# Learning Mathematics with Digital Resources: Reclaiming the Cognitive Role of Physical Movement

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## Contents



**Abstract**

In 1837, Fröbel introduced a pedagogical regimen focused on a set of simple tangible objects, beginning with a yarn ball, that children were invited to

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manipulate in various ways. Montessori and other educational luminaries followed this tradition of designing instructional manipulatives. Later, when digital technologies were invented for information management, computation, and telecommunication, these new media were adopted by educators eager to offer individualized learning, reach remote students, and create multimedia experiences that would augment on traditional classroom resources, such as textbooks, writing instruments, and authentic objects of inquiry, including natural phenomena and cultural artifacts. While early electric technologies privileged the visual and auditory sensory modalities and were not interactive (e.g., television), human–computer interaction innovations and the advent of personal platforms increasingly evolved toward state-of-the art devices, content, and activities offering immersive multimodal experiences in imaginary landscapes (e.g., virtual reality). What might be the educational promise of these media? How do digital technologies serve mathematics students differently than yarn balls? How might theories of learning guide the design of digital environments?

To investigate these questions, we survey the history of digital resources for mathematics education through the prism of philosophical and psychological theories – enactivist cognition and ecological dynamics – that look to capture the role of embodied interaction in cognitive development and conceptual learning. Then, through three case studies of contemporary digital educational resources, a proposal is put forth for how these embodied theories of learning could inform the design of educational technologies compatible with how people naturally learn. First, students should learn to enact new physical movement forms that have been designed to instantiate the targeted concepts. Students learn to move in these new ways by developing perceptual orientations that enable them to solve situated motor-control problems. Only then are these new cognitive skills formalized in disciplinary semiotic forms. Perhaps future technology can be as powerful a learning tool as the historical yarn ball.

**Keywords**

Ecological dynamics · Embodiment · Enactivism · Enactivist · Interactive · Manipulatives

> Recognition of the modes of existence of technical objects must be brought about through philosophical consideration; what philosophy has to achieve in this respect is analogous to what the abolition of slavery achieved in affirming the worth of the individual human being. (Gilbert Simondon 2017, p. 16)

## Introduction: In Search of a Practical Theory of Mathematics Learning

Digital technology is rapidly advancing. With each new invention of an interactive medium, educational practitioners, researchers, and commercial enterprises inspect how the medium might serve mathematics students. And yet, building effective educational resources depends on understanding how the mind functions, how people learn, and how individuals interact. There are quite a number of theoretical frameworks in the literature proposing to explain how people think and behave, and so, what philosophical scientific perspectives should we consider in evaluating the potential educational utility of some technological innovation? This chapter draws on embodiment philosophy and theories from the cognitive sciences that understand mathematical learning as grounded in new perceptual capacity for purposeful interaction with the environment. From that intellectual perspective, we will argue that digital resources for mathematics learning are useful to the extent that they can be applied in educating students' perception via engaging them in movement-based learning activities, i.e., *perception-for-action* (Varela et al. 1991). We will propose that the education of perception can be achieved by creating engaging contexts for students to learn how to physically move in new ways, even before learning to express their new skills in formal symbols, just as children learn to ride a bicycle long before they understand vector multiplication and the gyroscopic effect.

Thus, in this chapter we put forth a two-stage process for designing digital resources for conceptual learning: (1) identify how a perception-for-action instantiates and evokes the targeted concept; and then (2) build a learning environment that motivates students to develop that perception by making their actions constitute solutions to motor-control problems. This design framework taps our species' phylogenesis and tacit praxis: We learn to perform new skills by noticing and discussing structures in the sensory manifold that guide and promote our coordinated actions. While this ancient cultural practice has worked in learning to manipulate predigital artifacts, such as a plow, bow, or abacus, digital resources have the technological affordance (or liability) to skip physical interaction. Here, we argue that the temptation to bypass physical interaction is not necessarily (or always) a good thing for mathematics education.

Through a survey of the evolution of human–computer interaction, the chapter examines historical exemplars of mathematics learning environments to evaluate the extent to which those digital resources enabled students to embody, enact, and learn the concepts in question. We will discern a pendulum-like progression of educational media in terms of the physical movements they have solicited from students: Beginning with humble artifacts, such as a yarn ball, that invited bodily interaction, we swing down to the low-interaction nadir, epitomized by the television, then begin swinging back up through the ages of digital-technology interaction platforms – Command-Line Interfaces (CLI), Graphical User Interfaces (GUI), and Embodied and Tangible User Interfaces (TUI) – and through to current augmented-, virtual-, and mixed-reality devices (XR) that once again enable full-body movement, yet may come with their own interaction trade-offs. The remainder of this introduction steps back to the preelectric era, to revisit foundational pedagogical questions anticipating our field's persisting contemporary debates over which we should build digital artifacts to support students' learning.

In the beginning, there was a yarn ball. Yarn balls have a pedigree in the history of educational scholarship and practice. They were Gift #1 in the pedagogical regimen of Friedrich Fröbel (1782–1852), who invented kindergarten, child-centered education, and manipulatives (see Fig. 1; Fröbel, 2005). He need not have called his gifts

**Fig. 1** The yarn ball, Fröbel's Gift 1, with his original recommended activities. (Note: Plates 1a & 1b, from Ronge and Ronge (1858))

"interactive," because, centuries before digital technology, it would be strange to point out that concrete objects respond to handling in ways that disclose their essential properties and suggest their scope of functional utilities (cf. Bamberger and Schön 1983). Rather, staged in natural three-dimensional space, vested in material particularities, such as the elasticity of yarn, and governed by immutable regularities of terrestrial physics, such as gravity, the "interface" between a child and a yarn ball is skin alone, unmediated by hidden mechanical craft let alone software procedures. In the case of a yarn ball, physical actions of handling bear directly on the manipulated object, resulting in mutual deformation of organic fabric and palmar cuticle, and the child's proximal and distal actions blended indistinguishably.

Fröbel's educational program of basing curriculum in handling objects can be viewed as implementing Rousseau's Enlightenment edict, voiced a century earlier by the radical pedagogue of the eponymous Émile:

> What is the use of all these symbols; why not begin by showing him the real thing so that he may at least know what you are talking about? .... As a general rule—never substitute the symbol for the thing signified, unless it is impossible to show the thing itself; for the child's attention is so taken up with the symbol that he will forget what it signifies. (Rousseau, 1755/1979, Émile, Book III, p. 170)

Fast-forward through the annals of educational history, we find further resonance in the respective predigital design inventions of mathematics-education luminaries Maria Montessori (1870–1952), Hans Freudenthal (1905–1990), Caleb Gattegno (1911–1988), Zoltán Diénès (1916–2014), and Richard Skemp (1919–1995). Yet, a quarter of a millennium since Émile, here we all are still debating what good may come from showing children the thing itself, and from letting them handle the thing itself. Moreover, we are asking what this thing itself should even be, and what exactly it is we might want Émile to learn. How could grasping fabric possibly materialize as grasping ideas? What might we create for a child to grasp when, indeed, "it is impossible to show the thing itself"? How might a child's manual *know-how* become conceptual *know-that* (Ryle 1945)? In *Democracy and Education*, Dewey (1916) writes:

> [C]areful inspection of methods which are permanently successful in formal education, whether in arithmetic or learning to read, or studying geography, or learning physics or a foreign language, will reveal that they depend for their efficiency upon the fact that they go back to the type of the situation which causes reflection out of school in ordinary life. They give the pupils something to do, not something to learn; and the doing is of such a nature as to demand thinking, or the intentional noting of connections; learning naturally results. (Ch. 12, p. 154)

A century later, the field of educational research has yet to reach consensus over how students learn from operating on objects and, therefore, how optimally to design and facilitate learning experiences. Perhaps the efficacy of our educational technology will grow only with the quality of our educational philosophy and theory. That is, to build digital resources that offer quality learning, we really should know what it means to learn, for, as Kurt Lewin allegedly stated, "Nothing is as practical as a good theory." A good theory, we submit, could help educational designers take on practical questions such as the following: What do children learn from playing with a yarn ball? If we digitized the yarn ball, what might we lose? What might we gain? Could this digital experience somehow enhance conceptual learning?

Our introduction touched upon philosophy, theory, and technology, because the history and future of digital design for mathematics education braid these three pillars: (1) developments in philosophical understanding of epistemology and ontology respecting the nature and interrelations of cognition, perception, body, action, environment, artifacts, and discourse; (2) advances in cognitive-science research-based theories of conceptual development with respect to the interrelated roles of task, perception, action, and interpersonal interaction in personal experience; and (3) breakthroughs in technological engineering of computer-based tangible and virtual interaction platforms, artificial intelligence, and cognitive–affective biosensing measurement, analysis, and visualization. These philosophical, theoretical, and technological tributaries have been coevolving reciprocally as a self-organizing complex system, each prodded and swayed by the other's arguments and proofs-of-concept. The three tributaries, we put forth, all factor into the intellectual–technical nexus of practice that amounts to the design-based research of digital resources for mathematics education.

Section "Driving Intellectual Rationale: Recentering Situated Physical Action as Mathematics Learning", below, further sets this chapter as a critical examination of a century of digital resources in terms of the physical movements they have solicited from mathematics students. We capture this evolution as a pendulum that swung down away from movement, and then back up again, back to movement. Then section "Rethinking Mathematical Learning: Ecological Dynamics, 4E Cognition, and Embodied Design" examines this technological evolution in dialogue with educational research to suggest that movement is returning to the center of learning theories. Section "A Critical Analysis of the History of Digital Resources for Mathematics Education" surveys the evolution of digital resources for mathematics learning through the prism of developments in HCI platforms. Section "From Theory to Practice: Educational Design for Enactive Mathematics Learning" exemplifies three learning environments that foreground physical interaction in mathematical learning, and section "Concluding Remarks" offers summative comments moving forward.

