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Embodied Design of Digital Resources for Mathematics Education: Theory, Methodology, and Framework of a Pedagogical Research Program

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Abstract

Embodied Design is a long-term multi-project educational research program committed to advancing the field's understanding and improvement of teaching and learning processes. Operating in the design-based research approach, embodied-design investigators imagine, build, and evaluate problem-solving activities designed for students to get first grips on targeted concepts through physical interaction with dedicated technologies. The research program strives to produce an intellectually coherent paradigm, replete with theoretical models, a design framework, sample activities, and mixed-methods instruments and analytic techniques for capturing and interpreting students' multimodal sensorimotor behaviors and physiological responses as they engage with the activities. Embodied Design posits that to understand a new concept you must first learn to move in a new way that enacts the concept, and yet to move in a new way you must come to perceive the environment in a new way that affords new sensorimotor coordination. Designers therefore create motor-control problems whose perceptual solutions prospectively ground the target concept. Integrating radical enactivist philosophy, dynamic systems theory, and cultural–historical psychology, the paradigm explores how mathematical cognition of specific concepts emerges in perception–action loops and how students' new enactive capacity then becomes socially elaborated in disciplinary discourse through the mediated adoption of professional tools as pragmatic-cum-semiotic frames of reference. The chapter lays out the philosophical and theoretical foundations of the Embodied Design paradigm and explains the design rationale and research findings through discussing representative activities. The chapter further explains how teachers facilitate the activities through multimodal tutorial intervention, how teachers creatively apply the paradigm more broadly, how the research program seeks to serve students of diverse sensorial capacity and neural composition, and what theoretical and practical challenges lie ahead.

Keywords
Attentional anchor, Ecological dynamics, Enactivism, Perception, Special Education

Preamble and Structure

When Dor Abrahamson founded the Embodied Design Research Laboratory at Berkeley in 2005, he borrowed the phrase "embodied design" from a team of Dutch industrial designers bent on creating commercial products attuned to humans' tacit embodied phenomenology (van Rompay & Hekkert, 2001). The phrase "embodied design" appears also in the work of Thecla Schiphorst and collaborators, who examine HCI investigations of movement (Alaoui et al., 2015). As the embodiment paradigm in the cognitive sciences expanded to encompass embodied, embedded, extended, enacted, ecological, and emergent qualities of situated phenomenology (Newen et al., 2018), the meaning of "embodied design" complexified in dialogue with the growing literature.

Embodied design (ED, Abrahamson, 2009; Abrahamson & Lindgren, 2014; Abrahamson et al., 2020) is a design-based research program investigating the phenomenology of designing, teaching, enacting, and learning a culture's cognitive practices. ED's motivating *question* is, What would education look like if it were loyal to the most radical ideas from the embodied philosophy of the cognitive sciences? ED's *design rationale* is to leverage humans' innate biological capacities for enhancing their interactions with the environment and then surface these interactions for reflective dialogic reconfiguration through appropriated semiotic forms. ED's *theoretical* perspectives braid constructivism, enactivism, sociocultural theory, ecological psychology, and dynamic systems theory as these synergistically account for evidence of knowing, teaching, and learning. ED's *conceptual* reach is pan-curricular, albeit its current ambit primarily encompasses STEM domains, mostly mathematics. In *form*, ED's innovative educational resources are either material, digital, or hybrid. ED products are *accessible* to students of diverse sensory, motor, and neural constitution, while nurturing from investigations into *diverse* cultural epistemologies. ED *methods* include: quasi-experimental methodologies for gathering multimodal data from task-based semi-structured clinical interviews; collaborative micro-ethnographic qualitative analysis of teaching–learning interaction; ethnomethodology and conversation analysis, as combined with co-operative action; and multimodal learning analytics, such as cross-Recurrence Quantification Analysis. ED's *outreach* is to collaborate with teachers in developing applicable principles of instructional practice with embodied-design resources, prepare teachers to create their own design solutions to emerging pedagogical situations, and rethink the contents and nature of classroom discourse.

In all this, ED perceives the role of interactive digital resources as constituting *instrumented fields of promoted action* (Abrahamson & Trninic, 2015). These technological learning environments occasion opportunities for students to engage with challenging motor-control problems whose perceptual solutions foster new pre-semiotic cognitive structures at the core of reasoning about mathematical notions (Pirie & Kieren, 1989), such as vestibular feeling of balance to mobilize algebraic thinking or kinesthetic sense of covariation to mobilize proportional thinking.

ED's *claim* is that conceptual reasoning, such as in thinking through a mathematical problem, coopts the primitive biological resources humans tacitly use in engaging the natural and cultural environment, which are goal-oriented cognitive structures that enable perception to guide action (Abrahamson, 2021). These cognitive structures emerge developmentally from recurring sensorimotor patterns, as one figures out or is taught practical solutions to motor-control problems of enacting challenging movements under stable social, material, and task constraints. For example, you learn *where* and *when* to look and *what* to do as you're attempting to collaboratively equipartition a collection of objects or to rhythmically take turns in beating milk into butter. Then, furnished with supplementary semiotic artifacts provided by attentive facilitators, such as a grid, a ruler, a turn of phrase, or a tabular matrix of containers, we appropriate invariant figural features of these new cultural instruments as perceptual cues for new operational–discursive habits of being and acting. In so doing, we reconfigure our own movement forms, now utilizing the instruments as frames of reference, thus grounding normative disciplinary thinking, acting, and expressing in multimodal enactment.

Put simply, learning is moving is new ways: (a) to understand a concept, you must figure out how to move in a new way; and, yet, (b) moving in a new way depends on perceiving the environment in a new way; (c) cultural–historical artifacts can enhance our perception-for-action by highlighting features of the environment critical for motor control, thus outsourcing (distributing, extending) our perception–action loops; and, yet, (d) by endorsing artifacts into our perception–action loops, we appropriate their inherent mechanisms of disciplinary practice, including discourse and action routines. In sum, learning a concept is developing a new perceptual capacity by solving a motor-action coordination problem. *Per Embodied Design, if you want somebody to learn Concept X, build an interactive dynamical instantiation of Concept X whose normative operation is subordinated to the perceptual solution of a motor-control problem*. Perception is the key psychological construct in ED's inquiry, design, and theorization: perception is both the cognitive facilitator of movement and the epistemic mediator from action to symbol. Moving is cognitive activity, moving in a new way is perceptual problem-solving, and perceptual solutions drive conceptual insight.

The chapter elaborates on all the above through discussing examples of embodied designs for mathematics education. Ultimately, the chapter seeks to position Embodied Design as a coherent research paradigm integrating and implementing developments in theory, technology, and methods for evaluating conceptual learning as the multimodal sensorimotor grounding of cultural practice. The chapter is organized in the following sections:

1. Introducing the ED research program as situated in the rise of the embodied paradigm.
2. Framing the ED research program theoretically. Citing sample ED activities, we will highlight the function of movement in ED's design architecture and juxtapose it with the function of movement in other digital resources for discovery-based mathematics learning.
3. Reporting on findings from multimodal learning analytics of empirical data gathered in studies of students' perceptuomotor mathematical learning.Explaining SpEED—Special Education Embodied Design—and surveying its output to date.Explaining and demonstrating the roles that instructors play in supporting students' learning with embodied design.Offering a high-school mathematics-teacher's reflections on the classroom implementation of embodied design more generally in the form of enactivist mathematics pedagogy.Concluding with directions for future efforts of the ED research program.

1. Embodiment and Embodied Design: An Introduction

These are exciting times to be learning scientists with interests in cognition, learning, and teaching, because three major historical efforts—in *theory*, *technology*, and *methods*—are now converging, opening up new research possibilities that, until quite recently, were quite unimaginable. One new research possibility emerging at the intersection of theory, technology, and methods is using multimodal data in the empirical evaluation of action-based embodied design for mathematics education (cf. Arzarello & Robutti, 2010; Hackenberg & Sinclair, 2007; Leung et al., 2013). Each of these three lines of progress—in epistemological philosophy, embodied-interaction technology, and methodological instrumentation for monitoring multimodal learning processes—is relevant to educational research. Yet, as Venn diagrams often implicate, it is the intersection—the integration of theory, technology, and methods—that will draw our expositional attention in this section. This integration is generative for educational research. On the one hand, this section argues, the field's prior beliefs about the nature of teaching and learning have been implicitly shaped and constrained by what was then the field's state-of-the-art: now-obsolete cognitivist philosophies of knowledge; low-interaction digital technologies for building pedagogical activities; and barely nascent methodological instruments for capturing physiological and sensorimotor markers of cognitive activity. On the other hand, the section further argues, educational research in the embodied design paradigm coherently implements important breakthroughs in our field to investigate "the future development of technology that is sensitive to the principles of biological cognitive systems" (Glenberg, 2006, p. 271). In particular, this section outlines ED's perspectives on theory (1.1), technology (1.2), methods (1.3), and learning process (1.4), and then, by way of demonstration, asks *you* to perform a "low-tech" ED task (1.5) to experience the paradigm firsthand(s).

1.1 Theory: A Paradigm Shift Toward Embodiment

Recent decades have seen substantial efforts across the disciplines of philosophy and the cognitive sciences to revisit axiomatic tenets of phenomenology, epistemology, and ontology as these pertain to perception, action, and cognition. The 20th century staged a paradigm war between traditional Cartesian conceptualizations of the human mind and post-Cartesian alternatives. If we have long resigned ourselves to metaphorizing the mind as a computational device of sorts that inputs signals from multiple sensory organs, encodes and processes this information in a central unit in the form of abstract symbols, and outputs action commands to the actuating organs, many of us are now seriously considering that what we call cognition is not boxed away from the embodied modalities of lived experience but, rather, is an inherently modal activity, even when this modal activity is not readily evident to an onlooker (Varela, 1999). Knowledge is not a noun—it is a verb, it is knowing, a mode of doing (Núñez & Freeman, 1999). There are no internal vs. external facets of intelligence—only perceptually guided actions embedded, enacted, and extended in the actual or imagined phenomenal world (Malafouris, 2020); for example, manual gestures are not in or out of the head—they are thinking incarnate (Nemirovsky et al., 2012). Granted, everything we learn to do may be sustained as residual organic changes in the neural composition of cerebral tissue. It is, in a sense, stored in chemical potentialities. But the lived actualization of doing—to wit, be*ing*—is necessarily, and by our very evolutionary biological constitution, modal experience (Gallagher, 2015). In particular, the doing of thinking actuates our sensorimotor capacities, and these capacities, in turn, are iteratively honed through our socio–material interactions with the natural and cultural ecology, both nurturing from and enhancing our participation in the collective enactment of artifact-based practices (Vygotsky, 1926/1997). Thinking simulates perceptually guided action, and, so, if we want people to think in new ways then we need to create conditions for them to operate in new ways in the world. In more sense than one, learning is moving in new ways.

1.2 Technology: Innovation in Embodied-Interaction Interfaces

Embodied-interaction environments solicit the user's naturalistic multimodal forms of situated engagement. Working with embodied-interaction systems, people express knowledge as immersed sensorimotor know-how, and they learn through engaging with regulated feedback regimens (Dourish, 2001). Embodied-interaction designs cater to our mind's ancient object-oriented cognitive architecture—we orient to the world by way of seeking to manipulate—to handle situations by operating objects, whether these be concrete, virtual, or imaginary (Abrahamson, 2021). Yet, whereas commercial designers seek to create NUI (natural user interface) where embodied interaction is seamless, unreflective, and "invisible"—aspiring, thus, to minimize and even remove any learning curve—educational designers may choose to create situations where users (viz. students) struggle to figure out forms of engagement that enable the manipulation of a situation according to task specifications. In a sense, we can design for students to *learn* a new natural user interface, that is, to make a new form of embodied interaction familiar and, eventually, seamless (Black, 2014; Marshall et al., 2013). These new forms of embodied interaction could be dynamical instantiations of ideas, such as mathematical concepts (Hackenberg & Sinclair, 2007; Shvarts et al., 2021). As such, learning to operate the world in a new way is learning to think in a new way, where new mathematical concepts can be grounded in the phenomenology of enacting new perceptually guided actions on given situations (Abrahamson & Bakker, 2016; Arzarello et al., 2005). The ED research program seeks to articulate for educational practitioners a set of theoretically based and empirically validated guidelines for selecting technological platforms and creating interactions that foster these new ways of moving, new ways of perceiving, new ways of thinking.

