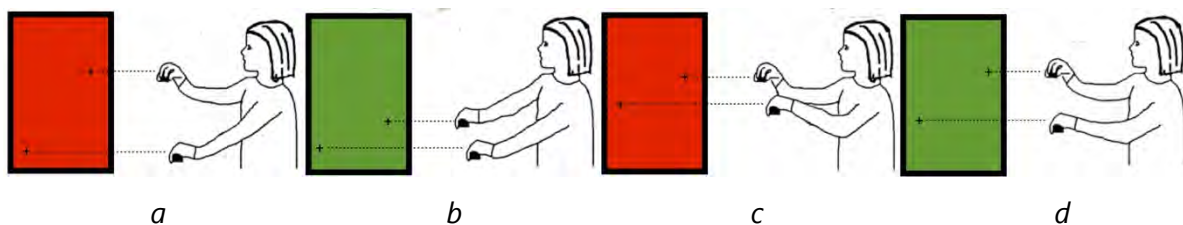


Fig. 7. Solving a movement riddle in the Mathematics Imagery Trainer, a child learns to move physically in a new way; she then articulates her movement formally as governed by a proportional function

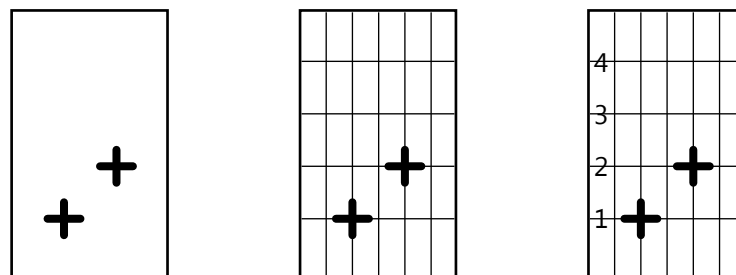


Fig. 8. Three configurations of the Mathematics Imagery Trainer's computer interface. From left: cursors only; with a grid; and with numerals

students had identified in the figural features of these didactically interpolated symbolic artefacts certain relevant utilities for enhancing either the enactment, evaluation, or explanation of their extant control strategy; in so doing, they implicitly assimilated the features as frames of reference, thus shifting from naïve to cultural forms of organizing and understanding their situated actions. We noted in particular that introducing the grid caused students to change their bimanual strategy from moving their hands simultaneously through the continuous space and using qualitative language to describe their strategy (e.g., “The higher I go, the bigger the gap”) to moving their hands sequentially through the discretized space and using quantitative language (e.g., “For every 1 I go up on the left, I go up 2 on the right”). As such, the students developed the activity’s intended psychological–discursive forms of acting and reasoning without the researchers offering any direct instruction, demonstration, or formatting [Abrahamson et al., 2011].

Participants were also able to coordinate among polysemous visualizations of the environment, for example explaining why raising their right hand 2 units every time they raise their left hand 1 unit (Fig. 9iii) means that the spatial interval between their hands should steadily increase (Fig. 9ii). As such, we noted the pedagogical potential of the activity design and, specifically, of the Trainer environment, to foster important conceptual reasoning using non-inscriptional media, that is, even before pen is set to paper [Abrahamson et al., 2014].

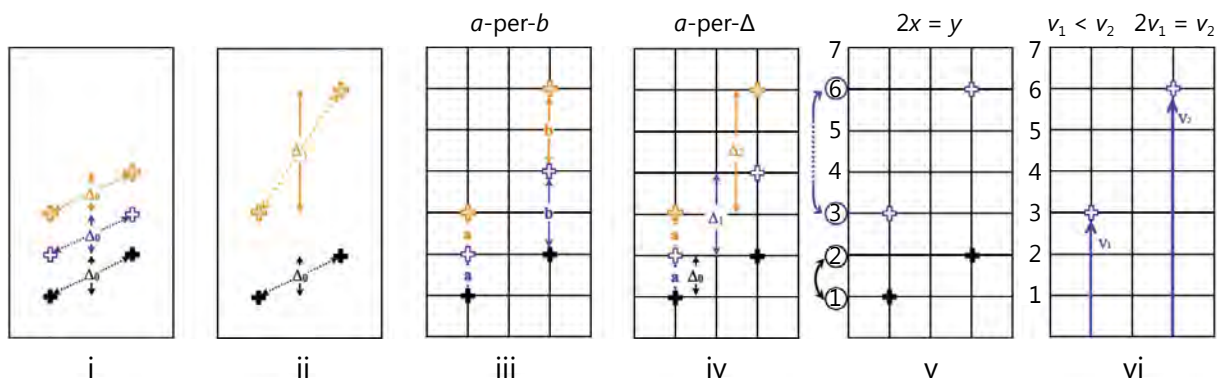


Fig. 9. Student generated solution strategies for the make-the-screen-green problem (the case of a 1:2 ratio)