## Driving Intellectual Rationale: Recentering Situated Physical Action as Mathematics Learning

As we look back at the evolution of digital resources for mathematics education, we might discern a "swing of the pendulum" away from the body and then back (cf. Allen and Bickhard 2013; see Fig. 2). Initially, under the exacting intellectual reign of Behaviorist and Cognitivist regimes, and given the modest dawn of Human–Computer Interaction (HCI) platforms, the body was elided from the discussion, constituting at best a requisite organic conduit between symbolic information on the screen and symbolic processing in the brain. Yet, over the mid-twentieth century, the philosophy–theory–technology nexus was defiantly morphing into what would eventually become known as the *embodiment turn* in the cognitive sciences (Nagataki and Hirose 2007) – a phenomenological reconsideration of the mind not as a computer-like central processing unit but, rather, as a form of situated activity that is embodied, enacted, and extended in natural and sociomaterial ecologies (Anderson 2003; Kiverstein 2012).

In parallel, cognitive anthropology studies of human–machine interactions (Suchman 1987) emphasized the importance of design catering to the user's cognitive ergonomics. When users become enskilled in operating a technology, it is not so much that they come to know –about the artifact than to know – how to use it (Dourish 2001; Hansen 2004; Heidegger, 1962). Thus, the design of a digitally enhanced technology is judged not by its intact functionality or esthetics but by its capacity to solicit from users' task-appropriate actions: While the objective functional grammar might be one of human-operating-on-a-machine-that-is-operating-on-the-world, the user's phenomenology is that of human+machine operating on the world and, with practice, just human–world engaged in tight perception–action loops (or utilization schemes, Vérillon and Rabardel 1995; see Gibson, 1977; Malafouris 2020; cf. Black 2014). Philosophical and theoretical discussions of technology, such

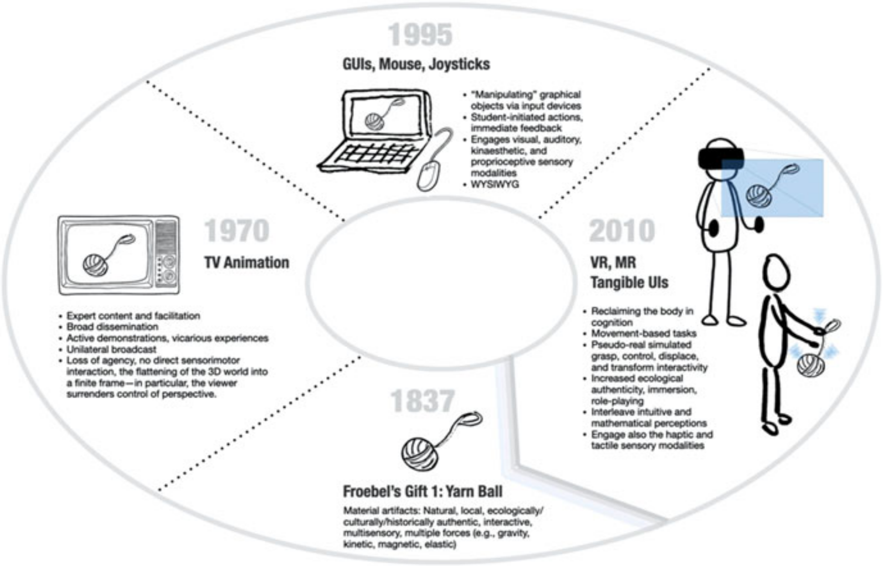


**Fig. 2** Modal engagement with educational media: evolution over two centuries. Note. A view from embodied-interaction on the historical evolution of digital media as a quest to simulate, regulate, and expand on unmediated multimodal experience

as the above, stoked and steered the ongoing pursuit of engineering breakthroughs, so that throughout the twentieth century HCI was fast evolving from: (1) Command-Line Interfaces (CLI); to (2) Graphical User Interfaces (GUI); and onto (3) Embodied and Tangible User Interfaces (TUI), including augmented and virtual reality (XR) (see section "Rethinking Mathematical Learning: Ecological Dynamics, 4E Cognition, and Embodied Design" for an elaboration on this evolution).

Meanwhile, *educational* researchers inspired by various postcognitivist philosophies of cognitive science were seeking how best to consider corporeality, perception, and action into the field's discourse on mathematics teaching and learning (e.g., Artigue 2002; Forman 1988; Nemirovsky et al. 1998; Núñez et al. 1999; Pirie and Kieren 1989; Sarama and Clements 2009; Sinclair and de Freitas 2014). An interest in the embodied qualities of mathematical cognition resonated with increasing realizations from archeology (Schmandt-Besserat 1992), science studies (Latour 1987), cognitive psychology (Greeno 1998), sociology (Barnes et al. 1996; Livingston 1999), and cognitive anthropology (Lave and Wenger 1991; Rogoff 1990; Saxe and Esmonde 2005; Urton 1997) that human techno–scientific capacity, including mathematical reasoning, is socio-materially constituted through situated collaborative practice enmeshed in cultural–historical artifacts. Educational researchers attentive to this cultural–material–linguistic turn in the cognitive sciences found new intellectual footing from which to resign a former theory of mathematical knowledge as information in the head in favor of considering mathematical knowing as multimodal enactment (de Freitas 2016; Nemirovsky and Ferrara 2009; Pirie and Kieren 1989; Petitmengin 2007; Roth 2009).

As educational researchers turned to consider situated perceptuomotor activity in sociocultural contexts as the very stuff of learning, new methodologies were called for that could capture and analyze the multimodality of human behavior, including sensorimotor and neural traces of action, utterance, and reasoning (Worsley et al. 2016; Worsley and Blikstein 2014). From this instrumented vantage point, the field is now technologically equipped to reevaluate and support central theoretical claims concerning the formative role of embodied participation in the enactment of cultural practice, such as Piaget's construct of reflective abstraction (Abrahamson et al. 2016) or Vygotsky's work on the zone of proximal development (Shvarts and Abrahamson 2019). Furthermore, the field can dialogue directly with enactivist philosophy and dynamic systems theory on the emergence of cognitive structures from perceptuomotor problem-solving (Boncoddo et al. 2010; Hutto et al. 2015; Ross and Vallée-Tourangeau 2021). Specifically, mathematical cognition is grounded in percepetuomotor activity (Abrahamson and Abdu 2020; Arzarello et al. 2005; de Koning and Tabbers 2011; Hackenberg and Sinclair 2007; Lakoff and Núñez 2000; Price and Duffy 2018). By reclaiming the body into cognition (Freeman and Núñez 1999), we have redeemed the yarn ball.

We therefore propose to parse the evolution of digital resources for mathematics education by implicating the cognitive function they allege to the student's physical *movement* and, in particular, the opportunities these resources create for students to figure out how to move in new ways that the designers have built as enacting the target curricular concepts. Learning to move in new ways may present substantial cognitive demands, and yet this embodied mental effort, empirical and ethnographic literature suggests, is precisely the praxis of mathematics learning and problem solving. In particular, we look to understand how students devise and sustain emergent multimodal perceptions of the situation that facilitate the coordination of motor actions that solve the situated tasks. When the history of interaction technology (see, earlier, Fig. 2) is revisited through highlighting students' *opportunities for conceptually meaningful movement* (see Fig. 3), we gain new insight into the interaction crisis, revival, and unknown future of digital resources for mathematics education.

Below, we continue the dialogue between the literatures of educational research and HCI engineering by introducing the theory of *ecological dynamics* and the affiliated pedagogical framework of *embodied design*.

## Rethinking Mathematical Learning: Ecological Dynamics, 4E Cognition, and Embodied Design

Scholars of HCI look to model humans' functional relation with technology as an interaction loop between a user and a system (see Fig. 4). The user performs an action on/to the system (i.e., the *do* input), and the system responds by providing feedback (i.e., the *feel* output). It is the interaction designer's purview to implement into the technologies sufficient cues that the user can perceive as the system's various affordances (e.g., textual, graphical, tangible UI design). To this interaction, the user brings their understanding of the world (see the "Know & Understand" bubble) and, with that, their tacit assumptions for the output of their actions.

