

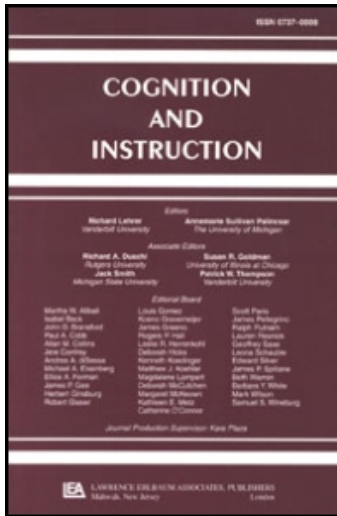
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Cognition and Instruction

Publication details, including instructions for authors and subscription information:

<http://www.informaworld.com/smpptitle-content=t775648096>

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Online Publication Date: 01 July 2009

To cite this Article Abrahamson, Dor(2009)'Orchestrating Semiotic Leaps from Tacit to Cultural Quantitative Reasoning—The Case of Anticipating Experimental Outcomes of a Quasi-Binomial Random Generator', *Cognition and Instruction*, 27:3, 175 — 224

To link to this Article: DOI: 10.1080/07370000903014261

URL: <http://dx.doi.org/10.1080/07370000903014261>

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Orchestrating Semiotic Leaps from Tacit to Cultural Quantitative Reasoning—The Case of Anticipating Experimental Outcomes of a Quasi-Binomial Random Generator

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This article reports on a case study from a design-based research project that investigated how students make sense of the disciplinary tools they are taught to use, and specifically, what personal, interpersonal, and material resources support this process. The probability topic of binomial distribution was selected due to robust documentation of widespread student error in comparing likelihoods of possible events generated in random compound-event experiments, such as flipping a coin four times, for example, students erroneously evaluate HHHT as more likely than HHHH, whereas in fact these are 2 of 16 equiprobable elemental events in the sample space of this experiment. The study's conjecture was that students' intuitive reasoning underlying these canonical errors is nevertheless in accordance with mathematical theory: student intuition is couched in terms of an unexpanded sample space—that is, five heteroprobable aggregate events (no-H, 1H, 2H, 3H, 4H), and therefore students' judgments should be understood accordingly as correct, for example, the combination “3H, 1T” is indeed more likely than “4H,” because “3H, 1T” can occur in four different orders (HHHT, HHTH, HTHH, THHH) but “4H” has only a single permutation (HHHH). The design problem was how to help students reconcile their mathematically correct 5 aggregate-event intuition with the expanded 16 elemental-event sample space. A sequence of activities was designed involving estimation of the outcome distribution in an urn-type quasi-binomial sampling experiment, followed by the construction and interpretation of its expanded sample space. Li, whose experiences were typical of a total of twenty-eight Grade 4–6 participants in individual semi-structured clinical interviews, successfully built on his population-to-sample expectation of likelihood in developing the notion of the expanded sample space. Drawing on cognitive-science, sociocultural, and cultural-semiotics theories of mathematical learning, I develop the construct *semiotic leap* to account for how Li appropriated as a warrant for his intuitive inference an artifact that had initially made no sense to him. More broadly, I conclude that students can ground mathematical procedures they are taught to operate even when they initially do not understand the rationale or objective of these cultural artifacts (i.e., students who are taught a procedure can still be guided to re-invent the procedure-*as-instrument*).

Experience can never be equated with concepts. But experience is not “undefined” either. It is more organized, more finely faceted by far, than any concepts can be. And yet it is always again able to be lived further in a new creation of meaning that takes account of, and also shifts, all the earlier meanings. (Gendlin, 1982, p. 166)

OVERVIEW AND OBJECTIVES

Can students construct meaning for the mathematical content they learn in school? What cognitive, technological, and social mechanisms might enable this learning process? In particular, what roles may semiotic tools—material objects, diagrams, symbols, speech, gesture, and so on—play in guiding students to appreciate mathematical procedures as enhancing their intuitive inferences? And what would be the implications of such a grounded developmental trajectory for the adequacy of theories of learning that are based exclusively either on the cognitive sciences or on sociocultural theory?

These are heady, persistent questions pertaining to ontological development in the social context—questions that frame a whole cross-disciplinary research program, if not the lion’s share of mathematics-education research. What I *will* attempt here, though, is to communicate one case study, in which Li, a sixth-grade student, was guided to engage his intuition of likelihood as he took first steps in learning fundamental notions of probability within the context of a single interview-based tutorial interaction. Li’s reasoning, I will claim, can be broadly characterized as a struggle to reconcile tacit knowledge and disciplinary practice. At the outset Li was shown a situation involving material objects and operations upon these objects, and he was asked about quantitative properties of this situation. Li offered a mathematically correct proposition that he inferred intuitively from perceptual judgment of the situation. Subsequently, Li was asked to warrant this proposition and was encouraged to do so by using a variety of analytic tools, and Li complied by engaging in relevant construction. Consequently, though, Li renounced his initial assertion, because his naïve reading of the mathematical artifacts suggested to him alternative inferences regarding the source situation. Next, Li’s attention was redirected to this source situation so that his initial inference was re-triggered, and Li was guided to bring this initial inference to bear vis-à-vis the artifacts he had created. Li was then first able to perceive the artifact he had constructed as meaning his intuitive inference. Thus reconciling perceptual constructions of a source phenomenon and its cultural reformulation, Li performed the essential cognitive work relevant to reinventing the topical disciplinary notions. That is, the contingencies that Li was impelled to articulate so as to align his immediate and mediated judgments were couched in the form of the very mathematical rules underlying the targeted content. Just what it is that Li experienced along the way, and how the instructional materials and interviewer contributed to this experience, is the subject of this article.