Research efforts are still underway to evaluate how sensorimotor competence, which students develop through participating in activities using the Trainer, could possibly be cast as constituting forms of knowing that the community of mathematics-education researchers and practitioners would recognize as relevant to normative disciplinary practice in educational settings. A logical and anticipated way forward here is to demonstrate how these ways of knowing play out, when study participants, who have engaged with the activities, then set to engage in solving problems that the field generally appreciates as constituting measures of domain-specific subject-matter content knowledge. This approach could potentially translate to methods of assessing what students gain conceptually through developing new movement forms for manipulating objects according to specified task requirements.

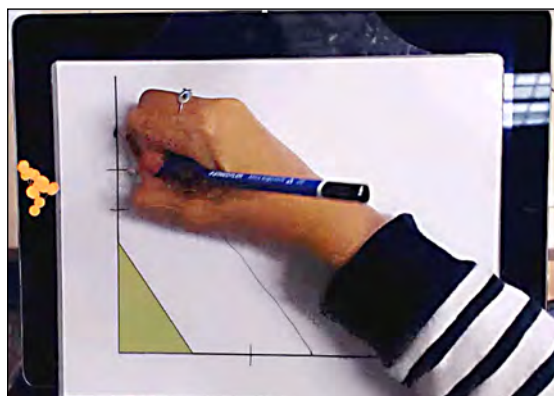


Fig. 10. A student who had imagined a diagonal line on a tablet, to solve a 3 : 2 orthogonal ratio problem, now recreates this line on paper, using measurement

Some studies are showing promise. A multi-classroom-based doctoral dissertation [Petrick, 2012] tentatively concluded that students who engaged with the Trainer and other analogous activities advanced conceptually more than a comparative group (see also in [Abrahamson, 2012]). Related studies resulted in similar empirical findings, suggesting the pedagogical potential of Trainer activities in the domains of proportionality [Bongers, Alberto, Bakker, 2018], the Cartesian coordinate system [Duijzer et al., 2017], geometrical area [Shvarts, 2017], and parabolas [Shvarts, Abrahamson, 2019]. For example, Bongers et al. [2018] demonstrated effective semiotic transitions from description to inscription (see Fig. 10): Students who had invented and manipulated imaginary sensorimotor perceptual structures on a tablet, as their pragmatic solutions to the problem of coordinating the enactment of solution movements, then recreated and thus materialized these percepts using pencil and paper; what more, they spontaneously used multiplication to measure proportional segments of these constructions, such as building a set of dilated similar right triangles. For further review of empirical findings from the project, see Abrahamson and Bakker [2016].

ATTENTIONAL ANCHORS INTO THE STUDENT'S NEW WORLD

As educational *designers*, we were thus learning more about the action-based activity genre of embodied design [Bakker, Shvarts, Abrahamson, 2019]. Yet as design-based *researchers*, we hoped to get a tighter theoretical grip on the evolution of sensorimotor perceptual structures: we wanted to witness and monitor the micro-process of a new structure emerging into students' interaction routines to become an object of reflection and mathematical modeling. This is where eye-tracking technology offered opportunities. Our study participants were seeing a new world by constructing sensorimotor perceptual structures, and now we, too, were about to see a new world by capturing the students' embodied learning process.

The emergence of a new world means that objects that had been latent to the environment have become salient to the human subject. Yet how might we theorize a phenomenal object that comes forth from the background? Enactivism suggests that

perceptual structures emerge from the background when they repeatedly facilitate our engagement in tasks requiring the performance of new movement forms. Yet how precisely is this happening? How should we operationalize the micro-process of perceptual emergence? To answer this question, we sought more radical stances.

Hutto and Sánchez–García [2015] propose a radical-enactivist interpretation of skilled athletic performance. They interpret skilled performance as utilizing specialized action-oriented relations with the environment. These relations are perceptual “anchors” into the environment that determine attentional routines guiding effective motor action. These *attentional anchors* are thus action-oriented sensorimotor perceptual constructions—the perceptual components of affordances. Abrahamson and Sánchez–García [2016] borrow the construct of attentional anchors to refer to the structures that study participants purportedly constructed to solve Trainer motor-control problems, such as using the spatial interval between the cursors to facilitate bimanual coordination.