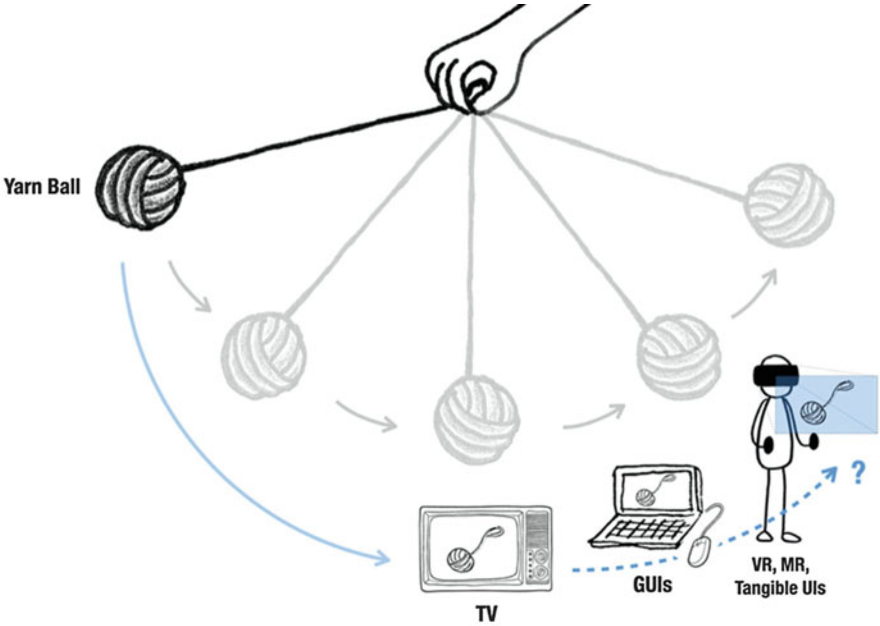


**Fig. 3** *Reclaiming the Body in Educational Experience Through Embodied-Interaction User Interfaces*. Note. The historical loss and gain of opportunities for movement-based multimodal interaction

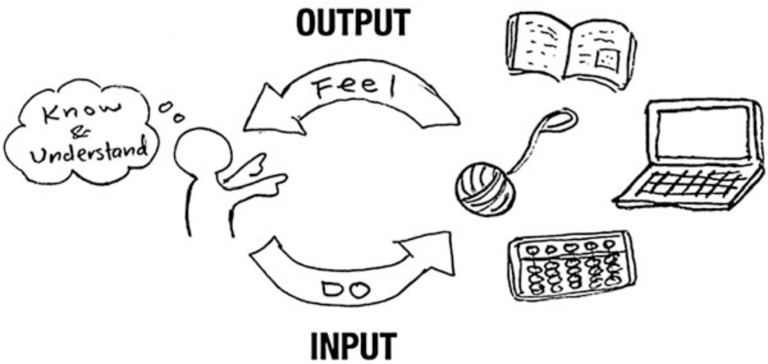


**Fig. 4** An interaction loop in HCI. (Adapted from a diagram by Verplank, in Moggridge (2007, p. 126))

By marking *do* and *feel,* we also wish to endorse *extended* and *distributed* conceptualizations of human cognition that foreground humble interactions with the concrete or digital environment as formatively constitutive of thinking, including conceptual reasoning. For example, physically rearranging material objects in a workspace may change its cognitive landscape (Schwartz and Martin 2006). Some scholars of mathematics education (e.g., de Freitas and Sinclair 2014; Moon and Lee

2020) go further to posit an *inclusive materialist* theory, where they attribute equal agency to material objects as to humans who wield the objects. In like vein, cognitive paleoanthropologists (e.g., Donald 2010) blur epistemological and ontological distinctions between the organic body and the material artifacts it manipulates in enacting cultural practices, thus conceptualizing enculturated cognitive activity not as thinking but as *thinging* (Malafouris 2020).

From an educational interaction perspective, the Verplank diagram (Fig. 4) pictures skill learning as the process of increasing one's capacity to effectively manipulate external resources; one learns on the job, i.e., we become better at doing something by doing it, all along tuning our actions to feedback from the objects themselves and, concurrently, their unfolding effect on the environment. What we thus come to know is not stored in the brain as content about the technology that is divorced from our actions and their mediated environmental consequences. Rather, tool-knowing is an inherently situated and enactive capacity to perceive and responsively apply cultural resources to the environment as a means of accomplishing a task of relevance and value. The diagram implies that *our own capacities* shape our learning trajectories, as do the *specific tasks at hand* and the *general sociomaterial circumstances of the situations* wherein these tasks are embedded. We now turn to a model that might be construed as expanding on the Verplank scheme, by way of foregrounding these triadic factors as organismic, task, and environmental constraints on learning outcomes.

*Ecological dynamics* is a theoretical framework for investigating and informing social practices involving the teaching and learning of physical movement, such as sports and rehabilitation (Araújo et al. 2020; Chow et al. 2016). As its name might suggest, the ecological dynamics framework draws on two intellectual traditions concerned with understanding human behavior – *ecological psychology* (Gibson 1977) and *dynamic systems theory* (Thelen and Smith 1994). Gibson believed that embedded in the natural and cultural environment are *affordances*, information structures that constitute opportunities for action as appropriate for an individual with relevant capacity. For example, a step affords stepping for individuals with typical use of their legs, but not for those who use a wheelchair, for whom a step may constitute an impediment (Heft 1989; Turvey 2019, esp. Lecture 22). When we look to understand the environment as affordances, we are rejecting objectivist ontologies of things being definitive entities (e.g., a step is an engineered elevation constructed by diverse materials, etc.) and, instead, we are adopting a phenomenological stance on things being what they are used for by some person (organismic) in some context (environment) for some purpose (task) (e.g., a staircase railing becomes a slide for an advanced skater eager to execute a trick).

Affordances need not be immediately detected. Situations can be designed (e.g., by teachers, coaches, and parents) to create conditions for learners to be perceptually attracted to task-promoting affordances in the course of *exploratory* doing and feeling (acting and gathering information). Through interaction, particular affordances become promoted that appear to facilitate an individual's enactment of movements appropriate to the task at hand. Shaping this process of dynamical self-assembly into functional stability are three types of constraints (see Fig. 5):

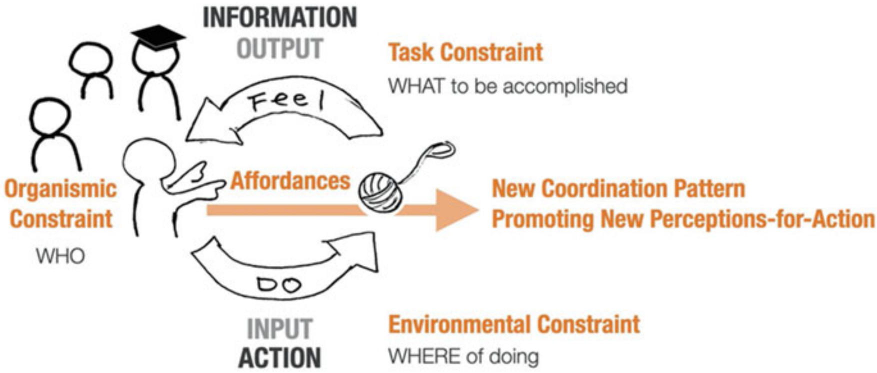


**Fig. 5** An Ecological dynamics view on learning through embodied interaction with educational media. Note. An ecological-dynamics analysis of learning mathematics through engaging with embodied-design educational media: grounding new concepts in perceived affordances for enacting movement forms that solve motor-control problems

(1) *organismic*, what learners themselves bring to the scene, e.g., their sensory, cognitive, and motor capacities; (2) *task*, what needs to be accomplished, e.g., kicking a ball into a goal; and (3) *environment*, e.g., qualities of the terrain, ambient light, the size and heft of the ball, and the repercussions of missing the goal. By positing that individuals detect affordances to facilitate their sensorimotor accomplishment of goal movements, we are necessarily positing that individuals' perception is given to change – perception adapts to tighten our grip and actions on the world (see Dreyfus and Dreyfus 1999, on the phenomenological philosophy of Merleau–Ponty). Centering perception, thus, as the key construct for investigating how individuals learn to move in new ways, is strongly supported by cognitive psychology research on how we learn to coordinate the performance of challenging bimanual forms (Mechsner 2003, 2004; Muraoka et al. 2016).

The notion of affordances captures an ecological epistemology that critiques the human–environment duality as fallacious (Heft 1989), an artifact of Subject–Object grammar deeply engrained in many languages (Barton 2008). In a sense, educators' quest to ground mathematical concepts in perception-for-action is to go behind language to presemiotic phenomenology. In this antediluvian psychic state, dualities melt down: I–thou, I–material. As such, embodiment theories resonate also with critical scholarship. Particularly relevant to our examination of educational technologies is Feminist postcolonial rejection of traditional ontological dualities, such as human–machine or concrete–abstract, in favor of various functional imbrications and assemblages enabled by technological breakthroughs (Haraway 1991). We thus witness the emergence of a broad interdisciplinary change of mind about the mind.

We are discussing here how people learn to move in new ways. But how might this bear on discussing how people learn new mathematical concepts? The answer is suggested by a rising paradigm within the cognitive sciences that conceptualizes all knowing, including of would-be abstract ideas such as mathematics, as necessarily shaped by our biological constitution and our engagement in the natural and cultural ecology (Hutto 2019). Sometimes dubbed *4E* to denote cognition as embodied, embedded, enacted, and extended (Newen et al. 2018), this embodied turn in the cognitive sciences has found eager readers among mathematics-education researchers (e.g., Pirie and Kieren 1989), who were bolstered to promote radical readings of Jean Piaget's genetic epistemology (Arsalidou and Pascual-Leone 2016; Di Paolo et al. 2014; Steffe and Kieren 1994; see also Allen and Bickhard 2013). By this view, mathematical ideas are grounded in perceived dynamic images that first develop through sensorimotor interaction and then rise to consciousness through languaging. As such, perceptual capacities constitute the referential grounding (Harnad 1990) of the various semiotic forms employed in the mathematical practices.

Theories of 4E cognition and learning, such as ecological dynamics, bear practical implications for how we design learning environments. These theories suggest that students can develop new mathematical knowledge by first learning to move in new ways and only then signifying this emergent capacity in normative semiotic forms of the discipline. Drawing on ecological dynamics and 4E cognition, *embodied design* (Abrahamson 2014, and see Abrahamson et al., Ch.?? in this volume) is a research paradigm that seeks to understand how students ground new mathematical knowledge in their naturalistic capacity to apply and develop perceptions of the environment in acting upon it.