1.3 Methods: Sensory Measures and Learning Analytics

We now have a vast range of instruments to measure and monitor biological indicators of physiological and neural activity. Consider commercial products, such as the various sports gear for quantifying our cardiovascular activity as we jog. This general type of equipment, which turns our body into a moving laboratory, is useful for educational research, too, because it can capture activity and changes in biological measures believed to indicate not only physical effort to accomplish athletic feats but also cognitive effort to solve problems (Lee et al., 2021). That is, we can monitor the biology of learning, sometimes even as it is happening. Moreover, not only do we capture information from multiple modalities of sensation and action (e.g., Ferrari, 2020), we can analyze and visualize this information in search of patterns (Worsley & Blikstein, 2014). For example, we can triangulate data from: (1) performance measures of a person's manual actions in some motor-control task; (2) eye-tracking instruments that capture and monitor where they are looking and what gaze paths they are developing; and (3) clinical data of what they are reporting about their experience, attention, and thoughts (Shvarts et al., 2021). Thus, biosensors may inform us not only *that* a person is learning but even *what* a person is learning; and they reveal *how* new perceptuomotor capability emerges through goal-oriented embodied interaction, that is, the multimodal nuts-and-bolts of how learning transpires (Tancredi, Abdu, et al., 2021).

1.4 The ED Learning Process

An *action-based embodied design* engages students in an activity centered on using available personal capacities, material resources, and, possibly, fellow participants in coordinating the motor-control of a situation (Abrahamson, 2014). In the course of tackling this challenge, students eventually figure out the movement they should enact in order to satisfy the task, and yet, initially, they are not able to organize their sensorimotor activity to enact this movement. In addressing this emergent problem, students experience opportunities to build upon yet modify what they know to do. They solve this motor-control problem in the form of a new perceptual orientation to the situation. This enacted solving of the task problem constitutes the design's core learning outcome. The problem may come in the form of an apparent need either: (a) to exercise, warrant, and argue for intuitive inferences; or (b) to enact a particular movement form, which, in turn, requires planning, scaling, coordinating, or predicting the consequences of actions; both goals that may require greater precision and new semiotic forms (Abrahamson, 2009b, 2014). The task itself initially features little to no symbolic notation and is as generic as the context enables (Rosen et al., 2018). Only once students have developed a solution, that is, they have learned to move and think in a new way, are they offered symbolic artifacts, usually a variety of tools for measuring and otherwise organizing and monitoring their solution operations (Chase & Abrahamson, 2015). Students are prone to adopt these artifacts, because they identify in them potential utilities for enhancing the enactment, evaluation, or explanation of their earlier solution. Yet in the course of adopting the tools as available means for these pragmatic, epistemic, or discursive ends, students surreptitiously elaborate their pre-symbolic orientation toward the situation into normative disciplinary practice pertaining to this class of situations—they see, discuss, and signify the situation in a new way, perhaps modifying their solution strategy in assimilating the new tools (Abrahamson et al., 2011).

We are about to explain the positionings and objectives of our ED work. Yet, to do so, it would help if you, our gentle reader, partake in a small motor-control activity that will not demand any digital resources. The activity will hopefully begin to illustrate how moving in new ways is related to thinking in new ways. We will be inviting you to introspect into the phenomenology of your cognitive activity to experience how perceiving emergent imaginary structures in the environment equips you to move and think in new ways that may lead to a mathematical understanding.

1.5 Interlude: A Brief Participatory Demonstration of Embodied-Design, With a First Explicit Experience of Attentional Anchors

We are asking you to kindly do the following. Seat yourself comfortably at a desk. Place both hands palm down in front of you on the desk, fists closed, but for the index fingers, which should be extended sagittally, as if pointing ahead. Now slide your right-hand index finger under your left-hand index finger, so that the two fingertips are stacked on the desk in front of your left shoulder. Let this point in space, where the two fingertips are stacked, be the origin of a Cartesian coordinate system, whose first quadrant lies on an imaginary plane rising from the desk to stand vertically in front of you. Your left-hand index fingertip may move up and down along the y-axis rising from this origin, and your right-hand index fingertip may move right and left along the x-axis extending to the right of this origin. You are further asked to move your hands at the same time along their respective axes, such that the right-hand index fingertip is always twice as far from the origin as is the left-hand index fingertip. Thus, we are asking you to enact a bimanual movement form that is simultaneous, continuous, orthogonal, proportional.

Most people find this task quite daunting. They report that their movements are halting and sequential rather than simultaneous, their attention keeps darting between their two fingertips, their vertical axis tilts to the right or left, and even that they experience slight nausea, vertigo, or vestibular discomfort! Apologies for the inconvenience. Hang on in there…When we presented this task to young children in an interactive tablet application, many spontaneously devised the following means of coordinating their hands, which we are inviting you to assay.

Position your fingertips at any pair of locations along their respective axes where their distances from the origin correspond to a 1:2 ratio per task specifications. Now, imagine a line beginning at your left-hand index fingertip and running down *diagonally* to the right, ending at your right-hand index fingertip. This imaginary diagonal line will always be subtended between the two index fingertips. Attend to this line and try to "see" it vividly. This line is a "thing"—an imaginary auxiliary construction—that you will soon be manipulating directly, as though it is an elastic thread stretched between your fingertips. Now move this diagonal line to the right, keeping constant its specific slope (an acute angle just under 27°, the arctan of 0.5). To emphasize, you are now attending to the line, not to your hands. You are operating *this line*, sliding *it* laterally. In so doing, maintaining the geometrical properties of this construction invariant in turn constrains you to maintain the specified 1:2 ratio. With some practice, you should become quite facile at performing this modest party trick.

The imaginary line is an interesting phenomenon. Its very existence is predicated entirely on your consented concerted perceptual effort to evoke it into your engaged phenomenology. The subjective appearance of this line may feel more "private" than the sight of your fingers—the fingers are actually there, whereas the diagonal is only make-belief there. And yet this line, if not literally palpable, presents itself as a *bonafide* percept to be reckoned with—an *ad hoc* though perhaps *ex nihilo* thing you conjured out of negative space to become an object you are manipulating. This voluntary percept, which popped into your lived experience, facilitates the enactment of the task's bimanual, continuous, orthogonal, proportional movement form by way of consolidating two independent hand motions onto a single object that the hands are co-manipulating.

That this imaginary line appeared spontaneously for our young study participants (Shayan et al., 2015) suggests that human neural architecture is inclined to discern and generate invariant action-oriented perceptual forms as its operative means of assimilating dynamical information structures in the environment (Piaget, 1971). What you just experienced is the birth of a new sensorimotor schema grounded in new neural potentialities. The schema is your new cognitive capacity. It "lives" only inasmuch as you evoke it, as a relational self–world suture enhancing your purposeful grip, enabling you to move in a new way. Later, we will refer to the diagonal line as an *attentional anchor*—in the sense that the line anchored your visual–haptic attention to the environment (Hutto & Sánchez-García, 2015)—and we will discuss the philosophical motivation, theoretical framing, methodological approach, empirical evidence, and pedagogical reach of grounding mathematical concepts in these spontaneous Gestalts (Abrahamson et al., 2016; Dessing et al., 2012; Hutto et al., 2015; Mechsner, 2003). At this point, we will highlight only that this emergent, portable, and dynamically extensible diagonal line constitutes a cognitive ontology instantiating the 1:2 mathematical relation as a diagrammatic invariant of a potential new conceptual category. Moreover, your rule-based manipulation dynamics created a dilating set of geometrically similar right-angle triangles with the subtended axes as orthogonal legs and the line as a diagonal hypotenuse. All these perceptual elements, both actual and imaginary, can be reproduced on paper and formally analyzed (Bongers, 2020).

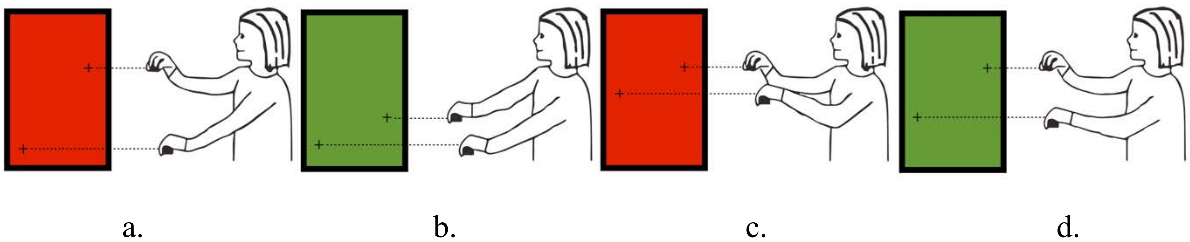


Figure 1. The embodied-design Mathematics Imagery Trainer for Proportion—the Parallels motor-control problem: schematic depiction of four key events in learning to enact the proportions conceptual choreography. (Abrahamson et al., 2019; artwork: Virginia J. Flood)

Our exposition of embodied design, above, referred to the Orthogonals motor-control problem as a context to offer the reader a firsthand(s) experience of the activities, explain the framework's rationale, and introduce the idea of an attentional anchor. The remainder of the chapter, however, will often allude to the Parallels problem, a geometrically simpler scenario, where the right- and left hand move up and down along parallel vertically oriented trajectories (see Figure 1). Receiving a red screen (Figure 1a), the student is tasked to make the screen green by moving two cursors up and down, one with each hand. Through exploration, she happens to produce a green screen (Figure 1b), because she has placed the cursors at locations where their respective heights above the bottom of the screen relate by the yet-undisclosed ratio, here 1:2. Trying to keep the screen green while moving both hands, she maintains the two cursors in fixed formation, but she thus violates the ratio and so the screen turns red again (Figure 1c). Eventually, she figures out that the higher her hands go, the bigger should be the distance between the cursors (Figure 1d).

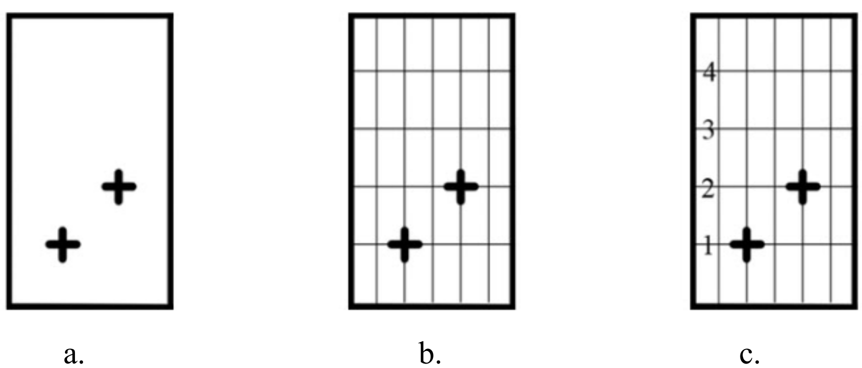


Figure 2. Three schematic interface modes of the Mathematics Imagery Trainer for Proportion—the Parallels motor-control problem: (a) cursors only; (b) grid; and (c) grid and numerals. (Abrahamson et al., 2011)

Once students who are tackling a problem such as the Parallels have succeeded in moving their hands fluently "in green," the instructor introduces into the working space supplementary resources—symbolic artifacts, such as a grid and numerals (see Figure 2). Initially, students adopt these semiotic resources as means of enhancing the enactment, explanation, or evaluation of their strategy. For example, the horizontal grid lines constitute convenient perceptual targets for raising their hands. Yet, in so doing their perception of the environment changes in line with normative disciplinary practice. As the continuous space becomes discrete (compare Figures 2a & 2b), the form of students' hand movements shifts from simultaneous to sequential (e.g., for every 1 unit the left hand goes up, the right hand goes up 2), and their language shifts from qualitative to quantitative (e.g., from "higher" to "double"). For a survey of students' solutions to the Parallels problem, see Abrahamson et al. (2014).