SETTING THE MATHEMATICAL CONTEXT: A QUICK REFRESHER OF CLASSICIST PROBABILITY

The mathematical content constituting the focus of this article is the treatment of a probability situation roughly analogous to the experiment of flipping sets of four fair coins, where each coin

can land either on heads (H) or tails (T). The result of each of the singleton flips in a set of four flips is considered an *independent outcome*, because it is not affected by the other three flips in the set, and the collective result of four flips is called the *compound-event outcome*, for example, the chance of getting “HTHT” is the contingent chance of getting H AND getting T AND getting H AND getting T (providing we can monitor for order by pre-assigning the four coins ordinal identities). We can think of this four-coin flipping phenomenon—its structure, mechanism, and associated procedures—as a type of *random generator*, generally similar to dice and spinners. The analysis of this four-flip random generator—that is, its *combinatorial analysis*—identifies 16 unique and *equiprobable* compound outcomes: TTTT, TTTH, TTHT, THTT, HTTT, TTHH, THTH, THHT, HTTH, HTHT, HHTT, HHHT, HHTH, HTHH, THHH, HHHH. These discernable outcomes, which collectively constitute the *sample space* of the experiment, can be classified into subsets according to their exact number of heads, yielding five *aggregate events*: no heads, one head, two heads, three heads, and four heads, which contain 1, 4, 6, 4, and 1 *elemental events*, respectively. This particular classification system is based on the idea of a *combination*, a non-ordered set of independent outcomes, for example, “3H and 1T in any order,” and the variants on each of the combinations are called its *permutations*, for example, the aggregate event “3H, 1T in any order” (hence, “3H, 1T”) has the four possible permutations “HHHT, HHTH, HTHH, THHH.” According to the *binomial theorem* applied to the simple case of two equiprobable values, H and T, an aggregate event’s expected likelihood is indicated by the proportional part of its constituent elemental events within the sample space (attributed to Euclid, ~300 B.C.E. in Weisstein, 2008). Thus, for example, the aggregate event “2H, 2T,” which includes 6 of the 16 elemental events in the sample space (see earlier), is expected to occur in 6/16 of the experimental trials with this random generator—more often than any of the other aggregate events.¹

As the data will demonstrate, upon being asked to guess the most likely outcome in a concrete situation roughly analogous to the four-coins experiment, students who are not familiar with the above calculus nevertheless offer that “2 heads and 2 tails” would occur most frequently, relative to each of the other aggregate events, albeit they initially cannot rationalize this prediction mathematically. We shall also see that, whereas the experiment’s sample space is constituted

¹I have chosen to describe the sample space as a collection of *elemental events* (TTTT, TTTH, TTHT, THTT, HHHT, HHTH, HTHH, THHH, HHHH) parsed into a subsets of *aggregate events* (no-H, 1H, 2H, 3H, 4H) so as to eschew common terminology confusions around the term “outcome,” which is often used for both theoretical and empirical probability. Generally, there is much overlap and ambiguity, in mathematical and educational texts on probability, with respect to the precise meanings of the terms “event,” “outcome,” etc., and these terminology challenges are related to different epistemologies (i.e. classicist vs. frequentist probability). One concern is a confusion between potential (theoretical) and actual (empirical) outcomes, and another is with regards to what an “event” encompasses within the sample space. In fact, the ambiguity of outcomes is mathematical and not only psychological. Weisstein (2006) writes, citing Papoulis (1984, pp. 24–25), “Experimental outcomes are not uniquely determined from the description of an experiment, and must be agreed upon to avoid ambiguity.” My particular choice of the adjective “aggregate,” as opposed to the more conventional “class,” “set,” or “group,” is an intentional nod to educational research on the cognition and instruction of complex phenomena, wherein agent- and aggregate-based perceptual/conceptual frames play a key role in understanding emergence—these perspectives have been implicated as instrumental in understanding natural distributions, for example, the normal curve, as emerging from multiple instances of random agent-based interactions (e.g., see Jacobson & Wilensky, 2006; Wilensky, 1997). Also, by using the same noun—“event”—for both the naïve and the expanded sample spaces, I am preparing lexical grounds for a claim that the naïve sample space is as legitimate as the mathematical sample space—they are contingent on different ways of seeing the random generator, and these contingencies must be acknowledged, embraced, and leveraged in educational design.

of 16 equiprobable elemental events, students' intuitive inferences are couched in terms of the five psychological objects corresponding to the five mathematical objects—the unexpanded heteroprobable aggregate events. This difference between tacit perception and mathematical formulation and its implications for learning disciplinary notions is intriguing for research on cognition, yet it also bears questions for research on instruction. Namely, can students connect their intuitive sense of relative likelihood with the calculus of probability despite a discrepancy in how these two resources frame the experiment's sample space? What obstacles might students encounter and what work must they perform in order to *connect* (Wilensky, 1993, 1997) to the calculus of probability? What design and facilitation responses might support students in reconciling immediate and mediated inferences?