The embodied-design research paradigm includes a pedagogical framework for building environments where learners detect affordances through tackling motor-coordination problems. In turn, the working principles of embodied design can be used as a critical lens on how digital resources have been integrated into mathematics education. This framework gives us purchase on the reciprocal evolution of theory and media, namely, (a) researchers' conceptualizations of the body's function in models of thinking, learning, and teaching; in dialogue with (b) designers' engineering of digital resources conducive to learners' detection of conceptually meaningful interaction affordances through sensorimotor exploration.

The next section looks back at the history of digital resources for mathematics education through the lens of embodied design. We will revisit the historical evolution of interactive digital media as three-epoched, modeled on the engineers' zeitgeist, specifically their apparent or stated beliefs concerning the role of movement in interactions designed to promote mathematics learning. Through this survey, we seek to support our proposal that the design of digital resources for mathematics learning should center on first offering opportunities for students to move in new ways before they signify this new capacity.

## A Critical Analysis of the History of Digital Resources for Mathematics Education

We are surveying the evolution of digital resources for mathematics education, a historical process we view as a pendulum swing down to early technologies that excluded the moving body up through to contemporary devices that return the body by increasingly foregrounding physical movement in conceptual learning. Educational technology and educational theory evolve in mutual reciprocity. The gradual return of the body to mathematics education has been greatly enabled through engineering advances in interaction platforms. Yet each of these advances, in turn, has inspired educational researchers to imagine new design architectures and activity genres that would further implement the embodiment paradigm. Specifically, design-based educational researchers inspired by the embodiment turn in the cognitive sciences have sought to create digital learning environments that engage students' multimodal sensorimotor interaction as their cognitive grounding of mathematical concepts. By building these technologies, educational researchers can contribute also to cognitive science scholarship through empirical evaluation and refinement of the embodiment paradigm. As such, the availability of embodied-interaction learning technologies both realizes and, reciprocally, rallies greater interest in the embodiment paradigm, among mathematics education researchers.

Departing from symbolic interaction, this section surveys HCI evolution and its offerings for a mathematics pedagogy based on learning to move in new ways. For each HCI era, we will discuss educational exemplars and their design rationales.

## Symbolic Interaction

Early personal computers interfaced users with the software through keyboard input and screen output. Users typed keys to enter symbolically encoded alphanumeric information. The screen featured virtual objects with limited interaction functionality, such as moving up/down and left/right by keystrokes. During the 1970s and beyond, some educational scholars embraced computation media as heralding a paradigm shift in students' epistemic, cognitive, affective, and social relations with mathematics and science – software was the new construction material for exploring ideas (diSessa 2000; Noss and Hoyles 1996; Papert 1993; Wilensky and Reisman 2006). Others sought to scale optimized instruction by supplementing problem-solving tasks with software *cognitive tutors* that diagnose students' misconceptions by their errors and respond with customized explanations (J. R. Anderson et al. 1995). Yet, for most, mathematics learning in these early media did little more than lift the textbook onto the screen, the novelty being primarily inherent to the medium itself (Bergstrom and Lazar 1982). Traditional instructional practice was thus rendered, perhaps, more engaging, more efficient, and more assessable, yet without bringing about foundational transitions in the provision of cognitive activity. At its worst, automated instruction introduced serious compromises to the quality of educational offering (Erlwanger 1973). By way of analogy from media studies, the first use of movie cameras was to place them on static tripods opposite a theater stage – it took a while for someone to notice that the camera itself could be lifted and moved about, hence the dawn of cinema. Dyson (1996) stated that "great advances in science usually result from new tools rather than from new doctrines" (p. 805). To actuate these advances, though, tools must dialogue with said doctrines.

*Summary*. The role of physical movement in early educational HCI was little more than modest keyboard fingering. Where this manual digitizing engendered screen movement, opaque software procedures hampered any naturalistic experience of immersed perceptuomotor enactment. This gap between the embodied mind and disembodied interfaces is liable to introduce what Morgan et al. (2009) dubbed an *epistemological distance* wedged between organic and cyber activity (see also Meira 1998, and Rabardel 1993, on *transparency*; see Haspekian, this volume, on instrumental distance).

## Exploring Graphical Objects

With the introduction of graphical user interfaces (GUIs), pioneered by Sutherland (1963) and later commercialized as Xerox Star (1981), Apple Lisa (1983), and Microsoft Windows (1985), interaction took place in two-dimensional space rather than as a one-dimensional stream of characters. GUIs exploited more areas of human abilities, such as peripheral attention, pattern recognition and spatial reasoning, information density, and visual metaphors. The advent of GUI and continuous-input devices, such as the mouse and joystick, rendered the transposition of manual movement on the horizontal plane into congruent movement on the digital screen. Users could thus operate on 2D displays of virtual objects through interactive software, supporting a sense of spatial immersion. Students could add, remove, and rearrange graphical objects into goal collections, forms, or patterns (Olive 2000); operate on computationally linked mathematical forms, such as diagrams, tables, and formulas, to investigate conceptual models of quantitative relations (Moreno-Armella et al. 2008); or inquire into the objects' invariant properties under transformation, such as by altering a display's figural appearance without changing its cardinal or ontological qualities (Hohenwarter et al. 2009). In geometry studies, a conceptual shift away from paper math (Papert 1996, 2004) was here introduced, where each particular manipulated form, say a 3:4:5 right triangle of specific measured lengths, became a token of a mathematical type, here, all similar 3:4:5 right triangles (Leung et al. 2013; Yerushalmy 2013).

By some respects, new technological functions, such as the mouse, render learning more *efficient* by removing the motoric challenges and occupational tedium of traditional media. Others take pause to theorize how different media *structurate* concepts, such as how compass-and-straightedge construction activities, as compared to authoring a Logo procedure that builds a 360-sided polygon, foster particular understandings of the circle (Wilensky and Papert 2010). Similarly, using a mouse to continuously vary components of diagrams can foster a realization of mathematical objects as conceptual classes (Hackenberg and Sinclair 2007). Moreover, the dragging mode of dynamic geometry environments revolutionized how mathematics education researchers, teachers, and students could learn about mathematical argumentation (Arzarello et al. 2002; Baccaglini-Frank and Mariotti 2010; Hollebrands 2007; Laborde et al. 2006; Sinclair and Yurita 2008). In contrast to the static, quasi-formal algorithm of a two-column proof, Dynamic Geometry

Environments (DGEs, e.g., The Geometer's Sketchpad; Jackiw 1991) enabled students to build, explore, and discover theoretical relationships implicit to a diagram by varying its spatiographic configurations under programmed constraints that kept its defining properties intact (Laborde 1998). For example, a point on a curve could be dragged along the curve simply and directly by making smooth movements of a mouse with one's hand. As the point moves along the curve, any objects that depend on that point, such as a tangent line, are automatically and continuously transformed. As such, DGE let students learn a mathematical relationship through enacting it, thus "acknowledging the epistemological import of the body" (Hackenberg and Sinclair 2007, p. 13).

*Summary.* Graphical user interfaces elevated user experience from symbolic textual interaction to continuous 2D graphical movement analogs. Equipped with these cognitively ergonomic digital extensions, students could remotely control and displace graphical objects on the screen. Still, moving in the GUI era was based on remotely controlling pixels rather than moving one's body to directly manipulate physical objects. By and large, these innovative educational resources were implicitly modeled on a philosophical and theoretical conceptualization of moving as a means of inquiry into inherent properties of virtual systems. Moving was still viewed as pragmatic rather than epistemic (cf. Kirsh and Maglio 1994), where handling objects is conceptualized as subservient of cognition and its would-be intracranial conceptual models – movement was not yet cognition itself (Gallagher 2015; Sheets-Johnstone 2015).

## Moving Is Learning

With the arrival of Embodied and Tangible User Interfaces (TUI) for multimodal embodied-interaction learning environments, we are moving back up toward the yarn ball's multimodal interaction affordances. Only now, digital technologies aim to *expand* on concrete experience to enhance and diversify educational impact (Arroyo et al. 2017; Bock and Dimmel 2021; Dimmel et al. 2021; Lindgren and Johnson-Glenberg 2013; Marshall et al. 2013; Tomlinson et al. 2020). An optimistic view of the end-state of this continual development is an all-consuming digital world, parallel to our own, accessible via an invisible interface, wherein users generate, transform, and broadcast spatial representations of information through natural movements with material things – a seamless, synergistic blend of the virtual with the actual (Fishkin et al. 1998; Haraway 1991). The touchscreen interface that is now ubiquitous on phones, tablets, and other displays already speaks to the appeal and power of user interactions that blend materiality – the haptic contact with a screen – with digital representations.

The *TouchCounts* environment (Sinclair and Jackiw 2014; Sinclair et al. 2016), wherein children can learn about ordinality and cardinality through dynamic, touch-based interactions, is an exemplar of how TUIs can leverage haptic engagement to create digital representations that augment our sensory engagement with the material world. In the case of *TouchCounts,* touching the screen creates a visual representation (a colored dot) at the point of contact that is visually labeled (by a corresponding numeral) and whose name is read aloud (Sinclair et al. 2016). At the moment that a learner feels the haptic connection with the screen, the *TouchCounts* environment responds to the learner's touch by instantiating a bundle of semiotic representations (graphical, literal, and auditory). This blend of representations broadens the range of sensory modalities through which children can apprehend concepts of number, magnitude, and sequencing.