Having situated embodied design as operating at the intersection of recent theory, technology, and methodology, and having adumbrated how attentional anchors may constitute the perceptual interface between action and concepts, we are now prepared to state more directly what embodied design is. As you read the next section, may we suggest you bear in mind—that would be your embodied, enactive, extended mind, of course—our recent bimanual exercise as a schematic exemplar of ED activities

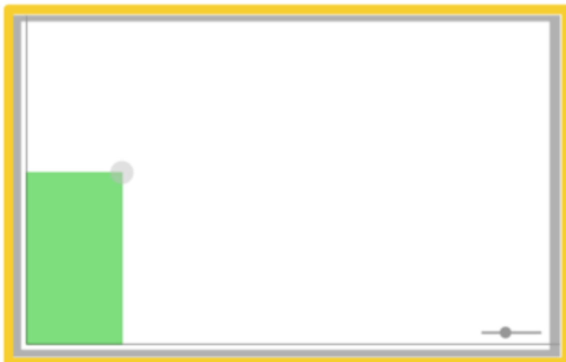
2. Overarching Rationale and Principles of the Embodied Design Research Program

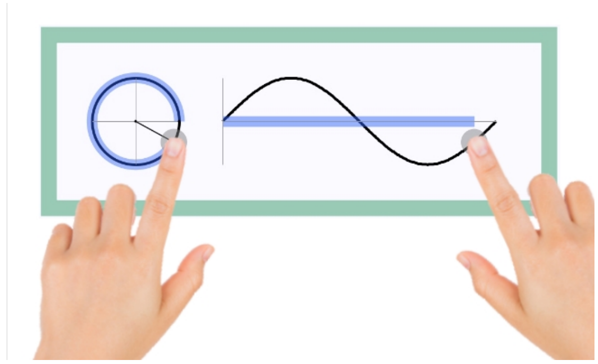
The embodied design program has been engaged in a systematic evaluation of embodiment scholarship—its philosophy, theories, and research—as potentially illuminating enduring conceptual and practical problems in the scientific field of mathematics education. This research program has centered on assessing the purchase of various embodiment frameworks on intriguing empirical data gathered in the context of developing experimental activities for the teaching and learning of mathematical concepts.

In more sense than one, we believe that learning is moving in new ways. We take movement to be the primordial function, driver, and expression of intelligence (Allen & Bickhard, 2013; Sheets-Johnstone, 1999). Phylogenetically, the capacity to enact movement brings our species in contact with nutrients, lets us escape threats, and enables us to consort with conspecifics (Maturana & Varela, 1992). Ontogenetically, motor and perceptual capacity develop in the service of purposeful locomotion (Adolph et al., 2018; Heft, 1989; Turvey, 1992). Building on enactivist premises (Varela et al., 1991), we look to understand how new perceptual capacity develops and how it can be fostered into mathematical knowing (Pirie & Kieren, 1989). If mathematical concepts are grounded in sensorimotor perception, and if perception be the capacity to move purposefully, then to develop new mathematical perceptions we should begin by setting up conditions for students to learn to move purposefully in some new way that would require perceiving the situation in a new way.

Viewed from the perspective of embodied design, mathematics education is, thus, the practice of creating and facilitating structured opportunities for students to develop new sensorimotor perceptions in the service of moving in ways that advance their contextual purposes and, in so doing, advance their cultural appropriations (Abrahamson et al., 2011). Structured opportunities for learning to move in new ways guided by perception necessarily involve some sensory manifold—an environment toward which one orients perceptually to guide one's movement-based actions (Abrahamson & Trninic, 2015). That is, perceptuomotor activity is intrinsically situated, in the sense that it is inherently coupled to an environment, whether actual, simulated, or imaginary (Thelen & Smith, 1994). Understanding a new mathematical concept is a perceptual achievement (Nemirovsky & Ferrara, 2009).

ED is a generative framework in the sense that it can guide fellow investigators in developing new experimental activities (Bakker et al., 2019). At Utrecht University, Anna Shvarts has been leading design-based research projects centered on a collection of new embodied designs for mathematical concepts. Figure 3 features two sample Mathematics Imagery Trainers implemented in tablets.





a. Drag the rectangle's top-right vertex (see grey circular cursor) to make the rectangle green, then continue dragging the vertex, keeping the rectangle continuously green. The emergent class here is equivalent area. What is the function of the emergent curved path of the green-conserving cursor? (Shvarts, 2017)

b. Simultaneously drag points along a unit circle (left hand) and its corresponding sine graph (right hand). An imaginary line between the fingertips (not shown here) can be lit to cue an attentional anchor. Students discover that this line should always be horizontal to make the frame green. (Alberto et al., 2019)

Figure 3. Two sample embodied-design activities from the Utrecht University team.

The ED educational program to ground mathematical concepts in movement bears ecological validity. Ethnographic testimonies from expert mathematicians suggest that their working process is an embodied and enactive search for a perceptual grip on their own emerging ideas—a perceptual grip that would serve to ground the meaning of the emerging ideas; a grip that may later be described in rich imagistic metaphorizing language; a grip that may then be revisited analytically through attention to frames of quantitative reference (Díaz-Rojas et al., 2021; Hadamard, 1945; Pallasmaa, 2017; Tao, 2016). As such, investigating perception appears to be profoundly important for both the science and practice of mathematics education.

A scientific inquiry into the development of mathematical cognition is, therefore, foremost an inquiry into the development of perception—where it comes from, how it changes. Perception is the core ontology under examination, the pivotal psychological construct at the center of our inquiry into learning (Abrahamson & Mechsner, 2022). We examine how mathematical perception could be motivated, nurtured, and signified in socio-cultural–material–epistemic–discursive contexts.

A stipulation of the action-based genre of embodied design is that students discover perceptual structures, such as the diagonal line, that enable them to coordinate the enactment of targeted movement forms designed to instantiate mathematical concepts, for example, bimanual proportional progression. These discovered perceptual structures constitute self-imposed task constraints—students figure out that they must move in a new way *even as* they abide with this imaginary structure, *given that* the obdurate world appears to function in some new unfamiliar way. This student-directed assimilation–accommodation process emerging from our activities is a hallmark of our design framework that distinguishes it from other interactive software for dynamical mathematics learning, where the environment relieves students from accommodating their coordinations, because in those environments students' non-conforming actions are digitally precluded (Abrahamson & Abdu, 2020).

In sum, if we are to take seriously the current paradigm shift towards an embodied view of human cognition, then we might think about mathematics education as following. Learning a mathematical concept is developing a new perceptual capacity by solving a motor-action coordination problem. The role of education is to engineer for students conceptually relevant motor-control problems and guide students' solution attempts by introducing productive constraints on their search for perceptual orientations enabling motor coordination (Araújo et al., 2020; Newell, 1986; Newell & Ranganathan, 2010). The educational designer perceives the student's manual solution actions as bearing semiotic potential to build meaning for particular target mathematical structures (Bartolini Bussi & Mariotti, 2008). In order to steer the student toward these structures, the instructor introduces into the problem space new frames of reference and symbolic artifacts. The student interprets these objects as means of better enacting, explaining, or evaluating their motor-action solution strategy. In so doing, the student describes their action in a progressively mathematical register. As new frames of reference and symbolic artifacts are introduced into the enactive–discursive space, the descriptions and re-descriptions cascade into signs whose figural forms increasingly differ from the original grounding actions they supposedly model, even as they may still evoke those enactive meanings. Coming full circle, students enact the grounding movements as a coordinated dynamical Gestalt, even as they are now able to attend to their motor actions analytically (for a review, see Alberto et al., 2021).

3. Multimodal Learning Analytics of Embodied Design: Recurrence Quantification Analyses of Motor and Eye Behaviors in Solving Mathematics Imagery Trainer Problems

If indeed learning is moving in new ways, learners' assimilation–accommodation processes not only *manifest in* but also *occur through* the development of new movement forms and the perceptual forms that facilitate them. From an embodied-cognition perspective, multimodal learning analytics take on new methodological potentials as means of inquiry into nascent conceptual understanding. Extant measurement technologies such as eye-tracking and touchscreen data streams allow for continuous monitoring of motor activity and eye movements, which offer unprecedented insight into the micro-processes of learning in progress.

The brunt of multimodal learning analytics of embodied-design work thus far has taken place in the context of an activity called the Mathematics Imagery Trainer for Proportion, wherein students learn to enact a new bimanual coordinated movement pattern that instantiates proportional transformation (e.g., see Figure 1). In this activity, learners manipulate two bars stretching vertically from the bottom of a tablet screen that turn green when fulfilling some hidden condition. The condition is programmed to some ratio of the left bar's length to the right bar's length (initially 1:2). Learners are tasked with finding green and moving their hands while keeping the bar in green, articulating along the way their rule for how to achieve these desired effects. The activity fosters moving in a new "multiplicative" movement pattern, whereby instead of staying constant (an "additive" pattern), the distance between the tops of the bars expands as the bars grow longer (and vice versa), establishing an embodied basis for reasoning and discourse about proportionality.[1]

Early multimodal learning analytics of Mathematics Imagery Trainer data triangulated for qualitative analysis the learner's solution strategies and verbal–gestural utterance (Abrahamson et al., 2014; Abrahamson et al., 2011), eye-tracking and touchscreen data streams (Abrahamson et al., 2016; Shayan et al., 2017), and tutor activity (Abrahamson et al., 2012; Flood et al., 2020; Shvarts & Abrahamson, 2019). Notably, eye-tracking analysis revealed that participants exhibited new gaze patterns as they progressed through the task, darting away from looking at their moving fingers towards fixating other areas of the screen (Shayan et al., 2015), a phenomenon then captured quantitatively (Duijzer et al., 2017). Later work applied machine-learning methods for strategy detection (Pardos et al., 2018) and intelligent-tutoring agent development (Abdullah et al., 2017), and applied statistical methods to model regimes within and between participants (Ou, Andrade, Alberto, van Helden, et al., 2020; Ou, Andrade, Alberto, Bakker, et al., 2020). These analyses establish the value of multimodal analytics to the embodied design enterprise.

The most recent development in multimodal learning analytics of embodied-design data is the usage of nonlinear methods to study learners-in-context as a complex system. Drawing upon the dynamic systems theory strand of embodied design's theoretical framework, we understand the new movement forms fostered in embodied-design activities as emergent rather than driven by a central controller. The process by which the separate activity of different modal systems become soft-assembled (Richardson & Chemero, 2014) into a new conceptual choreography is thus at the heart of the learning process. To model as a soft-assembly process the dynamical complexity present in continuous eye-tracking and activity-stream data, embodied design turns to a nonlinear method originating in physics, *Recurrence Quantification Analysis (RQA)*. RQA detects repetition and coupling in dynamical systems, including patterns in variability (Marwan et al., 2007; Webber & Zbilut, 1994) and has been applied to modeling dynamics in such diverse fields as physiology, joint action, and economics. As its input, RQA treats quantitative data sets, such as massive tables with entry listings of high-sampling location captures of the hands as they manipulate virtual objects vis-à-vis the eyes' contemporaneous foveal fixation points. We join a handful of RQA forays bearing upon questions of math and science learning (Fleuchaus et al., 2020; Stephen et al., 2009). To date, we have applied RQA to model the dynamics of both bimanual coordination and eye movements as learners develop a grip on a problem space.

We began by applying RQA to compare the coordination dynamics of participants' hands as they progressed through the different spontaneous phases of the Mathematics Imagery Trainer activity identified in prior analyses: initial exploration, discovery of some hand/cursor positions that elicit positive feedback, and, ultimately, a new fluent movement pattern. Across learners' specific idiosyncratic strategies and trajectories, we sought to model the dynamic change as learners achieved the enactment of the activity's targeted movement pattern. RQA enabled us to quantify dynamical features of bimanual coordination including the level of coupling, stability, determinism (predictability), entropy (level of disorder), and duration of connected states in the

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[1] There are a number of design variants on the Parallels task featured in Figure 1. In this section, we discuss a variant where two vertical bars, one for each hand, subtend between the horizonal screen base and the hands.

bimanual system (Tancredi, Abdu, et al., 2021). We found that participants' initial interactions with the embodied-design context were marked by relatively low determinism, reflecting exploratory movements with high variation. In contrast, hand movements were more deterministic once participants discovered positions that elicited green feedback. Eventually, as learners coordinated their hand movements to move-in-green, this new movement form exhibited high levels of coupling as well as high stability.

These findings offer a picture of enactive learning in an embodied-design environment as a process of exploration culminating in the stabilization of a novel coordination pattern. The findings also shed light on when and how this stable pattern arises: Discovering a rule for identifying successive green positions does not yet transition the learner–technology system into a new stability; only in continuous movement did bimanual dynamics exhibit the stability associated with a new attractor. Viewing movement as participatory in the state of the cognitive system, our data suggest that it is only in moving-in-green that the *proportional relation concept* solidifies. This highlights the importance of the *transitional* relation between each discrete green location and the next as critical to the concept of proportionality. The process of *dynamically* relating each of these locations, rather than enacting a sequence of static positions, gives rise to a new movement-qua-understanding. The RQA results corroborate that embodied-design activities can indeed foster the emergence of a new stable movement pattern.