PEDAGOGICAL AND EPISTEMOLOGICAL ORIENTATIONS

This study—and specifically its investigative focus on relations between intuition and mathematics—is aligned with the particular strain of constructivism whereby mathematical understanding is viewed as developing from naïve intuitions that are persistent yet become qualified, calibrated, refined, and reorganized, on the basis of feedback in diverse contexts, into the complex cognitive structures that are manifest as mastery in a domain. Thus, perceived at an appropriate granularity as deep-structure cognitive mechanisms, students' intuitions are the very stuff that educators should identify, embrace, and work with (Clement, Brown, & Zietsman, 1989; Confrey, 1991; diSessa, 1988, 1993, 2008a; diSessa, Hammer, Sherin, & Kolpakowski, 1991; diSessa & Minstrell, 1998; Smith, diSessa, & Roschelle, 1993; Wilensky, 1993, 1997). Accordingly, the learning materials, activity sequence, and facilitation of the tutorial interview at the core of this study as well as the analysis of data from the implementation of this interview are all oriented toward treating mathematical learning as an individual's negotiation of personal quantitative schemata and the mediated operation of cognitive artifacts that evolved historically to enhance these personal schemata (Abrahamson, 2004; Gelman, 1993; Greeno, 1998; Saxe & Esmonde, 2005; Schliemann & Carraher, 2002; Sfard, 2002; Stetsenko, 2002; Stevens & Hall, 1998).

A research program investigating routes to ground mathematical concepts in intuitive perceptions of problem-based situations is related historically to literature on insight, going back to *gestalt* theory (e.g., Wertheimer, 1938) or even much further back to Aristotle's *De Anima* (~350 B.C.E.). In a related vein, the American semiotician Charles S. Peirce regarded the production and elaboration of inscriptions, such as diagrams, as the essential activity of mathematical reasoning (see in, e.g., Bakker & Hoffmann, 2005; see also Rotman, 2000). Yet, more recently, many researchers have pointed to the double-edged sword of perceptual intuition or, more broadly, embodied perceptual simulation, in the solution of mathematical and other problems (Abrahamson, Berland, Shapiro, Unterman, & Wilensky, 2006; Barsalou, 1999; Barwise & Etchemendy, 1991; Brown, 1997; Cobb, 1989; Davis, 1993; Fauconnier & Turner, 2002; Fischbein, 1975, 1987; Glenberg, 1997; Lakoff & Núñez, 2000; L. C. Martin, 2008; Pirie & Kieren, 1994; Schwartz & Black, 1999; Sellarès & Toussaint, 2003). Namely, solution processes that largely depend on perceptual organization of a problem space, for example, the various picture-based proofs for the Pythagorean Theorem, may lend a strong sense of coherence—a synoptic view—to what

might otherwise remain a concatenation of propositions that are locally meaningful but globally incoherent. Yet, these intuitive problem-solving processes might nevertheless delimit the scope of possible solutions, which rigorous analysis would reveal, or introduce unfounded inferences, which scrupulous exploration would preempt. Notwithstanding this potential tradeoff introduced by engaging intuitive resources in working on problems, mathematicians appear to rely on a variety of undertheorized multimodal strategies and heuristics, in addition to symbolical formulations (Arnheim, 1969; Fauconnier & Turner, 2002; Hadamard, 1945; Lakatos, 1976; Polya, 1945/1988; Presmeg, 2006; Schoenfeld, 1985; Wilensky, 1991). Accordingly, this article treats imagistic reasoning as a commonplace, legitimate, and even necessary mechanism of mathematical problem solving.

From this perspective, the very act of incorporating percepts into mathematical reasoning is hardly too striking in and of itself. Rather, the research questions addressed herein are with regard to the particular personal, interpersonal, and technological factors engendering this process and how these factors might team so as to enable meaningful mathematical learning. Specifically, I will examine how a cognitive artifact supports the merging of two gestalts—two vying perceptual disambiguations of the artifact, which, I will propose, are complementary in the construction of the targeted concept as framed by the activities of the particular design.