Attentional anchors (hence AA) might originate in a gaze that is strategically cast between two or more manipulated objects, such as the cursors, so as to maintain them in peripheral vision, similar to a juggler who gazes not at the balls themselves but at an empty spot above her [Hutto, Sánchez–García, 2015]. That is, *students discover and use an AA, because it enables them to perform movements that conserve a select dynamic stability of an emergent system they are thus building and transforming.*

The function of perception in organizing motor action is possibly more critical than the literature has surmised. Empirical findings from studies of perception and action suggest that AAs can be generated independent of sensory access to one’s actuating limbs. Thus, perception takes the lead. When participants cannot see their hands, still,

voluntary movements are organized by way of a representation of the perceptual goals, whereas the corresponding motor activity, of sometimes high complexity, is spontaneously and flexibly tuned in [Mechsner et al., 2001, p. 69].

As such, generating AA may be a natural inclination of biological organisms’ embodied cognitive architecture. Perception is sentient enactment [Noë, 2006].

Led by collaborating researchers at Utrecht University, the next study applied eye-tracking methods to monitor the sensory behavior of students engaged in the solution of Trainer motor-control problems. Corroborating and expanding on our earlier clinical findings, we now had a new form of empirical data that we could put forth as evidence supporting our hypothesis that students’ task-effective bimanual coordination is associated with changes in the composition of their perceptual orientation toward the sensory display (see Fig. 11). We concluded that attentional anchors serve a vital function in the accomplishment of coordinated bimanual action. Moreover, our study participants’ mathematical discourse *about* these perceptual structures suggested that they constitute important cognitive pivots from unreflective engagement to disciplinary reasoning. These findings recur across a set of variants on the original Trainer task [Abrahamson et al., 2016].

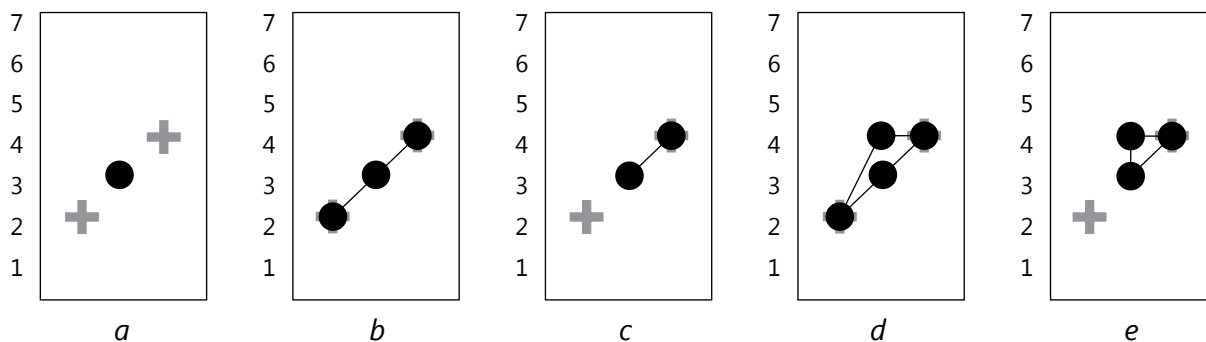


Fig. 11. Dynamical gaze patterns reveal a variety of attentional anchors. The focal gaze point between the cursors is at an un-manipulated location. Pattern *b* was the most prevalent among participants

NEW WORLD EXPANDING: THE TRAINER BEYOND PROPORTION

Now equipped with these new empirical data of students' combined multimodal problem-solving behaviors — both clinical and eye-tracking data — we felt more confident in claiming that: (a) the Mathematics Imagery Trainer environment realizes Abrahamson's action-based genre of embodied design; and (b) this genre achieves its objectives of fostering students' development of sensorimotor perceptions bearing semiotic potential as grounding new mathematical notions. Next we turned to apply these theoretical, pedagogical, and methodological ideas to the design of additional Trainer activities. Here I will briefly mention two more designs for grounding mathematical concepts in sensorimotor perception.

Parabolas

Figure 12 features two configurations of a Trainer for parabolas. Here, the triangle is green only when $BC = AC$. A is fixed at the parabola's focus, B runs along the horizontal dashed line immediately below C , and the student manipulates only Vertex C . By keeping the triangle green while moving Vertex C , the student effectively inscribes a parabola curve. (Note that Labels A , B , and C as well as the dashed lines in this figure are used only here to illustrate the design for readers of this text: these lines are never