Beyond bimanual *motor* coordination, RQA also sheds light on the development of new *perceptual* forms. In the Mathematics Imagery Trainer for Proportion activity, participants experienced moment-to-moment feedback on the quality of their bimanual coordination pattern by way of the color of the bars. There was no such direct guidance on how to orient their perception to achieve the new movement form. Gaze offers an entry point into the new perceptual forms spontaneously conjured by participants to achieve the new movement form, such as the imaginary projection of the shorter bar onto the middle of the longer one. With RQA analysis, we were able to compare gaze dynamics across the different phases of bimanual coordination to surface inter-participant trends across the idiosyncratic perceptual solutions of each participant (Abdu et al., under review). We found that discovery of greens was associated with a reduced level of disorder in gaze activity. Additionally, the accomplishment of fluent hand movement was associated with greater repetition, determinism, and stability of gaze patterns. Thus, as new bimanual coordinations stabilized, gaze patterns also stabilized, suggesting that learners were attending to and perceiving the problem in a new way, which triangulates with learners' verbal–gestural reports.

Taken together, our findings are consistent with the view that moving in a new way—a conceptually-instantiating mathematical way—is achieved through the soft-assembled activity of multiple modal systems. Indeed, if cognition can be conceptualized as inherently modal, we propose that it is furthermore inherently *inter*modal, structured through the evolving dynamical relations among modalities (Tancredi, Abdu, et al., 2022). Theorizing cognition as a complex dynamical system has implications for multimodal learning analytics methods; from this view, modal streams cannot be modeled as simply-cumulative linear contributions each causing some outcome, but, instead, as mutually affecting components exhibiting nonlinear interactions. The relations among sensory/perceptual/motor streams give rise to emergent structure greater than the sum of its parts. An embodied design approach to multimodal learning analytics, then, is one that treats multimodal data streams not as mere imprints of the activity of a would-be central controller, but rather as interacting components in a self-organized dynamical assemblage.

4. SpEED—Special Education Embodied Design: Rationale and Research

The burgeoning embodied paradigm in the cognitive sciences maintains that our bodily engagements with the world shape our cognitive capacity (Fincher–Kiefer, 2019). These embodied engagements often involve a vast range of cultural–historical artifacts, including material forms, such as various utensils, as well as intangible forms, such as language. Cultural mediators of knowledge—such as parents, teachers, peers, and educational designers—may not be aware of the embodied constitution of cognition. Nevertheless, embodied aspects of their pedagogical behaviors, such as the choices they make in shaping the interactive properties of students' opportunities to participate, still bear direct impact on the quality of students' learning.

In the historical occupations, such as agriculture, hunting, navigating, baking, and artisanal fabrication, skill pedagogy evolved over the millennia, unhampered by lack of formal cognitive theory or educational design frameworks. In this section, however, we note that historical pedagogy's pragmatic agenda often did not consider members of the population with diverse sensory, neural, and physical constitutions. These sectors engage with the world differently, so that their equitable inclusion requires suitable learning environments. But how does one go about designing learning environments for neuro–physical diversity? To begin with, we might consider an epistemological theory by which to understand how a person's embodied composition bears on their skill learning.

The theory of embodied cognition argues that conceptual learning is shaped through sensorimotor engagement. Accordingly, ED contends that the body itself—more precisely, embodied movement—becomes a primary instructional resource: learning occurs when the student agentially and intentionally achieves the enactment of new movement forms (Abrahamson, 2014). It follows that students of *diverse* sensorimotor composition may not ideally engage in conceptual learning using digital interactive resources designed for neurotypical students. Informed by embodiment theory, we could anticipate where conventional resources fall short in serving diverse students and how we should think about making education more equitable. We were thus motivated to further develop the embodied design framework so as to better serve special-education students, hence *SpEED*—*special-education embodied design* (Tancredi, Chen, et al., 2021).

SpEED draws on ED's intellectual foundations in enactivism to explore new directions for accessibility theory, research, and practice. As such, embodied cognition dialogues with Universal Design for Learning. *Universal Design for Learning* (UDL) roots theoretically in cognitive neuroscience to present a set of design principles defined according to the three primary sets of brain networks: action, representation, and engagement (CAST, 2018; Rose & Meyer, 2002). Per enactivism, however, these networks are not independent but, rather, neurally, evolutionarily, and phenomenologically intertwined and irreducible (Abrahamson, 2021; Hutto & Myin, 2013; Varela et al., 1991). That is, *how* you interact with a learning environment is not a mere perfunctory portal into "the content itself," rather, the "how" of engaging itself constitutes the learning. As such, educational design for neuro-physiologically diverse students should understand, valorize, and cater to their diverse perceptuomotor constitutions (Abrahamson et al., 2019).

With a commitment to bridging theory and practice, SpEED sets forth with the following theoretical and ideological principles (Tancredi, Chen, et al., 2021):

1. *Learning happens through the body's sensorimotor engagement with the world.* SpEED roots in embodied theories of cognition and learning, which posit that the nature of

sensorimotor engagement fundamentally shapes the learning that takes place.

2. *Learning begins from learners' existing embodied resources.* Embodied resources include prior sensorimotor experiences, practices, processes, and abilities.
3. *Instruction must flexibly adapt to learners' sensorimotor diversities.* This principle takes up disability studies' commitments to embrace human variation, challenge notions of normalcy, reject deficit ableist models, and recognize the social nature of disability (Ferguson & Nusbaum, 2012). SpEED actively centers learners whose educational potential could be further targeted in the general education classroom. It requires attention to how learners vary in their sensorimotor experience and how such diversities give rise to different cognitive architectures.

The SpEED framework guides the development and empirical evaluation of new learning resources that implement embodied cognition theory so as to serve diverse students. As such, SpEED expands upon the embodied-design framework (Abrahamson, 2009, 2014) by opening new research horizons and foci for the study of sensorimotor engagement (Abrahamson et al., 2019). Literature on multimodality and embodiment (Kress, 2001; Streeck et al., 2011) informs SpEED's structured attention to three interdependent key parameters: media, modalities, and semiotic modes:

1. *media* denotes natural and cultural material/virtual artifacts, such as pen-and-paper or dynamic mathematics environments;
2. *modality* delineates the sensorimotor system recruited by a task, such as the tactile, visual, or vestibular systems; and
3. *semiotic modes* refers to meaning making, which involves different kinds of sign systems (Kress, 2001), such as Sign Language and spoken language

SpEED proposes that media, modalities, and semiotic modes constitute interdependent constraints on students' perception–action loop. Created and managed by design, these constraints could serve students productively in shaping their opportunities to move in new ways undergirding targeted mathematical and other content.

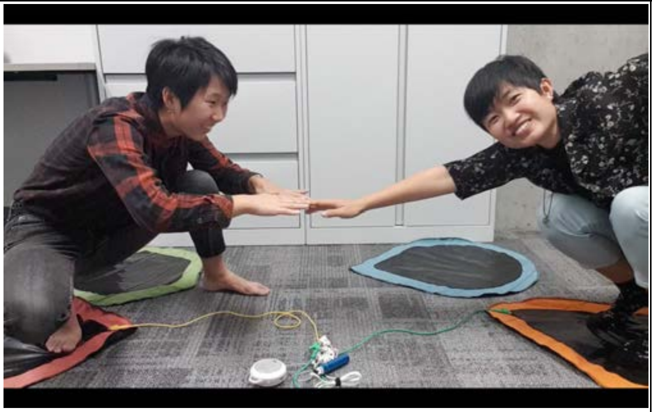
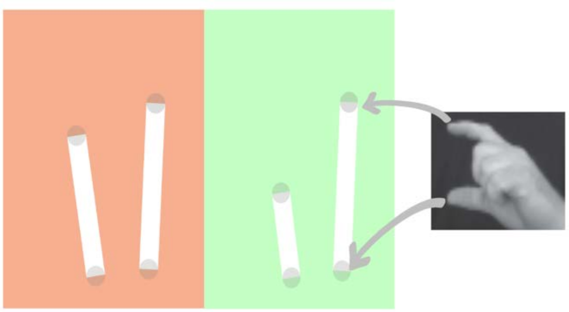
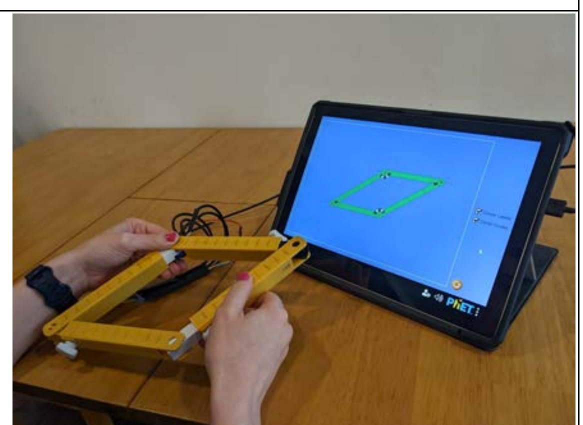
	
a. Magical Musical Mat—an interactive interface for non-speaking Autistic children and their families to produce sounds collaboratively through modulated touch (Chen, 2021)	b. Balance Board Mathematics—an interface for sensory-seeking students to explore mathematical representations through whole-body "fidgeting," e.g., rocking to generate sine graphs (Tancredi et al., 2022)
	
c. Sign|ED Math—fostering for deaf students modal continuity from manipulation to expression by engineering physical interactions that require manual formations signifying conceptually appropriate grammatical meaning (Krause & Abrahamson, 2020)	d. The Quad—a digitally enhanced haptic device for blind and visually impaired students learning geometry in inclusive classrooms (Lambert et al., 2022)

Figure 4. Four sample SpEED designs

Let us illustrate the SpEED principles and parameters through four design-based research projects, each focused on a different population of learners. Chen (2021) develops digitally infused environments, where non-speaking Autistic students and others can learn to interact through non-speech modalities, such as touch-based interaction (see Figure 4a). Participating in her activities, Autistic children's symptomatic repeated motor behaviors (stimming) implicitly become an interactional modality for coordinating emergent joint action with peers and family members. Tancredi designs for vestibular-seeking students by incorporating vestibular stimulation, such as rocking on a balance board, into mathematical learning activities (Tancredi et al., 2022; see Figure 4b). Krause and Abrahamson (2020) build digital resources for Deaf learners, where the shape of the manipulating hand has been designed to incorporate semiotic conventions of the signing hand, thus facilitating students' conceptual understanding through doing–signing modal continuity in social interaction (see Figure 4c). That is, the design recruits the modal affordances of signed languages, so that signing, as a semiotic mode, becomes a resource for mathematical meaning-making that is referentially grounded in a community's shared collaborative practice (Krause, 2017). Finally, Abrahamson et al. (2019) have offered a reconceptualization of equitable design for blind and visually impaired students in inclusive classrooms. Their design enables sensorially diverse students to collaborate on a joint-action task, even as different students experience feedback in different sensory modalities. The lab's collaboration with PhET at Colorado University led to building an accessible online resource (PhET-Interactive-Simulations, 2021) as well as a new line of accessible educatoinal products for haptic geometery learning (Lambert et al., 2022; see Figure 4d).

In mainstream education, populations with diverse sensory and neural constitutions daily encounter *modalism* (Tancredi, Chen, et al., 2022)—the prejudiced attitude of a cultural system that cannot tolerate the modal variety by which people engage with the world. SpEED advocates the design of learning processes grounded in all learners' embodied practices by building media that cater explicitly to diverse modalities as ways of knowing and links these actions and expressions to conventional semiotic articulations. The above-cited SpEED projects each build novel digital and material artifacts that solicit and incorporate diverse modalities—kinesthetic, auditory, tactile, vestibular, proprioceptive, to name a few—into conceptually formative sensorimotor interactions, so that students' different needs become essential in their phenomenology of enacting and articulating novel instantiations of traditional disciplinary content (cf. Turkle & Papert, 1991; Wilensky & Papert, 2010). To work together, we need not experience the world the same way.

By designing for sensorimotor diversity, SpEED offers a means to re-evaluate embodied theories of cognition. Theories of learning have been mostly, if not entirely, developed from studying the sensorimotor capacities of neuro-normative individuals. As a consequence, these theories have unwittingly studied a narrow subset of learning qualities and processes that, in turn, have established in the field of education a set of uniform prescriptions. Uniform prescription for diverse learners is discriminatory. Designing for a variety of learners' sensorimotor capacities opens avenues for expanding upon theories on how people learn, by allowing us to discover how diverse people learn (Tancredi, Chen et al., 2022). Arguably, modal variety is not only interpersonal but intrapersonal, so by discovering how diverse people learn we discover how all people learn.