The artifact in question is the expanded sample space of a four-coin-like experiment, and the vying disambiguations of this space pertain to its perceptual construction as a collection either of 16 equiprobable elemental events or as five subsets of heteroprobable aggregate events. An aggregate parsing of the sample space—that is, into the categories no-H, 1H, 2H, 3H, and 4H—undergirds the mathematical inference regarding the expected relative frequencies of these five aggregate events—that is, an expectation of a 1:4:6:4:1 ratio, respectively. This analytic quantitative inference could thus potentially enhance students' intuitive qualitative inference, which is based on perceptual judgment of structural–interactive properties of the random generator in question, because that inference, too, would favor “2H, 2T” as the most probable event. However, students may be reluctant to appreciate that the aggregate view of the experiment's sample space resonates with and enhances their intuitive analysis of the random generator, because the expanded sample space includes additional elemental events—the permutations—that may appear to the student as redundant and therefore arbitrarily imposed on the analytic process. Namely, the student may wonder why on earth an instructor is bothering to include the permutations in a listing of “things one could get,” when the student does not see the permutations as contextually discernable from each other. Hence, the theoretical research question of how students may ground a mathematical concept is cast in this study as a question of how students may be guided to successfully negotiate tacit and mediated objectifications of phenomenal elements in a mathematical concept's situated embodiment.

An argument rising from this study is that, at least for a certain class of mathematical constructs, the mediated objectification of phenomena cannot directly articulate or elaborate their tacit objectification—rather, students connect to mathematical concepts by *synthesizing* (Schön, 1981) the tacit and mediated (and see Fischbein, 1987, on “secondary intuition”). Such synthesizing, I will demonstrate, transpires when students are guided to appropriate the mediated objectification of a mathematical phenomenon as a semiotic means of warranting their tacit judgment for that phenomenon. In particular, I will propose the construct *semiotic leap* to explain this process of synthesizing or connecting the phenomenal and the cultural as a form of argumentation.

UNDERSTANDING INTUITIONS OF LIKELIHOOD

Research on probability intuitions and their educational implications is by no means a recent endeavor (for overviews, see Jones, Langrall, & Mooney, 2007; Shaughnessy, 1992). Specifically for this study, it has been robustly demonstrated that people tend to expect that a coin flipped four times will more likely land on “HHHT” than on “HHHH,” whereas the theory of probability construes these outcome sequences as equiprobable (adapted from Kahneman, Slovic, & Tversky, 1982; Tversky & Kahneman, 1974). One plausible instructional response might be that such a mathematically incorrect judgment should be eradicated and replaced, because it is inherently flawed and thus could not possibly contribute toward a productive path for learning probability (Cox & Mouw, 1992; Shaughnessy, 1977). Yet, in accordance with the work of diSessa and collaborators cited earlier, I entertain an alternative response, namely that we should interpret students’ “ $P(\text{HHHT}) > P(\text{HHHH})$ ” statements as resulting from a “legitimate reconstruction of the problem” (Borovcnik & Bentz, 1991; Chernoff, 2009), and that we should therefore seek to understand the intuitive mechanisms underlying students’ framing of the problem and explore learning pathways that may work *with* these intuitive mechanisms, rather than *against* them (see also Gigerenzer & Brighton, 2009).²

Several studies have explored students’ ability to generate the expanded sample space of compound-event random generators, and these studies evaluated students’ combinatorial reasoning as unsystematic, noting that students neglect to discern among permutations on a given combination (canonically, students do not see “2, 3” and “3, 2” as different rolls of a pair of dice; see in Jones et al., 2007). For the most part, though, research studies that incorporate investigations of combinatorial reasoning do not appear to create opportunities for students to analyze, build, and/or operate the random generator (but see Amit & Jan, 2007; Drier, 2000; Iversen & Nilsson, 2007; Kazak & Confrey, 2007; Pratt, 2000; Wilensky, 1995, 1997). We are thus left with the question of why people believe that “ $P(\text{HHHT}) > P(\text{HHHH})$.”

Tversky and Kahneman (1974) have suggested that people perceive HHHT as more likely than HHHH because HHHT—with its more balanced distribution of heads and tails—better captures characteristic aspects of the random generator, that is, HHHT appears more representative of, or better aligned with, structural properties of a fair coin that is equally likely to fall on heads and tails. Indeed, Xu and Vashti (2008) have elicited responses analogous to “ $P(\text{HHHT}) > P(\text{HHHH})$ ” from 8-month-old infants. This early or perhaps innate nature of the tendency to draw a “ $P(\text{HHHT}) > P(\text{HHHH})$ ” judgment may explain why the intuition is so robust and durable. But still, what is the epistemological nature of this intuition, and, therefore, how might we conceptualize this mathematically false intuition with respect to a program aimed at improving the teaching of probability?

Intuitions of the “ $P(\text{HHHT}) > P(\text{HHHH})$ ” type, which appear to govern humans’ natural filtering and construction of perceptual information, have been termed *enabling constraints* (Gelman & Williams, 1998, pp. 600–601):

²There is no broad agreement over the precise semantics of “intuition” (e.g., whether it is fixed or can develop). For example, Fischbein (1987) maintains that whereas “primary intuitions” are stable, “secondary intuitions” can develop with learning, such that initially fragmented or counterintuitive situations may become patterned schemes of expert practice (see also Dreyfus & Dreyfus, 1986, 1999). Interestingly, such learned skills are colloquially termed “second nature.”