Recent innovation in digital resources – including remote-action sensors, augmented reality (AR), virtual reality (VR), real-time computational movement analytics, and artificially intelligent feedback responses – stand to reduce a phenomenological and epistemic gap between embodied proximal action and electronic distal effect, practically removing the "inter" of "interaction" to leave only visceral, immersive dynamic coupling (Hansen 2004, p. 167). Drawing on radical-constructivist and enactivist philosophy, some of these contemporary digital resources have been theorized as occasioning opportunities for students to learn new concepts by learning to move in new ways, where doing so is predicated on learning to perceive the situation in new ways. In the *embodied design* framework (Abrahamson 2014), specifically, when students engage digital resources, they first solve motor-control problems through developing new perceptual forms guiding the enactment of movement – i.e., *perceptions-for-action* – and only then they appropriate mathematical symbolic artifacts to signify these enactments in formal semiotic registers. In turn, the embodied-design framework is enabling researchers to rethink the offerings of digital resources to students with sensory differences (Tancredi et al. 2021b).

*Summary*. Embodied-design activities conceptualize moving as the instantiation of knowing, so that mathematical learning is the cognitive process of figuring out how to move in a form that accommodates a given set of constraints imposed by one's own cognitive capacity as well as task demands and environmental contingencies. Moving in a new way is considered as developing an enactive grip on new conceptual understanding (Hutto 2019). Embodied design thus takes inspiration from dynamic mathematics environments (e.g., Arzarello et al. 2002; Hackenberg and Sinclair 2007; Nemirovsky et al. 1998; Yerushalmy 2013), yet its philosophical commitment to radical enactivism centers its discovery-based activities on educating students' perception-for-action (Abrahamson and Mechsner 2022). This line of research, as we will soon demonstrate, is exemplified by a technological learning environment called the Mathematics Imagery Trainer.

## From Theory to Practice: Educational Design for Enactive Mathematics Learning

What do mathematics learning activities look like that aim to foster new conceptual understandings and procedural skills by way of first engendering new perceptions-for-action? How can we make sense of learning processes in these environments through the theoretical lenses of enactivist cognition and ecological dynamics?

This section overviews three projects designed to engage students in operating an unfamiliar system where they are tasked to manipulate virtual objects so as to effect and maintain goal qualities of feedback from their environment. The projects were selected to exemplify a variety of interactive technological resources for different mathematics content that all center on enacting movements through digitally fabricated space. As such, these cases enable us both to demonstrate how general design principles obtain across educational projects, while highlighting several parameters that distinguish the potentials of these environments. Across all projects, students' task-concordant performance hinges on coming to perceive the situation in a new way that enables enacting specific movements that solve emergent problems of spatial orientation and object transformation. In turn, these new perceptions constitute the cognitive and discursive kernels of the designs' targeted mathematical concepts. To promote the mathematization of "experiential reality" (Gravemeijer 1999, p. 156), the designs all include a social contingency, by way of interacting with a collaborating partner or a guiding tutor. These interactions are designed to solicit students' reflection on their own perception to explicate their perspectives, actions, judgments, and inferences as well as consider differing perspectives and come to view them as complementary.

In formulating their multimodal interlocutions, students draw on available *semiotic means of objectification* (Radford 2013), such as frames of reference, measurement instruments, turns of phrase, symbolic notations, or kinetic input patterns, all predetermined and provided by the designers as potential utilities embedded in the activity space. Students recognize these embedded utilities and engage them to better enact, evaluate, or explain their solutions. In so doing, these available means of discourse and action that extend students' grip on the situation lend disciplinary form to students' emergent perceptions, thus grounding the target notions in their perceptual solutions to the motor problems (Shvarts et al. 2021). That is, means of engagement become means of thinking, per the Vygotskian principle of cognitive development as internalizing heritage routines through the social enactment of cultural practice (Stetsenko 2002).

Following the three project sections, below, a cross-project comparison will emphasize their differences as attributed to unique characteristics of the content in question as well as to general pedagogical objectives shaping students' task-selection.

## The Mathematics Imagery Trainer

The Mathematics Imagery Trainer is a type of interactive learning environment that implements enactivist theory by way of a particular pedagogical framework. The *action-based genre of embodied-design* (Abrahamson 2014) seeks for students to ground formal disciplinary concepts in their natural perceptuomotor phenomenology. Learning environments centered on the Mathematics Imagery Trainer (hence, referred to as "Trainer") constitute digital versions of what ecological psychologists Reed and Bril (1996) call *fields of promoted action*. These are socially orchestrated motor challenges occasioning opportunities for novices to develop neural capacity for enacting culturally valorized movements. Students who engage in Trainer activities discover and exercise perceptual solutions to motor-control problems of enacting targeted movement forms. For example, students are to raise their hands simultaneously, in parallel, at different speeds (see Fig. 6a). These movement forms have been choreographed by 4E mathematics-education researchers so as to "elicit the gestures which allow access to the source experience that gives [curricular] contents coherence and meaning" (Petitmengin 2007, p. 79). Once they are: (1) able to enact a protomathematical movement form solicited by the Trainer task

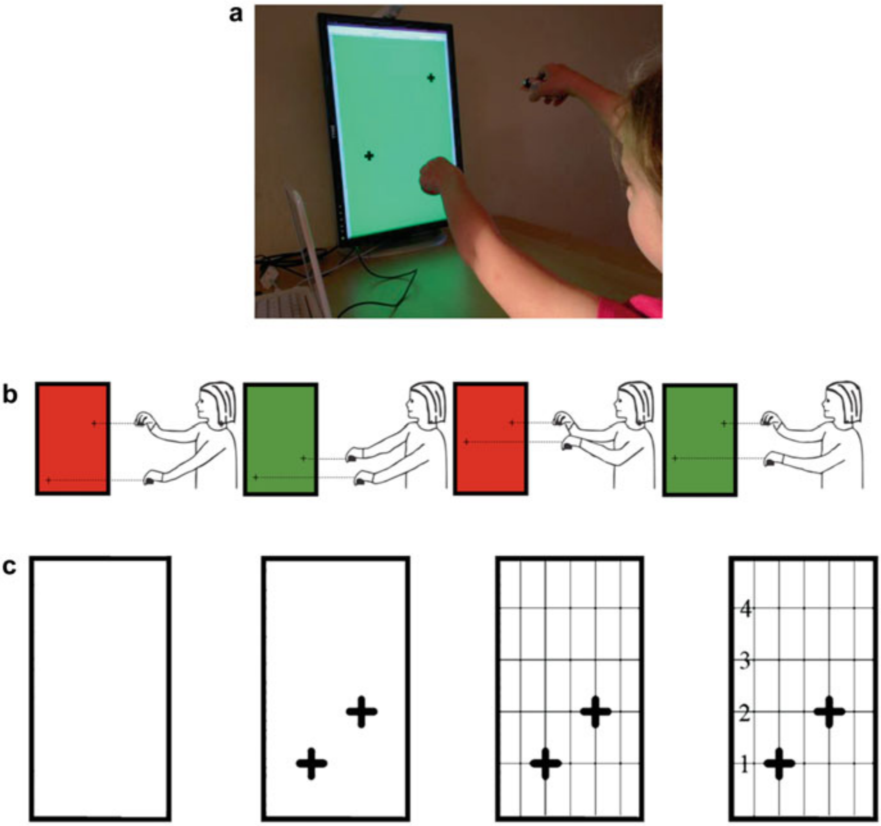


**Fig. 6** The Mathematics Imagery Trainer for Proportion: (**a**) A child using a remote-sensor version of the Mathematics Imagery Trainer has made the screen green by positioning the left-hand and right-hand cursors at respective heights above the screen-base as corresponding to a targeted mathematical relation, here a ratio of 1:2; (**b**) schematic sequence of exploring and discovering a set of two-cursor positions that effect a green screen – students learn to a move in a new way *between* these positions, in constant green, by keeping constant the hands' height ratios; and (**c**) symbolic artifacts laminated onto the activity space – cursors, a grid, and numerals – bring about spontaneous transitions in students' movement forms and language to include formal aspects of mathematical practice and representational systems

(Fig. 6b); students are (2) encouraged to articulate their strategies; and then they are (3) offered various mathematical instruments by which to enhance the enactment (Fig. 6c). As they incorporate these new resources as means of improving and regulating the enactment of the movement forms, students implicitly appropriate disciplinary perceptions mediated by these forms. As such, students engage the instruments' inherent semiotic systems of mathematical discourse (Arzarello et al. 2005; Bartolini Bussi and Mariotti 1999; Drijvers et al. 2013). Trainer activities span the K-16 curricular gamut (Alberto et al. 2021), have been implemented in a variety of interaction media (e.g., tablet), are suitable for child-centered tutoring (Flood et al. 2020), are geared for distance instruction (Shvarts and van Helden 2021), and cater equitably to the unique capacities of students with atypical sensory and neural constitution (Lambert et al. 2022; PhET Interactive Simulations 2021; Tancredi et al. 2021b).