SpEED aspires to promote a transformative agenda (Abrahamson, 2022; Stensenko, 2002) by challenging prevalent beliefs about learning competencies, expanding learning contexts towards sensory equitability, and empowering all students to learn in accord with their embodied constitution.

5. On Teachers' Multimodal Dialogic Work With Embodied Design

Vygotsky believed that social interactions with more culturally-competent others allow spontaneous embodied experience to grow together with more academic, culturally-specified ways of interpreting those experiences (Vygotsky, 1962). When experienced actors work with newcomers on embodied tasks—such as completing a surgery or using an embodied design—experts *inhabit* the actions of newcomers (Goodwin, 2018). They are able to anticipate a newcomer's embodied activity and perceptions, and make connections between naïve ways of seeing and acting and more professional forms of practice and perception. New meanings and intersubjective understandings emerge from the moment-by-moment interactional work necessary to navigate mutually intelligible courses of co-operative action (Goodwin, 2018) together. This *intercorporeal attunement* (Sheets-Johnstone, 2000) is not automatic, however, and must be constantly negotiated through *dialogic* embodied forms of social interaction (Flood, 2020). How educators attend to, interpret, and are responsive to learners' embodied activities are important aspects of *teacher noticing* (Sherin et al., 2011) and *responsive teaching* (Robertson et al., 2016) in STEM education that help support students' scientific and mathematical discoveries (Flood, 2021; Flood & Harrer, 2022; Flood et al., 2016; Flood et al., 2020; Flood et al., 2022).

Ethnomethodology and conversation analysis (EMCA), and co-operative action (CoA) provide useful frameworks for examining the interactional work that drives this process (Flood, 2018). EMCA attempts to identify the practical methods and resources people use to build, repair, and maintain a sense of shared meaning moment-by-moment in their interactions with one another (Schegloff, 1991). The CoA framework (Goodwin, 2018) enhances EMCA by examining dialogic embodied ways in which participants re-use, decompose, and transform each other's multimodal contributions (e.g., gesture, facial expression, prosody, talk, and so on) to build meaning and action. Students' multimodal utterances are *substrates* (Goodwin, 2018) that can be broken down, recycled, and retooled by educators to interweave and co-construct new ideas from old (Flood, 2020). Together, EMCA and the CoA framework help us appreciate meaning-making with embodied design as an *emergent, nondeterministic process* (De Jaegher et al., 2016) distributed across different individuals, their bodies, and the socio-material environment they interface with.

Examining interactions between educators and students in fine detail with EMCA and the CoA framework has recently brought to light a number of practices for attending and responding to learners' embodied ideas that help facilitate students' STEM discoveries (Flood, 2018, 2021). To support students working with embodied designs, teachers must not only continuously monitor and engage with what students say, but also with how students move and the idiosyncratic ways they make sense of their perceptuomotor activity (Abrahamson et al., 2014; Flood, 2018; Flood et al., 2020; Shvarts & Abrahamson, 2019). By carefully attending to learners' perceptuomotor activity and their multimodally-expressed ideas about it, teachers can highlight significant aspects of that activity, such as attentional anchors, center them as joint focuses of attention, and reframe them in terms of culturally-specified ways of seeing and feeling. In addition, teachers can encourage and support the use of cultural artifacts (e.g., the Cartesian plane, or a particular mathematical definition) as useful means to organize perception, interleaving cultural and disciplinary ways of perceiving with naïve ones (Abrahamson et al., 2012; Flood, 2018). We discuss three strategies of embodied responsive teaching for supporting students working with embodied designs below.

Figure 5. Teachers can use a number of embodied responsive teaching strategies when working with students with the MIT-P including eliciting and attending to students' gestures, repeating and reformulating gestures, and co-constructing gestures with students.

One strategy for embodied responsive teaching is to create opportunities for students to share and explain their ideas-in-progress using gesture (see Figure 5; Flood et al., 2020). Eliciting embodied ideas through the use of gesture is important, because students often know more than they can articulate in words. Educators must pay careful attention to STEM disciplinary potential in students' embodied activity, such as mathematical patterns realized through perceptuomotor activity, even when students can only vaguely refer to or describe those patterns with ambiguous language (Flood et al., 2016). Gesture and bodily activity are often non-redundant to, and sometimes, from adults' perspective, mismatched with what children say (Alibali & Goldin-Meadow, 1993). When working with embodied designs, the tactile and kinesthetic experiences of perceptuomotor activity contain complex, dynamic spatial information and sensation, and are especially challenging to articulate. In addition, children are often still making sense of these experiences as they try to express them (Crowder, 1996). Flood et al. (2020) describe the example of a student who, working with the Mathematics Imagery Trainer for Proportion (hence, MIT-P) says, "To keep it green you have to even them out." What does he mean by "even them out?" Taking this verbal statement at face value would seem to imply that the student believes his hands must stay at the same height as they move upwards. However, in this case, the tutors had observed the student's activity and noticed that he had figured out how to move his hands up the screen while keeping it green, that is, at different speeds. From the adults' perspective, there is a mismatch between the student's articulated strategy and his enacted strategy. To get to the bottom of this mismatch, the tutor asks the student to demonstrate with his hands (without using the device) how to make the screen green. The student illustrates, showing his right hand moving approximately twice as fast (and gaining height) as his left hand. However, he still describes the motion as "even a-paced." By attending to the student's gesture, the tutor can tell that by "even them out" and "even a-paced," the student means, in adult terms, something more akin to "both hands travel at different but constant speeds." The student eventually comes up with a useful mathematical analogy to explain: The "even" motion of his hands resembles two cars, each respectively staying at the same speed (e.g., 20 mph and 50 mph). Thus, by eliciting the gesture, the tutor is able to appreciate what the student means, despite his speech telling a different story, and together they explore this important idea about constant speed (Flood et al., 2020).

Another embodied responsive teaching strategy involves the repetition and reformulation of learners' gestures, as part of negotiating meaning together (Flood et al., 2020). Repetition and reformulation of gestures are two different types of what Shein (2018) first called *multimodal revoicing*. Different forms of multimodal revoicing can accomplish a number of different pedagogical functions (Flood, 2018, 2021; Flood & Harrer, 2022; Flood et al., 2022). First, repeating children's gestures back to them can be a helpful way for adults to make sure they understand what children mean, without introducing adult words and concepts prematurely. Adults can re-create children's gestures to check and make sure they understand them, and ask children to confirm or reject the interpretation (Flood et al., 2020). In conversation analysis, this is called a *candidate understanding* (Heritage, 1984). Second, repeating children's gestures can also be a helpful way to reflect children's ideas back to them for consideration (Flood et al., 2020). Third, it can also serve as a helpful way to connect students' gestures with new disciplinary words or concepts (Alibali et al., 2019; Arzarello et al., 2009; Shein, 2012). And lastly, educators can also go beyond simple repetition and actually reformulate and modify learners' gestures as part of revising and extending students' multimodally-expressed ideas (Flood, 2018, 2021). For example, Flood (2018) describes the situation where a student, working with the MIT-P, is asked to explain what it means for one remote to travel *faster* than another. The student re-enacts her experience using the device, creating a complicated, multi-part gesture about the situation as she thinks through a scenario out loud. The tutor reflects her idea back to her, but instead of repeating the complicated multi-part gesture, he reformulates it. By re-enacting only one of the movements the student made, he calls attention to a particular aspect of the gesture that is relevant to answering the question—namely the differential distance being traveled by each hand in the same amount of time. By reformulating gestures, educators can highlight and extend key information, refining multimodally-expressed embodied ideas, and help provide a bridge to a more disciplinary understanding of the situation (Flood, 2018, 2021).

A final embodied responsive teaching strategy is co-constructing gestures and multimodal explanations with students (Flood et al., 2020). Educators can directly intervene and interact with the gestures that learners produce as they explain their ideas while working with embodied designs. They can reach into students' gestures in progress to highlight particular parts, and/or they can reach in and contribute new imagery. Co-constructing gestures with students allows educators to physically steer gestures in productive new directions, while, at the same time, grounding new ideas in learners' initial embodied performances. Flood et al. (2020) describe the case of a student working with the MIT-P set to a 2:3 ratio, who has discovered that to "make green" she can move her right hand 1.5 units for every 1 unit she moves her left hand. She demonstrates her idea with her outstretched hands, lifting each hand incrementally 1 and 1.5 units. When asked to predict the position of the right hand from the left hand's position, her iterative gesturing method works well for smaller numbers, but breaks down for larger numbers like 10: She gets stuck. While her hands are still outstretched, the tutor reaches into her gesture with his own hands to show her a new way of moving and making sense of that movement. He uses his thumb and index finger to create a vertical pinch shape, highlighting the distance under the student's left hand. Then he slides this pinch shape under her other hand and shrinks it so it is half the height. As he says this, he explains that to predict the height of the right hand, she can imagine taking the height under her left hand, taking half of it, and then adding that half back in, to find the height of the right hand. After the tutor reaches into the student's gesture to show this to her, she is able to predict larger numbers while gesturing. Together, through this co-constructed gesture, they have created a dynamic embodied way of representing the proportional relationship between the hand heights.

Embodied designs pose unique challenges for instructional practice by making learners' hands and bodies the primary instruments of STEM learning. To support learning, educators must responsively guide learners towards disciplinary understandings starting with the *substrate* (Goodwin, 2018) of learners' perceptuomotor activity. In each case, they take up and transform this substrate as part of negotiating meaning with students. Eliciting students' gesture, repeating and reformulating students' gesture, and co-constructing gesture with students provide three useful dialogic embodied responsive teaching strategies for supporting learners' STEM discoveries with embodied design (Flood et al., 2020).

6. Practicing Embodied Design in the Classroom—A Teacher's Perspective

For embodied design to bear impact at scale, the framework's principles should be communicated to mathematics educators who would practice embodied design in the classroom, whether with or without dedicated digital resources. Yet, teachers, who themselves have studied mathematics in mainstream regimens, may find it difficult to become that proverbial teacher who patiently listens to students making sense of new ideas (Ma & Singer-Gabella, 2011), moreover attending reflectively to nuances of students' multimodal expression. What can be done about this? What forms of pre-service preparation or professional development might support teachers in developing an embodied-design perspective? As with the epistemological theory of embodiment, which—put simply—posits that students must *do* to *learn* content, teachers must *do* to *learn* practice. Teachers should themselves participate in embodied-design activities and, moreover, reflect with their cohort and teacher-educators on the educational theory underlying the activities' rationale and how to implement this theory in classroom practice. We thus require teacher-preparation courses and professional-development interventions in embodied design, where teachers engage in learning activities, wearing first a "student hat," then a "researcher hat," a "designer hat," and, finally, a "teacher hat."

What would such multi-hat teacher lessons entail? Essential to such a program would be the opportunity for teachers to *experience* the enactive essence of mathematical situations before discussing them. Having participated in such a course, I—Teacher Lizzy Dutton—can speak personally to the benefit of being grounded in a pre-conceptual perspective that forced me to *feel* the mathematical concepts that I had "mastered." For example, in one embodied-design lesson, a study-buddy and I explored what a circle is (!) by using an etch-a-sketch (a two-knobbed device for manipulating a single pen by separately yet simultaneously controlling its vertical and horizontal displacements) to make a circle collaboratively, where each of us operated one of the two knobs (see in Abrahamson & Bakker, 2016). In attempting to make the circle, we recognized that the left/right knob must slow down while the up/down knob speeds up, and vice versa. This realization prompted us to recognize we need to coordinate our actions *temporally* to execute the two contemporaneous rotations. We recruited another student, asking her to start us off together and clap her hands at a regular tempo while we rotate our respective knobs. Together, our cohort successfully made a circle. So doing, we *felt* the undulating trigonometric waves, as we navigated the device's constraints (cf. Petitmengin, 2017, p. 114).

This simple, yet powerful, activity offered more than a new lesson-plan idea for teaching sine and cosine waves. This activity, along with many other embodied-design activities I participated in during graduate school, called my attention to how I could bodily enact mathematical concepts I had previously held to be purely abstract. For this reason, teachers simply *reading* about embodied-design research in their teacher preparation programs, such as, perhaps, this chapter, is not enough. The very first step in teaching teachers how to facilitate students in connecting their sensory perceptions to mathematical symbols is to have teachers do this themselves. We would thus begin to diligently undo years of cultural messaging that mathematics is something purely abstract, untouchable, unfeelable.