Enabling constraints embodied in mental structure that support core domains . . . selectively guide the learner's *attention* to relevant inputs, and they selectively guide the learner's *interpretations* of those inputs toward accurate and adaptively useful ends. . . . The evolutionary history of our species has made us good learners in certain problem-relevant areas and poor learners in other areas. In general, learning in a noncore domain can be either facilitated or hindered by the degree to which it is consistent or inconsistent with a core learning mechanism that has evolved to selectively expect particular types of structure information (my italics. See also Gigerenzer, 1998; Zhu & Gigerenzer, 2006, on "ecological intelligence" and its implications for the representation of statistical information).

Humans, thus, appear to be "hard wired" with innate/early cognitive mechanisms governing perception. This assumption is helpful in explaining the "representativeness heuristic" (Tversky & Kahneman, 1974). Specifically, I propose, the intuitive expectation of structure information that frames the " $P(\text{HHHT}) > P(\text{HHHH})$ " judgment is callous to the linear-sequential order of outcomes (the placement of individual H and T within the four-outcome strings). Namely, the enabling constraint underlying our judgment of outcome frequency appears to privilege the *number* of occurrences of the binomial values H and T (or the ratio of Hs to Ts) at the expense of attention to their order, thus encoding "HHHT" as "3H, 1T." That is, attending to order within a collection of independent outcomes was not ecologically adaptive and therefore never developed as an enabling constraint or primary intuition. Therefore, a person who judges that " $P(\text{HHHT}) > P(\text{HHHH})$ " has interpreted the question—as though the author has asked them to compare the likelihood of "3H, 1T" and "4H." Moreover, the person is in fact giving a correct answer to a different question, because "3H, 1T" is indeed more likely than "4H"—it is in fact four times as likely (compare the four possible outcome sequences with exactly three heads—HHHT, HHTH, HTHH, THHH—with the single possible outcome sequence that has four heads, HHHH). Thus, if we do not attend to the order of independent outcomes in the two sequences, then it is subjectively true that " $P(\text{HHHT}) > P(\text{HHHH})$."

Given the previously conjectured diagnosis, how should we go about helping students attend to the order of independent outcomes?³ Plausibly, we might begin by emphasizing this property of order perceptually. As we shall encounter in the data analysis section of this article, however, making salient an implicit property of a mathematical artifact is complicated by the learner's

³I am by no means claiming that students cannot in principle attend to order. (Indeed, Tversky and Kahneman also documented assertions of type " $P(\text{HTHT}) > P(\text{HHTT})$ " that treat two different permutations on "2H, 2T." However, note that this comparison item makes the property of order salient because order is the only parameter distinguishing the two sequences—a pragmatic framing suggesting that the items are at least nominally distinct along the dimension of order and that this distinction is somehow meaningful to the task at hand. Also, note that the notion of "sequence" is not necessarily salient in a string of four symbols, such as "HHHT," if one is not privy to the process that is captured in the inscription (e.g., a temporal series of four coin flips). In like vein, when four coins are flipped simultaneously, the notion of order is complicated—order is rarely a spatial feature of the coins themselves and is, instead, an analytic construct.) Rather, I am suggesting that students do not attend to order because they do not conceptualize that property of the symbol sequence as relevant to the task of determining relative likelihoods, just as, say, one might not know to rap on a watermelon to hear if it is ripe—attending to this acoustic property of watermelons is not part of what Goodwin (1994) calls one's *professional vision* of this *domain of scrutiny*. Indeed, what people see when they look at objects is greatly contingent on the context and goals of their perceptual activity. This idea is probably self-evident and is certainly a tenet of phenomenology philosophy (e.g., Heidegger, 1962), but see, for example, Shinoda, Hayoe, and Srivastava (2001) for an overview of half a century of cognitive-science empirical studies relevant to this idea of goal-based selective attention.

intuitive ontology for the phenomenon that is being modeled, such that the learner's construction of the situation may still differ from the construction that the designer and facilitator are attempting to mediate through the artifact and activities. Indeed, as we shall see, even introducing into the random generator itself structural elements that might scaffold attention to the order of independent outcomes will prove pedagogically ineffective as long as students do not appreciate *that, how, and why* the order is relevant to the task with which they are charged. Thus, to iterate our focal question from the intersecting perspectives of sociocultural and cognitive theory, I ask: How might teachers *highlight* and *code* (Goodwin, 1994) order as essential to the task of comparing likelihoods, when students are intuitively disinclined to *attend to* let alone *interpret* (Gelman & Williams, 1998) order as pertinent to the task?

As I shall explain later in the article, a constructivist approach that distinguishes between tacit and cultural foundations of individual knowledge implies that educational design should create opportunities that both trigger tacit responses and help students coordinate these responses with relevant mathematical formulations, a process that diSessa (1993) describes as the bootstrapping of content triggered by top-down systemic coherence.⁴ Therefore, the design utilized in this study guided students from the phenomenal to the semiotic, and the study focuses on the impediments germane to this learning path from intuition to inscription. Specifically for the focal topic of binomial expansion, the meta-design decision to depart from the phenomenal implied that students' naïve sample space (5 events) would be invoked prior to having them engage in constructing the formal sample space (16 elemental events, cf. Shaughnessy, 1977).