In its conception, the Trainer design was inspired by 4E theories of cognition (Newen et al. 2018) as well as radical-constructivist frameworks for mathematics pedagogy (Steffe and Kieren 1994). Of particular pertinence to theorizing empirical results from Trainer research is a hypothetical construct from enactivist anti-representationalist analysis of athletic skill – the attentional anchor. An *attentional anchor* is expert athletes' heuristic perceptual orientation toward the environment that lends them an optimal grip on the enactment of a movement, for example, a juggler's soft gaze forward that monitors multiple moving props in peripheral vision (Hutto and Sánchez-García 2015; cf. Gigerenzer 2021). In Fig. 6, the spatial interval between the cursors comes forth from the background to anchor the student's attention, facilitating the otherwise challenging bimanual coordination: To raise both hands simultaneously, keeping their vertical displacements at a constant ratio, students spontaneously attend to an imaginary line subtending the cursors, and they experience *raising that line* (Abrahamson and Sánchez-García 2016). Unprompted, they say, for example, "The higher my hands go, the bigger the distance [between my hands]." As such, students tacitly tap their primitive knowledge to make and have new dynamical images serving vital interactions in their ecological niche (cf. Pirie and Kieren 1989). Teachers work with students to language these emerging percepts into mathematical assertions and to develop deeper mathematical understandings by comparing across their different solution strategies (for further references, see Abrahamson et al. this volume).

Researchers investigating the phenomenon of attentional anchors in the context of Trainer activities have analyzed data comprising: (a) video images and digital records of students' actions on virtual objects; (b) audio–video data of students' verbal–gestural utterances about these actions; and (c) eye-tracking data of students' gaze fixations and paths. These mixed-methods studies found attentional anchors ubiquitously as diverse students' perceptual solutions to a variety of Trainer motor-control problems spanning multiple mathematical concepts (Duijzer et al. 2017). As such, Trainer research both supports and applies an argument from empirical research investigating bimanual coordination, to wit, that physical movements are *perceptually* organized rather than learned as specific motor actions (Mechsner et al. 2001; Mechsner 2003, 2004). As enactivist cognitive scientists maintain, "(1) perception consists in perceptually guided action; and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (Varela et al. 1991, p. 173). That is, attentional anchors are the cognitive structures that emerge from the recurrent sensorimotor patterns that enable students' bimanual operation of the Trainer to be perceptually guided (Abrahamson and Sánchez-García 2016). And it is these attentional anchors that seed and ground conceptual sense-making and reasoning.

Using a methodology from coordination dynamics, Cross-Recurrence Quantification Analysis (Marwan et al. 2007), researchers of Trainer activities have further demonstrated that students' spontaneous development of new perceptuomotor capacity manifests as a complex dynamic system in flux transitioning in phases along students' solution progress: exploration, discovery, and fluency (Tancredi et al. 2021a). This finding, the researchers argue, could enable the prediction of mathematical insight based on hand motions (cf. Church & Goldin-Meadow, 1986, on the emergence of new gestures as heralding conceptual insight). These predictions of incipient understanding could draw on algorithmic processing of action telemetry not only for assessment and research but also for real-time teaching, whether teachers are colocated or remote, human or artificially intelligent (Pardos et al. 2021).

From an ecological-dynamics perspective, attentional anchors can be analyzed as self-imposed task constraints that students develop spontaneously as their means of adapting to new environmental constraints. For example, as a student begins realizing that raising her hands while keeping *constant* the spatial gap between them is not a viable solution to the Trainer proportions problem (i.e., an environmental constraint), she assimilates to the environment by way of accommodating the extent of the spatial gap corelative with the height of her hands (i.e., a self-imposed task constraint). Abrahamson and Abdu (2020) note that Trainer design architecture, in which students must figure out how to move in new ways, differs from the design architecture of other DME (dynamical mathematics environments), such as GeoGebra, where the software constrains permissible manipulations, and so students need not discover and develop new cognitive structures as self-imposed constraints. Further research is still needed to compare the cognitive effects of activities that center mathematical learning either on student discovery or direct instruction of movement forms that instantiate mathematical concepts (e.g., Walkington et al. 2022).

## HandWaver

*HandWaver* (Dimmel and Bock 2019) is an immersive virtual environment for inscribing and exploring three-dimensional diagrams via pseudonatural actions (Nicolas and Trgalová 2019), such as pointing, pinching, stretching, and spinning. HandWaver actions are *pseudo*-natural, because they must be executed within a range of movements that can be recognized by a head-mounted sensor. HandWaver was designed to be a proof of concept for how movements – both fine movements of immersed participants' fingers/hands and also gross movements of their heads, torsos, and bodies – could be used to create and explore mathematical diagrams that are realized in space. Figure 7 shows a first-person perspective of an immersed

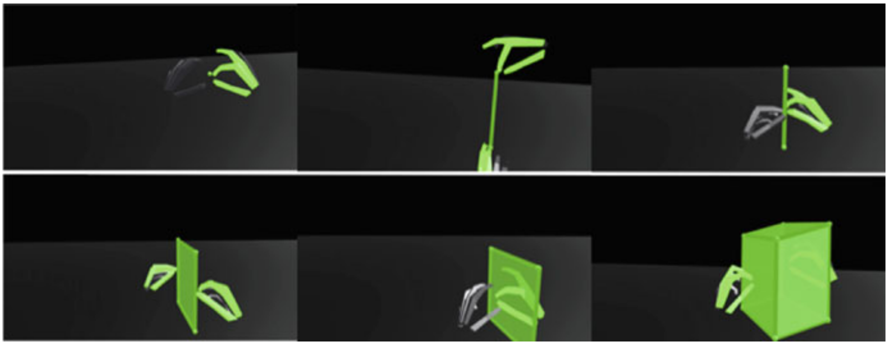

**Fig. 7** *HandWaver*. Note. The *stretch* gesture in *HandWaver* (from Dimmel and Bock 2019)

user transforming a point into a prism by successive applications of an action we refer to as *stretch*.

Figure 7 provides a flattened, first-person view of what an immersed participant sees as the *stretch* action is applied iteratively to a point, a line segment, and then a polygon. The hands in the frames are virtual versions of the immersed participant's real hands, as tracked and digitally approximated by a *Leap Motion* sensor mounted to the front of an HTC Vive head-mounted virtual reality display. Across the first row (from the left), the participant begins by pinching a point with both thumb and index finger on the right hand (top left) – the fingers have turned green, indicating that the gesture-based user-interface recognizes a pinching gesture. Next, with pinching gestures engaged in both hands, the user pulls the point apart, using a gesture like what one might use to pull apart a cotton ball (all fingers green, top middle); this is the *stretch* action, and its effect is to transform a zero-dimensional point into a one-dimensional line segment. The user next pinches the line segment with each hand (top right, left pinching gesture not yet recognized) and stretches it again, now transforming a one-dimensional line segment into a two-dimensional polygon (bottom left). Finally, the user pinches the face of the flat polygon (bottom middle) and stretches it into space (bottom right), thereby transforming a two-dimensional polygon into a three-dimensional prism.

The *stretch* technologically rendered gesture was designed to link the geometric concept of dimension to specific movements that were grounded in natural gestures the third author had used to explain the concept of dimension in the context of a geometry class for preservice K-8 teachers. But beyond this practice-based inspiration, the *stretch* gesture is an example of how perceptions-for-action can foster new forms of movement through interactions with a digitally rendered immersive environment. The iterative use of *stretch* to effect the point→line segment→polygon→prism transformation necessarily requires coordinated movements of an immersed user's hands along axes that span $R^3$ (Fig. 8). Further, positioning one's hands to enact these axes tends to involve arm, torso, and head movements – in practice, immersed users turn their bodies around the representations as they transform them, taking on new perspectives as they coordinate the movements of their bodies with the spatially extended diagram they are transforming with their hands. Students operating in

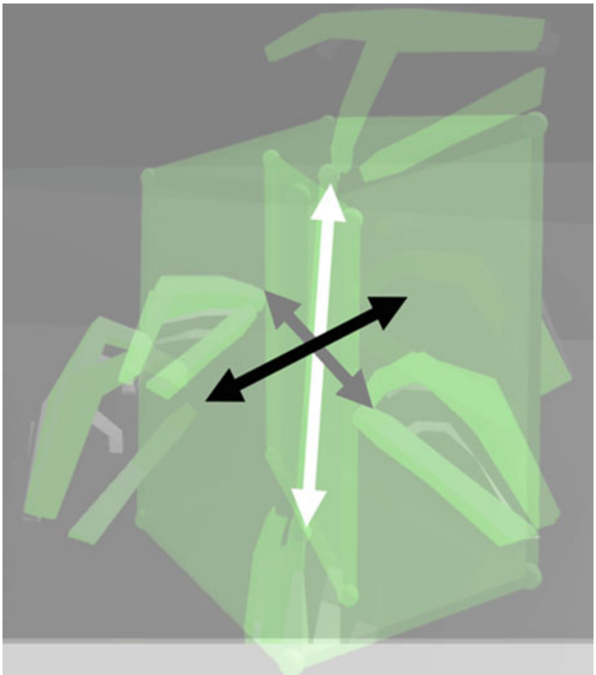

**Fig. 8** *Gesture-based axes that span* $R^3$. Note: In Fig. 8, frames from Fig. 7 (top middle, bottom left, and bottom right) have been augmented with axes and superimposed to illustrate how repeating the *stretch* action realizes a basis for $R^3$

*HandWaver* ground the textbook concept of dimension (noun) as an iterated sequence of dimensionalizing (verb) actions, where the constitution of each dimension $n$ is oriented perceptually on the $n$-1 dimension's generative form. The technological environment thus grounds the target concept in a new way of moving, where a set of enactively analogous perceptions-for-action serving the geometrical construction become schematized as a coherent class of structural affinities.