Practicing embodied design in the classroom also demands that teachers perceive the framework as a tool that can be readily accessed when a lesson doesn't go as planned. As most curriculum is not centered around embodied design, it is up to teachers to feel grounded in an embodied epistemology (hence, robust teacher preparation programs vs. a one-day training), so that they may readily access an embodied perspective at any given moment in a lesson. With a wealth of personal experiences in embodied design, teachers are able to improvise decisions that incorporate embodiment into even the most mainstream lesson. A teacher who has had ample time and support in intellectually grappling with the epistemological questions that embodied design demands is more prepared to bring forth embodied design as a resource.

I was personally able to improvise an embodied design into my classroom when my students struggled to understand slope and proportion. This ability to improvise only came from a deep understanding and belief that embodied design is not an extra gimmick to try out but a deeply important theoretical framework for appreciating and impacting how students learn. Thinking on my feet, and remembering Dor Abrahamson's design from graduate school on embodying proportion (the MIT-P Parallels activity), I brought my students into the hallway and had them walk out a proportional relationship. Only after physically moving in proportion to one another and naming that phenomenological experience as something proportional, were students able to make sense of the numerical relationships represented symbolically on the white-board (for details of that activity, see Abrahamson et al., 2022).

In my own teaching practice, I see practicing embodied design as a process of stripping away mainstream instructional regimens that detract from how we naturally embody mathematical phenomena. Too often, students are policed in schools for how they move their bodies, teachers are apt to rush students into worksheets or explaining their thinking before they've had a chance to feel out a concept, and schools are much too focused on how far students are from learning outcomes and targets than on the messy yet critical nuances of student sense-making. Embodied design boldly asks us to shift our focus to the way that students move their bodies as they make sense of mathematical concepts. As such, it gets us to slow down and listen to what students say and watch what they do, rather than anxiously demanding they arrive at a specific and shallow conclusion by the end of a lesson. Teaching in embodied design is a radical act, considering the immense pressure teachers face to have students almost immediately understand abstract mathematical notation. Yet, when we as teachers change our pedagogical values to consider how our own moving body is integral to the learning process, we are able to support students' attention to their own movements, ultimately connecting these intuitive movements to the formal mathematical notations we hope they learn. To make this happen, we must incorporate into classroom discussions how students are feeling the mathematical ideas, because doing so helps them ground and negotiate the new ideas we need them to understand.

7. Moving Forward

Kurt Lewin famously observed that "there is nothing so practical as a good theory." This oft-cited maxim well obtains for design-based researchers of mathematical cognition. Theories of embodiment have, indeed, been so practical for members of the Embodied Design Research Laboratory and their global collaborators by way of lending conceptual coherence to the design, evaluation, and analysis of digital resources. In turn, our practice-oriented research and development projects have created empirical contexts by which to corroborate, expand, and, at times, challenge these theories. Moreover, the theories have opened for us new intellectual horizons by charting the scope of potential research collaborators into a vital eclectic network including enactivist philosophers, cognitive developmental psychologists, contemplative movement trainers, gesture animators, somatic therapists, artificial intelligence experts, learning analytic methodologists, accessibility technologists, and coordination dynamicists. This motley ecumenical crowd, the Embodied Underground, weekly convenes and coheres industriously under the banner of embodiment theory.

Our chapter discussed the embodied design research program by explaining its philosophical and theoretical foundations, guiding research questions, activity architecture, instructional methodology, empirical evaluations, special-education application, and classroom practice, including implications for teacher preparation. As we look to the future of embodied design, we envision the continuous incorporation of ever-evolving technologies for learning, teaching, assessment, and analysis. Ambitiously, we foresee the embodied-design research program as contributing a radical reconceptualization of human psychology centered on theorizing how people develop and apply perceptual orientations to guide actual and simulated interaction (Abrahamson & Mechsner, 2022), and how cognition, language, and science itself, including the learning sciences, evolve from moving in new ways (Feiten et al., 2022).

An equally ambitious theoretical frontier for ED is to continue integrating dynamic systems approaches to movement sciences and sociocultural approaches to mediation (Abrahamson & Trninic, 2015), ultimately rethinking semiosis as the negotiated conventionalization of motor action (Shvarts & Abrahamson, under review). Dynamic systems theory and coordination dynamics share with genetic epistemology (constructivism), ecological psychology, and enactivism the fundamental tenets that innate sensorimotor capacity is adaptively shaped under ecological and cultural constraints into increasingly effective skill (but see McGann et al., 2020, on more refined distinctions). Borrowing from Reed and Bril (1996) the notion "fields of promoted action," ED scholars have argued for a motor-developmental view on mathematical learning (Abrahamson & Trninic, 2015), where inadequate operative schemes become reconfigured into new dynamical stabilities (Abrahamson, 2021; Abrahamson et al, 2016). Analyses of tutorial interactions around ED activities have demonstrated the critical pedagogical role of culturally mediated perception, whether this mediation is inherent to the available resources (Abrahamson et al., 2011) or is actively performed by an attentive tutor (Abrahamson et al., 2012; Flood, 2018; Flood et al., 2020; Shvarts & Abrahamson, 2019). Shvarts and Abrahamson (under review) are looking to blur the traditional ontological divides between action and symbol by theorizing multimodal linguistic expression as socially mediated constraints shaping perception-for-action.

A practical frontier for ED will be to integrate its activities into mainstream educational practice, a process that would require teacher preparation and professional development along with new classroom resources, lesson plans, and forms of assessment. The rapid development of artificially intelligent interactive tutors, along with the meteoric proliferation of personal digital devices and the advent of mixed-reality technologies, could play a pivotal role in fostering enactive understanding of mathematical concepts. Further empirical evidence for the conceptual advantages of grounded understandings could well serve the ED campaign.

Metaphysical claims about the embodied quality of human cognition, once the exclusive realm of philosophical inquiry, can now be pinned down for public inspection in the form of empirical data evidencing the micro-emergence of mathematical concepts constituted as perception-for-action. Scientists can literally witness, track, and anticipate how new sensorimotor patterns coalesce into dynamical stability as students solve motor-control problems of enacting mathematical concepts, how these patterns come forth into students' consciousness as they first express their own movement strategies, and how students appropriate conventional mathematical tools to stabilize, refine, and document their actions. As such, by concretely implementing the philosophical claim that conceptual learning is perceptuomotor activity, ED demystifies cognitive processes of sense-making: mathematical knowledge is not abstract, at least, learning a new mathematical concept is no more abstract than learning to flip a pancake.

References

Abdu, R., Tancredi, S., Abrahamson, D., & Balasubramaniam, R. (under review). A complex-systems view on mathematical learning as hand–eye coordination. In M. Schindler, A. Shvarts, & A. Lilienthal (Eds.), Eye-tracking research in mathematics education [Special issue]. *Educational Studies in Mathematics*.

Abdullah, A., Adil, M., Rosenbaum, L., Clemmons, M., Shah, M., Abrahamson, D., & Neff, M. (2017). Pedagogical agents to support embodied, discovery-based learning. In J. Beskow, C. Peters, G. Castellano, C. O'Sullivan, I. Leite, & S. Kopp (Eds.), *Proceedings of 17th International Conference on Intelligent Virtual Agents (IVA 2017)* (pp. 1–14). Springer International Publishing. https://doi.org/10.1007/978-3-319-67401-8_1

Abrahamson, D. (2009a). Embodied design: Constructing means for constructing meaning. *Educational Studies in Mathematics, 70*(1), 27–47. https://doi.org/10.1007/s10649-008-9137-1

Abrahamson, D. (2009b). Orchestrating semiotic leaps from tacit to cultural quantitative reasoning—the case of anticipating experimental outcomes of a quasi-binomial random generator. *Cognition and Instruction, 27*(3), 175–224.

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction, 2*(1), 1–16. https://doi.org/10.1016/j.ijcci.2014.07.002

Abrahamson, D. (2018). Moving forward: In search of synergy across diverse views on the role of physical movement in design for STEM education [symposium]. In J. Kay & R. Luckin (Eds.), *"Rethinking learning in the digital age: Making the Learning Sciences count," Proceedings of the 13th International Conference of the Learning Sciences (ICLS 2018)* (Vol. 2, pp. 1243–1250). International Society of the Learning Sciences.

Abrahamson, D. (2020). Strawberry feel forever: Understanding metaphor as sensorimotor dynamics. *The Senses and Society, 15*(2), 216–238. https://doi.org/10.1080/17458927.2020.1764742

Abrahamson, D. (2021). Grasp actually: An evolutionist argument for enactivist mathematics education. *Human Development*, 1–17. https://doi.org/10.1159/000515680

Abrahamson, D. (2022). Enactive perception as mathematics learning. In M.-C. Shanahan, B. Kim, M. A. Takeuchi, K. Koh, A. P. Preciado-Babb, & P. Sengupta (Eds.), *The Learning Sciences in conversation: Theories, methodologies, and boundary spaces* (pp. 153–170). Routledge.

Abrahamson, D., & Abdu, R. (2020). Towards an ecological-dynamics design framework for embodied-interaction conceptual learning: The case of dynamic mathematics environments. In T. J. Kopcha, K. D. Valentine, & C. Ocak (Eds.), Embodied cognition and technology for learning [Special issue]. *Educational Technology Research and Development*, *69*, 1889–1923. https://doi.org/10.1007/s11423-020-09805-1

Abrahamson, D., & Bakker, A. (2016). Making sense of movement in embodied design for mathematics learning. In N. Newcombe and S. Weisberg (Eds.), Embodied cognition and STEM learning [Special issue] [journal article]. *Cognitive Research: Principles and Implications, 1*(1), 1–13. https://doi.org/10.1186/s41235-016-0034-3

Abrahamson, D., Dutton, E., & Bakker, A. (2022). Towards an enactivist mathematics pedagogy. In S. A. Stolz (Ed.), *The body, embodiment, and education: An interdisciplinary approach* (pp. 156–182). Routledge.

Abrahamson, D., Flood, V. J., Miele, J. A., & Siu, Y.-T. (2019). Enactivism and ethnomethodological conversation analysis as tools for expanding Universal Design for Learning: The case of visually impaired mathematics students. *ZDM Mathematics Education, 51*(2), 291–303. https://doi.org/10.1007/s11858-018-0998-1

Abrahamson, D., Gutiérrez, J. F., Charoenying, T., Negrete, A. G., & Bumbacher, E. (2012). Fostering hooks and shifts: Tutorial tactics for guided mathematical discovery. *Technology, Knowledge, and Learning, 17*(1-2), 61–86. https://doi.org/10.1007/s10758-012-9192-7

Abrahamson, D., Lee, R. G., Negrete, A. G., & Gutiérrez, J. F. (2014). Coordinating visualizations of polysemous action: Values added for grounding proportion. *ZDM Mathematics Education, 46*(1), 79–93. https://doi.org/10.1007/s11858-013-0521-7

Abrahamson, D., & Lindgren, R. (2014). Embodiment and embodied design. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences (2nd ed.)* (pp. 358–376). Cambridge University Press.

Abrahamson, D., & Mechsner, F. (2022). Toward synergizing educational research and movement sciences: A dialogue on learning as developing perception for action.

Educational Psychology Review, 34(3), 1813–1842. https://doi.org/10.1007/s10648-022-09668-3

Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning [Original Research]. *Frontiers in Education, 5*(147) (29 pages). https://doi.org/10.3389/feduc.2020.00147

Abrahamson, D., & Sánchez-García, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *Journal of the Learning Sciences, 25*(2), 203–239. https://doi.org/10.1080/10508406.2016.1143370

Abrahamson, D., Shayan, S., Bakker, A., & van der Schaaf, M. F. (2016). Eye-tracking Piaget: Capturing the emergence of attentional anchors in the coordination of proportional motor action. *Human Development, 58*(4–5), 218–244. https://doi.org/10.1159/000443153

Abrahamson, D., & Trninic, D. (2015). Bringing forth mathematical concepts: Signifying sensorimotor enactment in fields of promoted action. *ZDM Mathematics Education, 47*(2), 295–306. https://doi.org/10.1007/s11858-014-0620-0

Abrahamson, D., Trninic, D., Gutiérrez, J. F., Huth, J., & Lee, R. G. (2011). Hooks and shifts: A dialectical study of mediated discovery. *Technology, Knowledge, and Learning, 16*(1), 55–85.