Granted, students can certainly learn the principle and procedure of expanding the full sample space from the naïve one, but they may not initially recognize that such expansion results in a structure that provides ready tools for warranting their intuitive sense of likelihood distribution. That is, students may work with the naïve sample space oblivious to the fact that each element can be expanded, let alone that such expansion would generate vital information for determining the expected distribution of frequency. Thus, the design problem becomes how to enable students to see the mathematical sample space as resonating with, and enhancing, their naïve expectations of distribution.

RESEARCH QUESTIONS

The epistemological conjecture underlying the rationale of this study is that students' " $P(\text{HHHT}) > P(\text{HHHH})$ " assertion, although mathematically incorrect, points to cognitive resources that could potentially be incorporated effectively into correct mathematical reasoning. That is, albeit the assertion is *prima facie* discordant with normative knowledge, some deep-structure cognitive mechanism that is agentive in generating this assertion could constitute one "piece" of what may become a "knowledge in pieces" that *is* aligned with probability theory. What, then, would

⁴"*Top-down coherence*: Symbolic and verbal propositions are prominent in instruction. It is possible to view these as being learned prior to the broader coordinations in intuitive knowledge that are eventually required. . . . The subtleties and reliability of top-down coherence generation as a developmental principle are important to understand. Most schooling seems to count heavily on explicitly and literally rememberable elements. My working assumption is that this only works well within subsystems that already involve a sufficiently rich and reliable network. . . . I would like the theory sketch developed here to be capable of expressing the difficulties in top-down development" (diSessa, 1993, pp. 115–116).

the other pieces be? What experiences, such as guided interactions with learning tools, might support students in coordinating all these pieces into a cognitive structure supporting the targeted knowledge? Approaching this study, I thus asked:

- What pedagogical resources, including mathematical problem situations, activities, semiotic tools, and mediation strategies, might enable learners to build on their intuitive sense of likelihood en route to understanding how attention to the order of independent-event outcomes complements and enhances this intuition?
- More broadly, what roles do artifacts play in the course of students' guided coordination of intuitive resources and newly introduced mathematical procedures?
- To what extent are mathematical symbols, such as numerals and operations, indispensable prerequisite tools for learners to construct meaning for the target mathematical content of sample space? Can students reconcile intuitive and mathematical knowledge through using non-numerical tools, such as a sample space consisting of set of iconic representations of the random outcomes?

The Methods section introduces the design of this study and then reformulates the aforementioned research questions in terms of this specific design. The subsequent section presents the case study, which treats the focal empirical data examined in this article. I conclude by developing a new construct, *semiotic leap*, with which I attempt to capture the nature of what I believe to be a central mechanism at play in students' guided learning of a certain class of mathematical constructs.

Briefly, a semiotic leap is a student's insightful appropriation of a cultural artifact as a means of warranting an intuitive inference based on perceptual judgment of quantitative properties in a phenomenal embodiment of a targeted concept. The leap thus bridges from the tacit to the semiotic. The pedagogical challenge underlying this and other research studies in mathematics education can now be defined as the teacher's need to orchestrate the vying agents of conceptual understanding: on the one hand, a student comes in to the lesson with robust intuitive framings and associated inference-making mechanisms for privileged domains, yet on the other hand these inferences cannot *directly* support an appropriation of the mathematical analytic formulations of the phenomena in question. The case study reported in this article is an example of how a tutor and student achieved a critical milestone in connecting from the tacit to the semiotic in the mathematical disciplinary content of binomial distribution.

METHODS: IMPLEMENTING AN EXPERIMENTAL INSTRUCTIONAL DESIGN SO AS TO CREATE AN EMPIRICAL CONTEXT FOR INVESTIGATING RELATIONS BETWEEN TACIT KNOWLEDGE AND SEMIOTIC FORMULATION

This study is part of a larger research project, *Seeing Chance*, which explores middle-school students' intuitions of probability as well as the potential of a set of mixed-media mathematical objects designed so as to support the learning of probability through a process in which students build on their intuitions and are guided toward meaningful construction of the targeted

mathematical concepts.⁵ These dual foci of the project—describing learners’ intuitions and designing artifacts conjectured to breach the gap between those intuitions and the targeted mathematical ideas—suggested *design-based research* as a potentially effective approach for addressing the research questions (e.g., Collins, Joseph, & Bielaczyc, 2004; Confrey, 2005). Thus, over a developmental research spiral of design–implement–analyze cycles, we have been crafting objects and planning activities envisioned to enable students to tap and assert their intuitive sense of likelihood, reflect on the implications of these intuitive inferences, and explore the relations of these intuitive inferences to mathematical concepts and procedures. Simultaneously, we have been seeking *ontological innovation* (diSessa & Cobb, 2004)—hypothetical constructs that researchers formulate on the basis of data analyses and put forth so as to illuminate the nature of phenomena observed in teaching and learning situations. For example, the construct of semiotic leap introduced in this article is an ontological innovation that emerged from a design-based research study.