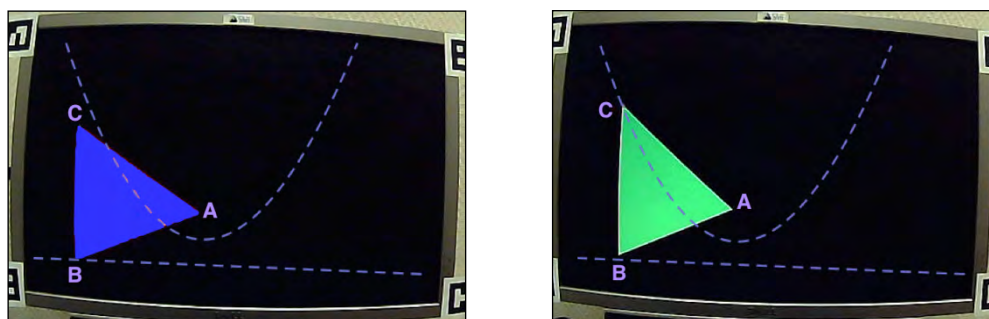


Fig. 12. An action-based embodied design for parabolas. By manipulating Vertex C , in an attempt to keep the triangle green, students find themselves inscribing a parabola

shown to the students, as they engage in the activity.) Participant college students learned to move in green, and then they were guided to derive a definition of the parabola from geometrical properties of the isosceles triangle and auxiliary constructions [Shvarts, Abrahamson, 2019]. The key cognitive event, along this solution process, was perceiving the isosceles triangle. Once they saw it, participants immediately became more fluent in operating the device according to task specifications.

Trigonometry

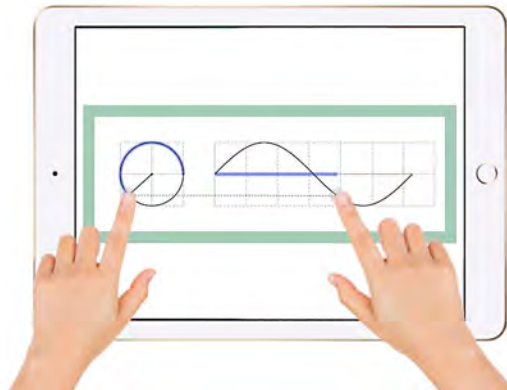


Fig. 13. An action-based embodied design for trigonometry. The frame is green, if the unit-circle's radian (on left) corresponds to the sine-graph's x -value (on right)

Figure 13 features a Trainer activity for trigonometry. Here, the student slides their left-hand fingertip on the perimeter of a unit circle, while sliding the right-hand fingertip on a sine graph. Whenever the radian value on the circle corresponds to the x -value in the sine graph, the rectangular frame around the interactive zone becomes green. The student needs to keep the frame green while moving both hands. Data analysis of a pilot study with participant college students suggests that they imagined a horizontal line segment connecting the two fingertips (not shown in Fig. 13).

The horizontal-line attentional anchor seemed to help participants keep the two fingers at the same height. Mathematized, this imaginary line then came to mean that the left- and right fingertip positions are equally high or low on the grid, thus sharing the same y -value, which is $\sin(x)$. This awareness appeared further to support the enactment of green-keeping movement [Alberto et al., 2019].

CONCLUSION: THE WORLD TODAY, THE WAY I SEE IT

Mathematical concepts are grounded in sensorimotor perceptions that emerge as practical solutions for the efficient enactment of goal-oriented ecologically coupled movements. Sociocultural reframing of these sensorimotor perceptions occurs through the timely mediation of symbolic artifacts, when learners participate in facilitated cultural practice. Students adopt these artifacts spontaneously as readily available means of enhancing their goal-oriented actions, yet in so doing they surreptitiously appropriate

heritage conceptual systems that enable them to participate in the discourse and social enactment of cultural–historical mathematical practice.

I believe it is important to help students maintain their original sensorimotor perceptions as their means of grounding mathematical concepts, even as we support them in appropriating and exercising the complementary powerful cultural devices of mathematical practice. Just how teachers should do this remains an open question. However, I further believe that the multimodal affordance of natural communication, particularly the combination of speech and gesture, bears promise [Abrahamson et al., 2012; Flood, 2018; Fuson, Abrahamson, 2005]. As such, one way forward is to study how teachers engage with students' grounded sensorimotor perceptions to sustain ecological meaning in mathematical concepts.

MOVING FORWARD: EXPLORING OUR NEW WORLD TOGETHER

It has been thrilling to discover the new world of sensorimotor perception that facilitates our engagement with the environment, orienting our every mundane operation, and to conjecture as to the horizons this discovery opens up for educational theory and practice. There are ethical issues at stake. For example, the action-based genre of embodied design is reshaping our approach to the mathematics learning of sensorily diverse students, such as those who are blind or visually impaired [Abrahamson et al., in press].

Like Columbus, however, it has been sobering to realize that the novelty of this new world is truly subjective. This new world as I found it had always been there, eons before my collaborators and I came along to first cast our eyes on it and claim it. Yet unlike Columbus, this new world that our research is now colonizing has truly always been our own — ours, yours, everyone's. This beautiful coral reef has been there forever below the surface of our own sensorimotor waters, waiting.

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