Adolph, K. E., Hoch, J. E., & Cole, W. G. (2018). Development (of walking): 15 suggestions. *Trends in Cognitive Sciences, 22*(8), 699–711. https://doi.org/10.1016/j.tics.2018.05.010

Alaoui, S. F., Schiphorst, T., Cuykendall, S., Carlson, K., Studd, K., & Bradley, K. (2015). Strategies for embodied design: The value and challenges of observing movement. In T. Mayer & E. Y.-L. Do (Eds.), *Proceedings of the 2015 ACM SIGCHI Conference on Creativity and Cognition* (pp. 121–130). ACM. https://doi.org/10.1145/2757226.2757238

Alberto, R., Bakker, A., Walker-van Aalst, O., Boon, P., & Drijvers, P. (2019). Networking theories in design research: An embodied instrumentation case study in trigonometry. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the 11th Congress of the European Society for Research in Mathematics Education (CERME 11) (Vol. TWG17: Theoretical perspectives and approaches in mathematics education research)* (pp. 3088–3095). Utrecht University.

Alberto, R., Shvarts, A., Drijvers, P., & Bakker, A. (2021). Action-based embodied design for mathematics learning: A decade of variations on a theme. *International Journal of Child-Computer Interaction*, 100419 (23 pages). https://doi.org/10.1016/j.ijcci.2021.100419

Alibali, M. W., & Goldin-Meadow, S. (1993). Gesture-speech mismatch and mechanisms of learning: What the hands reveal about a child's state of mind. *Cognitive Psychology, 25*, 468–523.

Alibali, M. W., Nathan, M. J., Boncoddo, R., & Pier, E. (2019). Managing common ground in the classroom: Teachers use gestures to support students' contributions to classroom discourse. *ZDM Mathematics Education, 51*, 347–360.

Allen, J. W. P., & Bickhard, M. H. (2013). Stepping off the pendulum: Why only an action-based approach can transcend the nativist–empiricist debate. *Cognitive Development, 28*(2), 96–133. https://doi.org/http://dx.doi.org/10.1016/j.cogdev.2013.01.002

Araújo, D., Davids, K., & Renshaw, I. (2020). Cognition, emotion, and action in sport: And ecological dynamics approach. In G. Tenenbaum & R. C. Eklund (Eds.), *Handbook of sport psychology (4th ed.)* (pp. 535–555). John Wiley & Sons, Inc.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*, 97–109. https://doi.org/10.1007/s10649-008-9163-z

Arzarello, F., & Robutti, O. (2010). Multimodality in multi-representational environments. *ZDM–The international Journal on Mathematics Education, 42*(7), 715–731.

Arzarello, F., Robutti, O., & Bazzini, L. (2005). Acting is learning: Focus on the construction of mathematical concepts. *Cambridge Journal of Education, 35*(1), 55–67.

Bakker, A., Shvarts, A., & Abrahamson, D. (2019). Generativity in design research: the case of developing a genre of action-based mathematics learning activities. In U. T. Jankvist, M. H. A. M. v. d. Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the 11th Congress of the European Society for Research in Mathematics Education (CERME 11)* (Vol. TWG17: Theoretical perspectives and approaches in mathematics education research, pp. 3096–3103). Utrecht University.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. D. English, M. G. Bartolini Bussi, G. A. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (*2nd ed.*) (pp. 720–749). Lawrence Erlbaum Associates.

Black, D. A. (2014, October 9, 2013). Where bodies end and artefacts begin: Tools, machines and interfaces. *Body & Society, 20*(1), 31–60. https://doi.org/10.1177/1357034x13506946

Bongers, T. J. D. (2020). *Transfer of embodied experiences in a tablet environment towards a pen and paper task*. Unpublished Masters thesis (Applied cognitive psychology). Utrecht University.

CAST. (2018). *Universal Design for Learning Guidelines (Version 2.2) [Study guides]. http://udlguidelines.cast.org*.

Chase, K., & Abrahamson, D. (2015). Reverse-scaffolding algebra: Empirical evaluation of design architecture. *ZDM Mathematics Education, 47*(7), 1195–1209. https://doi.org/10.1007/s11858-015-0710-7

Chen, R. S. Y. (2021). Embodied design for non-speaking Autistic children: The emergence of rhythmical joint action. In M. Roussou & S. Shahid (Eds.), *Proceedings of the annual confderence of the ACM special interest group on Interaction Design and Children (IDC)* (pp. 648–651). https://doi.org/10.1075/gest.7.3.02dic

De Jaegher, H., Peräkylä, A., & Stevanovic, M. (2016). The co-creation of meaningful action: Bridging enaction and interactional sociology. *Philosophical transactions of the Royal Society of London. Series B, Biological sciences, 371*(1693), 20150378. https://doi.org/10.1098/rstb.2015.0378

Dessing, J. C., Rey, F. P., & Beek, P. J. (2012). Gaze fixation improves the stability of expert juggling. *Experimental Brain Research, 216*, 635–644. https://doi.org/10.1007/s00221-011-2967-6

Díaz-Rojas, D., Soto-Andrade, J., & Videla-Reyes, R. (2021). Enactive metaphorizing in the mathematical experience. *Constructivist Foundations, 16*(3), 265–274. https://constructivist.info/16/3/265.diaz

Dourish, P. (2001). *Where the action is: The foundations of embodied interaction*. MIT Press.

Duijzer, A. C. G., Shayan, S., Bakker, A., van der Schaaf, M. F., & Abrahamson, D. (2017). Touchscreen tablets: Coordinating action and perception for mathematical cognition. In J. Tarasuik, G. Strouse, & J. Kaufman (Eds.), Touchscreen tablets touching children's lives

[Special issue] [Original Research]. *Frontiers in Psychology, 8*(144) (19 pages). https://doi.org/10.3389/fpsyg.2017.00144

Feiten, T. E., Holland, K., & Chemero, A. (2022). Scalar symmetries and strange tools: Learning analytics and embodied design as techno–cultural confluence. *International Journal of Child Computer Interaction*, *31*, 100452. https://doi.org/10.1016/j.ijcci.2021.100452.

Ferrari, G. (2020). Moving as a circle: Folds and nuances of a mathematical concept. *For the Learning of Mathematics, 40*(3), 3–8.

Fincher–Kiefer, R. (2019). *How the body shapes knowledge: Empirical support for embodied cognition*. American Psychological Association. https://doi.org/10.1037/0000136-000

Fleuchaus, E., Kloos, H., Kiefer, A. W., & Silva, P. L. (2020). Complexity in science learning: Measuring the underlying dynamics of persistent mistakes. *The Journal of Experimental Education, 88*(3), 448–469. https://doi.org/10.1080/00220973.2019.1660603

Flood, V. J. (2018). Multimodal revoicing as an interactional mechanism for connecting scientific and everyday concepts. *Human Development, 61*, 145–173.

Flood, V. J. (2020). *Gesture as a dialogic resource in STEM instructional interactions* (No. 27996938). [Doctoral University. University of California, Berkeley. ProQuest Dissertations & Theses Database].

Flood, V. J. (2021). The secret multimodal life of IREs: Looking more closely at representational gestures in a familiar questioning sequence. *Linguistics and Education, 63*, 100913. https://doi.org/10.1016/j.linged.2021.100913

Flood, V. J., Harrer, B. W. (2022) Teachers' responsiveness to students' gestured candidate responses in STEM whole-class interactions. *Classroom Discourse*, 1–21. 10.1080/19463014.2022.2080086

Flood, V. J., Harrer, B. W., & Abrahamson, D. (2016). The interactional work of configuring a mathematical object in a technology-enabled embodied learning environment. In C.-K. Looi, J. L. Polman, U. Cress, & P. Reimann (Eds.), *"Transforming learning, empowering learners," Proceedings of the International Conference of the Learning Sciences (ICLS 2016)* (Vol. 1, "Full Papers," pp. 122–129). International Society of the Learning Sciences.

Flood, V. J., Shvarts, A., & Abrahamson, D. (2020). Teaching with embodied learning technologies for mathematics: responsive teaching for embodied learning. *ZDM Mathematics Education, 52*(7), 1307–1331. https://doi.org/10.1007/s11858-020-01165-7

Flood, V. J., Wang, X. C., & Sheridan, M. (2022). Embodied responsive teaching for supporting computational thinking in early childhood. In C. Chinn, E. Tan, C. Chan, & Y. Kali (Eds.). *Proceedings of the 16th International Conference of the Learning Sciences (ICLS) 2022* (pp. 855–862). Hiroshima, Japan: International Society of the Learning Sciences.

Gallagher, S. (2015). Invasion of the body snatchers: How embodied cognition is being disembodied. *The Philosophers' Magazine, April*, 96–102.

Goodwin, C. (2018). *Co-operative action*. Cambridge University Press.

Hackenberg, A. J., & Sinclair, N. (2007). Talking about embodiment and caring in relation to computer use in mathematics education. *For the Learning of Mathematics, 27*(3), 12–16.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. Dover.

Heft, H. (1989). Affordances and the body: An intentional analysis of Gibson's ecological approach to visual perception. *Journal for the Theory of Social Behaviour, 19*(1), 1–30.

Heritage, J. (1984). A change-of-state token and aspects of its sequential placement. In J. M. Atkinson & J. Heritage (Eds.), *Structures of Social Action* (pp. 299–345). Cambridge University Press. https://doi.org/10.1017/cbo9780511665868.020

Hutto, D. D., Kirchhoff, M. D., & Abrahamson, D. (2015). The enactive roots of STEM: Rethinking educational design in mathematics. In P. Chandler & A. Tricot (Eds.), Human movement, physical and mental health, and learning [Special issue]. *Educational Psychology Review, 27*(3), 371–389. https://doi.org/10.1007/s10648-015-9326-2

Hutto, D. D., & Myin, E. (2013). *Radicalizing enactivism: Basic minds without content*. MIT Press.

Hutto, D. D., & Sánchez-García, R. (2015, 2014/08/17). Choking RECtified: Embodied expertise beyond Dreyfus. *Phenomenology and the Cognitive Sciences, 14*(2), 309–331. https://doi.org/10.1007/s11097-014-9380-0

Johnson-Glenberg, M. C., Megowan-Romanowicz, C., Birchfield, D. A., & Savio-Ramos, C. (2016). Effects of embodied learning and digital platform on the retention of physics content: Centripetal force [Original Research]. *Frontiers in Psychology, 7*(1819) (22 pages). https://doi.org/10.3389/fpsyg.2016.01819

Krause, C. M. (2017). DeafMath: Exploring the influence of sign language on mathematical conceptualization. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the 10th Congress of the European Society for Research in Mathematics Education* (pp. 1316–1323). DCU Institute of Education and ERME. https://hal.archives-ouvertes.fr/hal-01937152/document

Krause, C. M., & Abrahamson, D. (2020). Modal continuityin deaf students' signed mathematical discourse. In A. Isabel Sacristán & J. Carlos Cortés (Eds.), *"Entre Culturas / Across Cultures"—Proceedings of the 42nd annual meeting of the North-American chapter of the International Group for the Psychology of Mathematics Education (PME-NA)*. PME-NA (pp. 1448–1449). PME-NA. https://doi.org/10.51272/pmena.42.2020-228.

Kress, G. (2001). *Multimodal discourse: The modes and media of contemporary communication*. Arnold Publishers.

Lambert, S. G., Tancredi, S., Fiedler, B. L., Moore, E. B., Gorlewicz, J. L., Abrahamson, D., & Gomez Paloma, F. (2022). Getting a grip on geometry: Developing a tangible manipulative for inclusive quadrilateral learning. *Italian Journal of Health Education, Sports and Inclusive Didactics, 6*(1), 1–21. https://doi.org/10.32043/gsd.v6i1.604

Lee, V. R., Drake, J., Cain, R., & Thayne, J. (2021). Remembering what produced the data: Individual and social reconstruction in the context of a quantified self elementary data and statistics unit. *Cognition and Instruction*, 1–42. https://doi.org/10.1080/07370008.2021.1936529

Leung, A., Baccaglini-Frank, A., & Mariotti, M. A. (2013). Discernment of invariants in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460.

Ma, J. Y., & Singer-Gabella, M. (2011). Learning to teach in the figured world of reform mathematics: Negotiating new models of identity. *Journal of Teacher Education, 62*(1), 8–22.