Design Rationale⁶

The topic of probability appeared to be an appropriate choice for a study of relations between intuition and content, due to a literature describing the would-be counterintuitive nature of probability, even among experts (Kahneman et al., 1982; Wilensky, 1997).⁷ Furthermore, the particular instructional design developed for this project turned out to be inadvertently auspicious for exploring the emergent research questions guiding the current study (see diSessa & Cobb, 2004, on empirical data from educational research as points of departure for new “post facto” studies). Namely, the activity sequence implemented in this study was such that, against the background of students’ actions and utterances, the juxtaposition of naïve intuition and mathematical procedures was brought out in stark relief, perhaps more so than in traditional activities that aim to create opportunities for students to build on their intuitions, as the following explanation of the design will demonstrate (see an elaboration in Abrahamson & White, 2008).

The randomness process created for this study was designed to trigger the same cognitive mechanisms that I hypothesized underlie students’ “ $P(\text{HHHT}) > P(\text{HHHH})$ ” belief, yet I sought to create a process that would empower students to avail of this belief in addressing a mathematical problem. I thus aimed for a situation sufficiently analogous to the four-coin situation, yet one in which ignoring the order of independent outcomes would not undermine an alignment with

⁵Initial results appear in Abrahamson and Cendak (2006), implications for learning theory appear in Abrahamson (2008a, 2009b), and implications for design-based research methodology appear in Abrahamson and Wilensky (2007), Abrahamson and White (2008), and Abrahamson (2009a).

⁶It would be outside the focus of this article to survey the wealth of prior constructivist design for probability learning. Here, the design is taken mainly as setting the context for data analysis. Also, the learning activities that are the focus of this study are only the first of several activities in the full sequence that also includes computer-based interactive modules, all part of the *ProbLab* experimental unit (Abrahamson, 2009a, Abrahamson, Janusz, & Wilensky, 2006; Abrahamson & Wilensky, 2002, 2004a, 2005).

⁷The difficulty that probability concepts present to learners, and therefore its aptness for a study of intuition and learning, is perhaps augured by the tumultuous history of this mathematical topic, which was fraught with resistances from unlikely quarters going far beyond lay people’s naïve beliefs and including religious authorities (e.g., Hacking, 1975).

mathematical theory. Specifically, I sought a situation and activity sequence in which students would initially assert that “ $P(3H, 1T) > P(4H)$ ”—a mathematically correct statement—and subsequently view the combinatorial-analysis procedure as providing a mathematical warrant for this assertion. In particular, I aimed for a design that would ultimately engender the insightful proposition that, “‘ $P(3H, 1T) > P(4H)$,’ because there are more ways of getting ‘3H, 1T’ than there are of getting ‘4H.’”

I anticipated that students would face challenges in coordinating and reconciling two legitimate views of the sample space: (a) an intuition-based view consisting of five possible objects, “4T”; “1H, 3T”; “2H, 2T”; “3H, 1T”; “4H” (the five aggregate events); and (b) a mathematical view that expands these five objects into 1, 4, 6, 4, and 1 object(s), respectively, for a total of 16 objects (the elemental events). I aimed for students to ground fundamental notions of classicist probability by bringing to bear these two views of the sample space, considering each as viable, and therefore attempting to reconcile them (see Abrahamson, 2006a; Abrahamson & Wilensky, 2007, for an explication of the framework underlying this type of design). I now introduce the actual materials and procedures we designed.

Learning Tools

The marbles box (see Figure 1) contains a mixture of hundreds of marbles of two colors, with equal numbers of each color (green and blue), and the *marbles scooper* is a utensil for drawing out of this box samples of exactly four marbles.⁸ The marbles box and scooper thus constitute a random generator of type “urn” or “bag,” a mathematical artifact that is widely used and/or referred to in probability literature, only that: (a) the marbles box is an open urn, so that the color ratios are exposed for perceptual inspection; and (b) the scooper’s structural properties impose constraints on the possible spatial configuration of the independent outcomes.⁹ I assumed that affixing the locations of the independent outcomes might create opportunities for a student and facilitator to co-attend to the phenomenal property of order as a potentially meaningful parameter

⁸In this study, there were equal numbers of marbles of each of the two colors. The rationale was to begin the instructional sequence with the case in which it is equally likely to draw a marble of each color ($p = .5$), so that all 16 compound events are equiprobable (see also Falk & Lann, 2008). This scaffold would subsequently be removed, at which point the equiprobability would need to be qualified as obtaining only *within* aggregate-event classes—not *between* them (Abrahamson, 2009a).