Stretching a segment generates four-sided polygons that are parallelograms, and then stretching these polygons generates six-faced prisms that are parallelepipeds (evident in the upward lift of the prism in Fig. 8). Note that the gesture-based axes enacted when performing the sequence of stretch transformations depicted in the figures need not be orthogonal. However, for the task of inscribing a rectilinear prism, there is the possibility of a more focused perception-for-action: As the dimensions of the stretched figures increase, the space of allowable instantiations of the stretch gesture decreases. A point can be pulled apart along any line in $R^3$ to create a line segment, but the slope of the line then constrains how the line segment can be transformed into a polygon ($R^2$, i.e., along any line incident with a plane that is perpendicular to the center of the segment) and how the polygon can be transformed into a prism (R, i.e., along the line normal to the center of the polygon's face). In such cases, there is a natural correspondence between participants' movements around the diagram, the actions they take to successively transform it, the perspectives from which the diagram is viewed as those actions are performed, and the underlying mathematical concepts of dimensionality and degrees of freedom. Here is an instance where the design of the environment, the constraints of the gesture-based user interface, and the underlying mathematical concepts work concordantly to train immersed participants to move in new ways. The result is a three-dimensional movement-centric environment for spatially inscribing representations of mathematical figures.

HandWaver is an initial example of how the affordances of natural and gesture-based user interfaces – the third wave of the evolution of digital tools described in section "A Critical Analysis of the History of Digital Resources for Mathematics Education" – can weave together movement, perception, and mathematical activity. In the coming years, as the technologies that render spatial inscriptions and facilitate movement tracking become more reliable, portable, and widely accessible, there will be opportunities to realize digital tools that further develop the potential to use natural movement as a means for inscribing representations of mathematical figures that can fill three-dimensional space.

We are in the middle of a profound shift in how we represent and interact with information. The most familiar historical representations of mathematical figures are small, bounded, two-dimensional diagrams that are typically shown from a fixed third-person perspective, such as the diagram of parallel lines shown in Fig. 9 (left). But we are rapidly approaching a future where it will be routine to create large, apparently unbounded, spatial diagrams that can be viewed from continuously variable perspectives that are controlled by natural movements. This potential is illustrated in Fig. 9 (right), where the parallel lines from Fig. 9 (left) are viewed as if from a point on the transversal – a viewpoint made available to an immersed participant who stepped into the plane of the parallel lines and looked up. Learning scientists, mathematics educators, and technology designers collaborating on the design of immersive digital resources will shape this emerging future by combining their unique points of view.

## VR SandScape

*VR SandScape* (Ryokai and Li 2020; Ryokai et al. 2022) is a hybrid Spatial Augmented Reality (SAR) sandbox and VR system developed to support children's

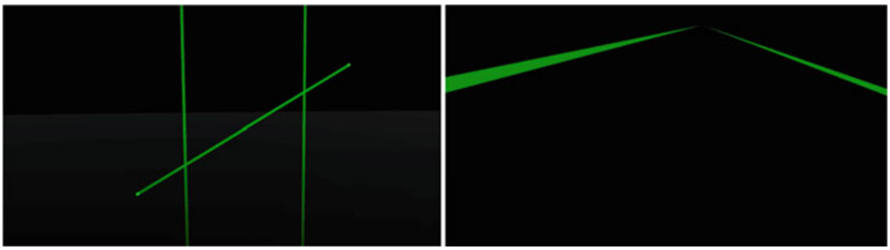

**Fig. 9** Parallel lines cut by a transversal: allocentric textbook view (left) and egocentric immersed view (right)

collaborative construction and evaluation of 3D volumetric designs. Using a depth-sensing camera installed above the sandbox, the system scans the surface of the sand in real time and generates a corresponding three-dimensional VR rendering of the sandbox topology that is constantly changing as one of the children physically sculpts the sandscape. In the corresponding VR world, the other child wearing a VR head-mounted display (HMD) can virtually walk through the mountains, valleys, etc., that were physically created in the sandbox, with a first-person point of view and at full scale. The physical sandbox is augmented with color projections from above to visually emphasize the sand's topographical contours such as lakes, peaks, etc. (see Fig. 10). The virtual model uses the same colors as the projection. A large external monitor displays the virtual explorer's view so that the child at the sandbox can access the virtual view.

VR SandScape was designed as a hybrid SAR & VR system to promote children's reflection on their own perspective as well as others' differing perspectives in

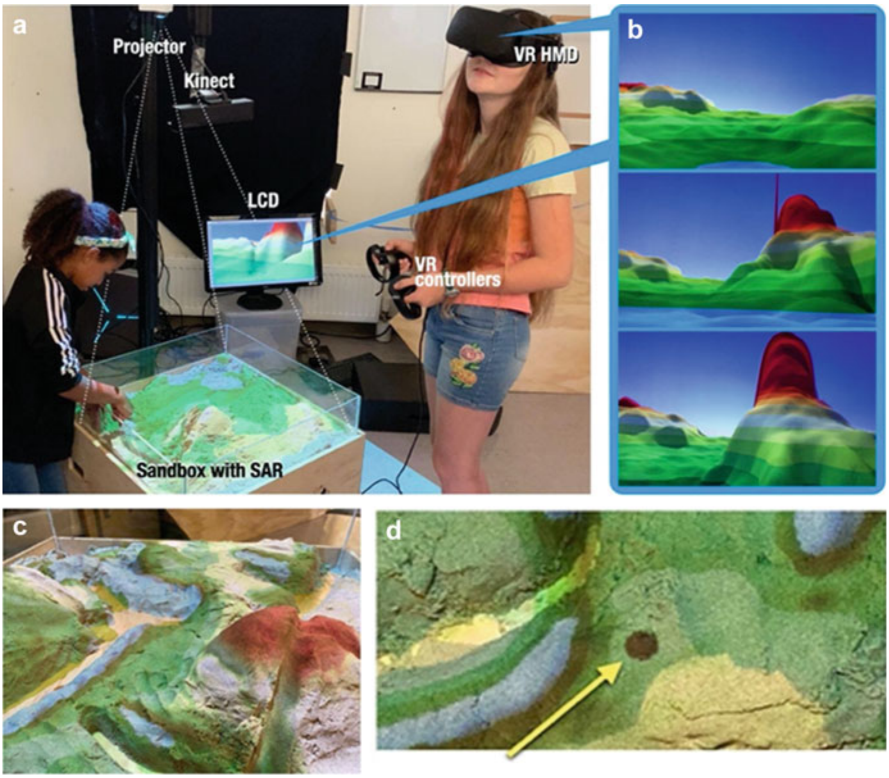


**Fig. 10** VR SandScape: (**a**) VR SandScape setup. A projector and kinect are mounted above the sandbox. A nearby LCD shows the VR child's view in real time; (**b**) VR view of the terrain. A hand of the child working at the sandbox appearing in the VR view; (**c**) a sand model by children shown with spatial augmented reality to emphasize topography such as peaks and valleys; and (**d**) the "dot" showing the location of the virtual player on the sandbox

collaboratively constructing and evaluating 3D volumetric models. The design of VR SandScape enables this by having children take turns serving different roles – being the designer of a 3D model in the physical world and being the explorer of the design in the virtual world – where each role requires considering differences in scale and movement due to the environmental affordances.

VR SandScape may be used in a collaborative design task, such as landscape architecture. For example, users are asked to construct an immersive maze with certain geometric requirements, such as three mountains of varying sizes (e.g., Mountain A is three times taller than Mountain C, and Mountain B is two times taller than Mountain C, where Mountain C can be any size). While the designer works on the physical sand model from a bird's-eye view (approximately 1:43 scale), the explorer wears a VR HMD to immerse themselves in the virtual model from the 1:1 scale to explore the virtual terrain. This results in two perceptually disparate yet structurally complementary perspectives and movements between collaborating partners. For example, for the designer at the sandbox, an apex is to be *looked down* on from the top, while for the explorer, the same apex is to be *looked up* to from the base of the model.

Table 1 illustrates a typical interaction between two middle school children designing and evaluating their maze with VR SandScape. Alex, donning an HMD, explores the maze virtually, while Lee, at the sandbox, guides Alex. In order to help Alex navigate the maze, Lee at the sandbox constantly shifts his perspectives between the sandbox where he sculpts and the LCD, which shows Alex's virtual view in real time.

Viewing features of the sandscape from allocentric (outsider) perspectives, Lee over time increasingly develops a sense of what it is like to be immersed as a user of the maze (evidenced in the change in his language, e.g., from "*You have to go over*

**Table 1** Developing cartographic fluency through coordinating interpersonal perspectives to collaboratively solve immersive navigation challenges

| | | |
|---|---|---|
| Alex, donning an HMD, explores the maze virtually. Lee sculpts the sandscape by looking at the sandbox. The blue "dot" on the sandbox indicates the virtual position of Alex as he explores the model. LCD shows Alex's VR view in real time | Lee often shifts his perspective to the LCD to check where Alex is in the model and what part of the model Alex is looking at in the VR world. Moving between different perspectives, from one's own to another's, becomes essential for two children coordinating and communicating their 3D navigation. They might say, "*jump over there,* ***right where you are looking***" | Time to time, Lee also shifts his perspective to Alex in person. Lee acknowledges asymmetry of access to different views. That is, Lee's multiple views (physical sandbox, VR view shown on LCD, and his physical partner) versus Alex who is completely immersed in the VR world. This results in Lee's explicit use of language, such as "*you have to go over here.* ***Where my hand is***" in concert with his deictic hand movement in the sandbox to connect directly with Alex's VR view |

*here*" (without any reference, spoken from an egocentric perspective) to phrases that acknowledge an allocentric perspective, e.g., "*right where you are looking*," to be explicit about coordinating locations in 3D space with his partner. By shifting his attention between the dynamic marker in the VR SandScape (i.e., a virtual moving "dot" showing where Alex is in the sandbox) and the LCD showing Alex's virtual view, Lee physically moves between two differing perspectives to navigate and design the maze. In the end, the two children collaboratively modify the 3D landscape with a trench that is manageable from the user's perspective. What both children accomplished, we believe, is developing new perceptions-for-*another-person's*-action of their respective displays, a necessary perceptual development enabling collaborative engagement within a negotiated frame of reference. Their new cartographical orientations thus emerged spontaneously as pragmatic solutions to the encountered problem of coordinating the enactment of situated movements.