Malafouris, L. (2020). Thinking as "thinging": Psychology with things. *Current Directions in Psychological Science, 29*(1), 3–8. https://doi.org/10.1177/0963721419873349

McGann, M., Di Paolo, E. A., Heras-Escribano, M., & Chemero, A. (2020). Editorial: Enaction and ecological psychology: Convergences and complementarities [Editorial]. *Frontiers in Psychology, 11*(3176) (5 pages). https://doi.org/10.3389/fpsyg.2020.617898

Marshall, P., Antle, A. N., van den Hoven, E., & Rogers, Y. (Eds.). (2013). The theory and practice of embodied interaction in HCI and interaction design [Special issue]. *ACM Transactions on Human–Computer Interaction, 20*(1) (3 pages).

Marwan, N., Romano, M. C., Thiel, M., & Kurths, J. (2007). Recurrence plots for the analysis of complex systems. *Physics Reports, 438*, 237–239.

Maturana, H. R., & Varela, F. J. (1992). *The tree of knowledge: The biological roots of human understanding*. Shambala Publications. (Originally published in 1987).

Mechsner, F. (2003). Gestalt factors in human movement coordination. *Gestalt Theory, 25*(4), 225–245.

Mechsner, F. (2004). A psychological approach to human voluntary movements. *Journal of Motor Behavior, 36*(4), 355–370.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. In L. Radford, L. Edwards, & F. Arzarello (Eds.), Gestures and multimodality in the construction of mathematical meaning [Special issue]. *Educational Studies in Mathematics, 70*(2), 159–174.

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2012). Gesture and imagination: On the constitution and uses of phantasms. *Gesture, 2*, 130–165. https://doi.org/10.1075/gest.12.2.02nem

Newell, K. M. (1986). Constraints on the development of coordination. In M. G. Wade & H. T. A. Whiting (Eds.), *Motor development in children: Aspects of coordination and control* (pp. 341–361). Martinus Nijhoff Publishers.

Newell, K. M., & Ranganathan, R. (2010). Instructions as constraints in motor skill acquisition. In I. Renshaw, K. Davids, & G. J. P. Savelsbergh (Eds.), *Motor learning in practice: A constraints-led approach* (pp. 17–32). Routledge.

Newen, A., Bruin, L. D., & Gallagher, S. (Eds.). (2018). *The Oxford handbook of 4E cognition*. Oxford University Press.

Núñez, R. E., & Freeman, W. J. (Eds.). (1999). *Reclaiming cognition: The primacy of action, intention, and emotion (Journal of Consciousness Studies 6[11–12])*. Imprint Academic.

Ou, L., Andrade, A., Alberto, R., van Helden, G., & Bakker, A. (2020). Using a cluster-based regime-switching dynamic model to understand embodied mathematical learning. In K. Verbert, M. Scheffel, N. Pinkwart, & V. Kovanonic (Eds.), *Proceedings of the 10th International Conference on Learning Analytics & Knowledge* (pp. 496–501). https://doi.org/10.1145/3375462.3375513

Ou, L., Andrade, A., Alberto, R. A., Bakker, A., & Bechger, T. (2020). Identifying qualitative between-subject and within-subject variability: A method for clustering regime-switching dynamics [Original Research]. *Frontiers in Psychology, 11*(1136). https://doi.org/10.3389/fpsyg.2020.01136

Pallasmaa, J. (2017). Embodied and existential wisdom in architecture: The thinking hand. *Body & Society, 23*(1), 96–111. https://doi.org/10.1177/1357034x16681443

Pardos, Z. A., Hu, C., Meng, P., Neff, M., & Abrahamson, D. (2018). Characterizing learner behavior from high frequency touchscreen data using recurrent neural networks. In D. Chin & L. Chen (Eds.), *Adjunct proceedings of the 26th Conference on User Modeling, Adaptation and Personalization (UMAP '18)*. ACM. 6 pages.

Petitmengin, C. (2017). Enaction as a lived experience: Towards a radical neurophenomenology. *Constructivist Foundations, 12*(2), 139–147. http://constructivist.info/12/2/139.petitmengin

PhET-Interactive-Simulations. (2021). *Ratio and Proportion*. https://phet.colorado.edu/en/simulation/ratio-and-proportion

Piaget, J. (1971). *Biology and knowledge: An essay on the relations between organic regulations and cognitive processes* (B. Walsh, Trans.). The University of Chicago Press.

Pirie, S. E. B., & Kieren, T. E. (1989). A recursive theory of mathematical understanding. *For the Learning of Mathematics, 9*(3), 7–11.

Reed, E. S., & Bril, B. (1996). The primacy of action in development. In M. L. Latash & M. T. Turvey (Eds.), *Dexterity and its development* (pp. 431–451). Lawrence Erlbaum Associates.

Richardson, M. J., & Chemero, A. (2014). Complex dynamical systems and embodiment. In L. Shapiro (Ed.), *The Routledge handbook of embodied cognition* (pp. 39–50). Routledge.

Robertson, A. D., Scherr, R. E., & Hammer, D. (2016). *Responsive teaching in science and mathematics*. Routledge.

Rose, D. H., & Meyer, A. (2002). *Teaching every student in the digital age: Universal design for learning*. Association for Supervision and Curriculum Development.

Rosen, D. M., Palatnik, A., & Abrahamson, D. (2018). A better story: An embodiment argument for stark manipulatives. In N. Calder, N. Sinclair, & K. Larkin (Eds.), *Using mobile technologies in the learning of mathematics* (pp. 189–211). Springer.

Shayan, S., Abrahamson, D., Bakker, A., Duijzer, A. C. G., & Van der Schaaf, M. F. (2015). The emergence of proportional reasoning from embodied interaction with a tablet application: An eye-tracking study. In L. Gómez Chova, A. López Martínez, & I. Candel Torres (Eds.), *Proceedings of the 9th International Technology, Education, and Development Conference (INTED 2015)* (pp. 5732–5741). International Academy of Technology, Education, and Development.

Shayan, S., Abrahamson, D., Bakker, A., Duijzer, A. C. G., & Van der Schaaf, M. F. (2017). Eye-tracking the emergence of attentional anchors in a mathematics learning tablet activity. In C. A. Was, F. J. Sansosti, & B. J. Morris (Eds.), *Eye-tracking technology applications in educational research* (pp. 166–194). IGI Global. https://doi.org/10.4018/978-1-5225-1005-5.ch009

Sheets-Johnstone, M. (1999). *The primacy of movement*. John Benjamins.

Sheets-Johnstone, M. (2000). Kinetic tactile-kinesthetic bodies: Ontogenetical foundations of apprenticeship learning. *Human Studies, 23*, 343–370.

Shein, P. P. (2012). Seeing with two eyes: A teacher's use of gestures in questioning and revoicing to engage english language learners in the repair of mathematical errors. *Journal for Research in Mathematics Education, 43*(2), 182–222. https://doi.org/10.5951/jresematheduc.43.2.0182

Sherin, M. G., Jacobs, V. R., & Phlipp, R. A. (2011). *Mathematics teacher noticing: Seeing through teachers' eyes*. Routledge.

Shvarts, A. (2017). Eye movements in emerging conceptual understanding of rectangle area. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (*Vol. 1*, p. 268). PME.

Shvarts, A., & Abrahamson, D. (2019). Dual-eye-tracking Vygotsky: A microgenetic account of a teaching/learning collaboration in an embodied-interaction technological tutorial for mathematics. *Learning, Culture and Social Interaction, 22*, 100316. https://doi.org/10.1016/j.lcsi.2019.05.003

Shvarts, A., & Abrahamson, D. (under review). Coordination dynamics of semiotic mediation: Bridging Lev Vygotsky and Nikolai Bernstein. *Constructivist Foundations*.

Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2021). Embodied instrumentation in learning mathematics as the genesis of a body-artifact functional system. *Educational Studies in Mathematics, 107*(3), 447–469. https://doi.org/10.1007/s10649-021-10053-0

Stephen, D. G., Dixon, J. A., & Isenhower, R. W. (2009). Dynamics of representational change: Entropy, action, and cognition. *Journal of Experimental Psychology: Human Perception and Performance, 35*(6), 1811–1832.

Streeck, J., Goodwin, C., & LeBaron, C. (Eds.). (2011). *Embodied interaction: Language and body in the material world*. Cambridge Academic Press.

Tancredi, S., Abdu, R., Abrahamson, D., & Balasubramaniam, R. (2021). Modeling nonlinear dynamics of fluency development in an embodied-design mathematics learning environment with Recurrence Quantification Analysis. *International Journal of Child–Computer Interaction*, 100297 (15 pages). https://doi.org/10.1016/j.ijcci.2021.100297

Tancredi, S., Abdu, R., Balasubramaniam, R., & Abrahamson, D. (2022). Intermodality in multimodal learning analytics for cognitive theory development: A case from embodied design for mathematics learning. In M. Giannakos, D. Spikol, D. Di Mitri, K. Sharma, X. Ochoa, & R. Hammad (Eds.), *The multimodal learning analytics handbook* (pp. 133–158). Springer International Publishing. https://doi.org/10.1007/978-3-031-08076-0_6

Tancredi, S., Chen, R. S. Y., Krause, C. M., Abrahamson, D., & Gomez Paloma, F. (2021). Getting up to SpEED: Special education embodied design for sensorially equitable inclusion. *Education Science and Society, 12*(1). https://doi.org/10.3280/ess1-2021oa11818

Tancredi, S., Chen, R. S. Y., Krause, C. M., & Siu, Y. T. (2022). The need for SpEED: Reimagining accessibility through Special Education Embodied Design. In S. L. Macrine & J. M. Fugate (Eds.), *Movement matters: How embodied cognition informs teaching and learning* (pp. 197–216). MIT Press.

Tancredi, S., Wang, J. X., Li, H. L., Yao, C. J., Macfarlan, G. L., & Ryokai, K. (2022). Balance Board Math: "Being the graph" through the sense of balance for embodied self-regulation and learning. In M. Horn, M. Giannakos, & T. Pontual (Eds.), *Proceedings of IDC '22: Interaction Design and Children* (Vol. "Full papers," pp. 137–149). https://doi.org/10.1145/3501712.3529743

Tao, T. (2016). Thinking and explaining. *mathOverflow*. https://mathoverflow.net/questions/38639/thinking-and-explaining

Thelen, E., & Smith, L. B. (1994). *A dynamic systems approach to the development of cognition and action*. MIT Press.

Turkle, S., & Papert, S. (1991). Epistemological pluralism and the revaluation of the concrete. In I. Harel & S. Papert (Eds.), *Constructionism* (pp. 161–192). Ablex Publishing.

Turvey, M. T. (1992). Ecological foundations of cognition: Invariants of perception and action. In P. J. Herbert L., P. W. P. van den Broek, & D. C. Knill (Eds.), *Cognition: Conceptual and methodological issues* (pp. 85–117). American Psychological Association.

van Rompay, T., & Hekkert, P. (2001). Embodied design: On the role of bodily experiences in product design. In *Proceedings of the International Conference on Affective Human Factors Design* (pp. 39–46). ASEAN Academic Press.

Varela, F. J. (1999). *Ethical know-how: Action, wisdom, and cognition*. Stanford University Press.

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. MIT Press.

Vygotsky, L. S. (1926/1997). *Educational psychology* (R. H. Silverman, Trans.). CRC Press LLC. (Original work published 1926).

Vygotsky, L. S. (1962). *Thought and language*. M.I.T. Press. (Original work published 1934). (Thinking and Speaking, published 1934)

Webber, C. L., Jr., & Zbilut, J. P. (1994, Feb). Dynamical assessment of physiological systems and states using recurrence plot strategies. *Journal of Applied Physiology, 76*(2), 965–973. https://doi.org/10.1152/jappl.1994.76.2.965

Wilensky, U., & Papert, S. (2010). Restructurations: Reformulations of knowledge disciplines through new representational forms. In J. Clayson & I. Kallas (Eds.), *Proceedings of the Constructionism 2010 Conference (The 12th EuroLogo conference)*. Library and Publishing Centre, Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava, Slovakia. https://ccl.northwestern.edu/2010/wilensky_restructurations_Constructionism%202010-latest.pdf

Worsley, M., & Blikstein, P. (2014). Using multimodal learning analytics to study learning mechanisms. In J. Stamper, Z. Pardos, M. Mavrikis, & B. M. McLaren (Eds.), *Proceedings of the 7th International Conference on Educational Data Mining* (pp. 431–432). Institute of Education.

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