In our study with middle school children, in order to coordinate their actions toward their design goal, the children discussed explicitly their respective frames of reference (Ryokai et al. 2022). As children came to recognize differences in their perspectives and actions, they increasingly moved between allocentric and egocentric perspectives and attended to details of 3D geometry that otherwise go unnoticed. Through the process, they seemed to expand their perception of the 3D geometry in a 3D space in a new way to gain geometric literacy.

## Summary

The three educational designs surveyed in this section – the Mathematics Imagery Trainer, HandWaver, and VR SandScape – each creates opportunities for students to develop new spatial reasoning skills through forming new perceptions enabling the actions that solve activity tasks. Our characterizations of the learning processes sought to highlight the spontaneous emergence of the designers' targeted notions as students' embodied solutions to situated problems of handling space. In the Mathematics Imagery Trainer, students figure out a new way of manipulating a spatial interval that rises to their multimodal attention as a perceptual means of coordinating the enactment of a bimanual movement, leading to proportional reasoning. In HandWaver, immersed construction mechanics foster an enactive conceptualization of geometrical dimensions as dimension*alizing*, where selected features in each $R^n$ afford generating $R^{n+1}$. In VR SandScape, the design's deliberate cross-perspectival dyadic architecture engenders reciprocal coconstruction of topographical literacy as discursive solutions to collaboration challenges. These three designs exemplify how interleaving embodiment theory, tangible user interfaces, and multimodal methods is scoping new horizons for conceptual learning grounded in sensorimotor exploration.

Yet even as these three showcased interfaces are virtually tangible, by and large they elide actual tangibility, such as what a yarn ball might afford, including tactility of texture and interoception of skin deformation. And yet multimodal educational activities that include actual touch may be more effective than those with virtual handling alone (Chettaoui et al. 2022; cf. Lauwrens 2019). As we embrace and implement the "magic future of interaction design" (Kirsh 2013), we must seek to reintroduce actual haptic–tactile sensation into learning experiences, so that students keep in touch with their natural ways of learning and knowing (Lambert et al. 2022; Price et al. 2022).

## Concluding Remarks

With an eye on the future of interaction design for mathematics learning, this chapter surveyed the evolution of its digital resources. Beginning with simple material objects and analog devices, then advancing through the annals of computational technology, and finally culminating with three contemporary designs, our account was set through the prism of the enactivist thesis from the philosophy of cognitive science. From that perspective, we have examined the idea that conceptual learning begins from developing new perceptions-for-action.

Per enactivism, conceptual knowledge is grounded in multimodal action-oriented perceptions – these perceptions emerge from figuring out how to enact purposive movement in natural and cultural ecologies (Varela et al. 1991). For example, the arithmetic operation of addition could be grounded in multimodal perceptions supporting the enactment of motor actions that produce aggregated portions of material substance, such as stacking clumps of clay, that constitute *protoquantities* in children's conceptual development (Resnick 1992; Silverman 2022). This view on the grounding of mathematical concepts in dynamical imagery (Pirie and Kieren 1989) partly aligns with other theoretical perspectives (e.g., *conceptual metaphor*, Lakoff and Núñez 2000; *concept images*, Tall and Vinner 1981; *grounding metaphors*, Presmeg 1992).

At the same time, mathematics is obviously more than stacking clumps of clay – it is an academic enterprise that is documented, expanded, and conveyed in cultural–historical inscriptional forms, such as diagrams, tables, graphs, and alphanumerical symbolic notation, each with its established structural, procedural, and linguistic conventions (Bartolini Bussi and Mariotti 1999; Duval 2006; Ernest 2008; Sfard 2002). As such, for the mundane skill of stacking stuff to become the disciplinary notion of addition, children need to quantify the material magnitudes and calculate their aggregation, beginning with counting and measuring.

As educational designers, we are thus dealing with two different ways of knowing (Drury and Tudor 2023; Ryle 1945): Whereas perceptuomotor enactment of movement is tacit phenomenology that cannot be directly articulated, the formalization of these intimate experiences as specified cultural ontologies, measurements, and algorithms is an interpersonal practice with strictly regulated semiotic rules (Shvarts and Abrahamson in press). These theoretical distinctions between ways of knowing bear direct practical implications for creating educational resources serving the grounded learning of mathematical content. We have surveyed the evolution of interaction design as modeled on the types, mechanics, and cognitive functions that digital resources have allocated to physical movement, even as we evaluated how the resources enable students to model their movements in normative disciplinary forms toward achieving grounded fluency in mathematical practices.

Educational designers constantly seek to leverage technological innovation as learning media. Whereas we do not presume to predict the evolution of new human–computer interfaces, this chapter has attempted to offer timeless heuristics for "the future development of technology that is sensitive to the principles of biological cognitive systems" (Glenberg 2006, p. 271). At the same time, we have saluted the sage words of Dyson (1996) on the transformative theoretical power of new technology. We thus stand by to constantly query our theoretical assumptions in light of the ever-surprising empirical data we gather as we implement new tools in the service of teaching and learning mathematics.

Yet even as emerging technologies may stimulate us to query our theoretical assumptions pertaining to the human mind and how artifacts become entangled in cognitive practices, being educational designers, we are morally obligated to query our axiological assumptions regarding the ultimate valorization of curricular objectives. Why teach what we teach? What ethos imbues our telos of algebra, geometry, and calculus? These digital resources for mathematics education – are they ultimately means of procuring our students' prospective gainful employment? Because, for some concerned citizens, this fiscal promise of high-tech salaries at once also bears intimations of lurking cybermalfeasance that would only enhance the insidious reach of surveillance, warfare, and exploitation. Educational design is never ethically agnostic, because the didactic "what" is inherently parceled in the political "why." As we harness enactivist philosophy to build technologies of mathematical entrainment, we must stand guard to vouchsafe humanity's historical capacity for practicing and transmitting down the generations our "many forms of knowledge (knowing how to live, knowing what to do, knowing how to think [savoir-vivre, savoir-faire, savoir-théorique])" (Stiegler 2010). Thus, witnessing the human species veer on climatic catastrophe, one might take pause to ponder whether educational designers are implicitly complicit to global determinantal trajectories; one might take initiative to consider how education might subvert the terminal juggernaut of extractive hedonistic consumption (Petitmengin 2021) – how educating a youth who can critique and undo our praxis could serve as society's doomsday means of saving itself from itself (Stetsenko 2017). As such, school curriculum itself should be vigilantly questioned in dialogue with designers of tools that school systems may appreciate and adopt.

The field is now at a juncture, where the yarn-ball pendulum is swaying yet again, perhaps along a different axis, as researchers struggle to pin down the precise cognitive role of motor action in conceptual learning (Abrahamson et al. 2020). In that vein, some researchers claim that what counts for grounding new mathematical concepts through embodied interaction is not a new motor capacity to move per se but, rather, the new *perceptual* capacity that enables one to move in this new way (Abrahamson and Mechsner 2022). The perception of new Gestalts in the environment is our cognitive means of operating on it physically (Mechsner 2003, 2004). From that perspective, the objective of digital resources should be to foster opportunities for students to develop new situated perceptions that ground and mobilize the prospective enactment of mathematical practices (Abrahamson and Sánchez-García 2016). It could be that these attentional anchors are what enable flexible detection of mathematical meanings in novel contexts, that is, to transfer knowledge (q.v. Nemirovsky 2011).

We are often mesmerized by the interactive capacity of digital resources as compared to concrete objects. Should we aim for resources that digitally substitute their concrete sources? Or should we build digital resources that expand on the interactivity of concrete objects? Do we lose anything when our simulations select or otherwise privilege certain sensory modalities over others, and might we, thus, exclude some students with different sensory capacities (Lambert et al. 2022)? Are we liable, by virtue of designing novel artifacts, to disenfranchise ancient cultural epistemologies that differ in perceptual–linguistic practices (Barton 2008; Benally et al. 2022; Urton 1997; Verran 2001)? As we scramble to program, rig, and infuse digital objects with haptic, tactile, auditory, and other simulated responses, are we simply coming back full circle to yarn balls, or does our technology reach farther? Would Rousseau consider contemporary design a triumph, or would he grimace and reiterate, "Why not begin by showing them the real thing?"

The real thing, we suggest, is *any activity where you learn to move in a new way that would support the development of new perceptions-for-action grounding the target concepts*. We still have much to learn from the yarn ball. In particular, the yarn ball inspires us not to lose sight of the *action* component of human–computer interaction and to align the designs of digital resources for mathematics education with how these resources might afford new modes of enacting movements that instantiate, and thus potentiate, mathematical reasoning. In this way, Froebel's first gift endures as both a reminder of where we have come from and also an inspiration for the future we can collectively design.

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