⁹The scooper’s spatial template is an *indigenous-structural* format for inscribing the outcomes, similar to the left-to-right order of independent events in Vegas-style fruit machines. In the case of coin flipping, however, one must apply an *exogenous-analytic* format. Note that the “HHTH” left-to-right linear textual inscription commonly used to designate discernable compound-event outcomes does not directly translate spatial/physical features of the actual coin-flipping experiment. Rather, the spatial configuration of the actual four flipped coins that have landed on a desk is a necessary, if prosaic, mechanical feature of this random generator that is irrelevant for our current probability analyses. Finally, note that, strictly speaking, the marbles-box experiment is only a hypergeometric approximation of the binomial. That is, a single scoop is in fact the result of four dependent without-replacement trials and so is not precisely commensurate with the concurrent flipping of four independent coins or a sequence of four coin flips. Thus, the true expected outcome distribution from the experiment has lower variance than the binomial distribution (e.g., it is even more difficult to draw a 4H scoop than in a truly with-replacement experiment). That said, the ratio of the scooper sample size (4) to the population (hundreds of marbles) renders this issue practically negligible for the purposes of simulating the binomial. Also, our computer modules implement truly binomial experiments. Henceforth in this article, I shall treat the marbles-box experiment as though it were truly binomial.



FIGURE 1 A 2-by-2 marbles scooper—a utensil for drawing out ordered samples—displaying a sample drawn out of a box full of marbles of two colors.

in the problem space, that is, as bearing on the question of anticipated outcome distribution. Moreover, the built-in 2-by-2 configuration could serve as a ready template for creating the set of “what we could get” (the sample space). As a means to further scaffold students in constructing the complete sample space of the marbles-scooping experiment, we provided the students with an ample stock of cards, each bearing an empty 2-by-2 matrix (see Figure 2) as well as two crayons (green and blue). Furthermore, we marked one edge of the matrix with a thicker line, as a means of cueing the idea that a rotation permutation might be interpreted as a discernable outcome. Yet, as we will see in the data, participants actually attended to this order only when they perceived that doing so served a goal, namely, either when the interviewer explicitly instructed a participant to attend to order or when the participant inferred that doing so might support the pragmatic objective of warranting an earlier assertion pertaining to the expected outcome distribution in the marbles-scooping experiment.

Participant and Procedure

Li, a sixth-grade student (aged 11.5), was one of 28 Grade 4–6 participants, selected from a pool of 46 volunteers, in a study conducted at a suburban school in the San Francisco East Bay area (Abrahamson & Cendak, 2006). Li, who was rated by his mathematics teachers as a high achiever, was loquacious, articulate, and argumentative. His generally engaged disposition throughout the interview as well as the contents of his observations were not atypical with respect to the majority of participants in this study, yet Li’s comfort and confidence in expressing his beliefs, even as these were shifting, were especially illuminating of the reasoning processes that possibly underlie most participants’ assertions and actions. Li’s disposition is therefore helpful for the current expository objective of presenting and explicating the participants’ assumed trajectories—their

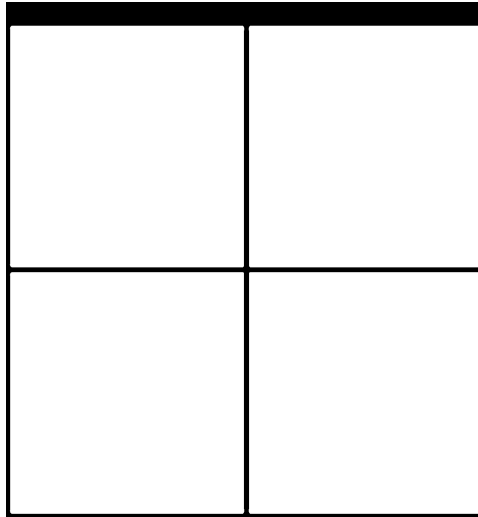


FIGURE 2 A card for constructing the sample space of the marbles-scooping experiment (a stack of these is provided).

“hypothetical genetic learning paths” (diSessa et al., 1991)—through the interaction (see also Simon & Tzur, 2004, on “hypothetical learning trajectories”). Crucially, the interview protocol by-and-large led all the participants to comparable milestones along their respective learning paths, albeit we witnessed natural variability in the particular order, duration, and frequency of the dyad’s occasions at each of these milestones. In any case, the objective of this article is not so much to offer empirically incontestable generalizations for what an educator might expect when implementing this imperfect design with Grade 4–6 students, but rather to richly describe one case study that we view as epitomizing a need, within the learning sciences, to persist in developing comprehensive theoretical models of mathematical learning that are geared to embrace a constructivist pedagogical program (for further descriptions of individual students’ experiences in this activity sequence, see Abrahamson, 2007a; Abrahamson, 2007b, 2009a; Abrahamson, Bryant, Howison, & Relaford-Doyle, 2008).

The study consisted of conducting a one-to-one semi-structured clinical interview, which implemented a prepared protocol that included activities and questions as well as follow-up questions, which had been anticipated through earlier pilot studies and were selected in real time as contingent on students’ responses (diSessa, 2007; Ginsburg, 1997). Li’s interview took place in a quiet room on the school premises, and the interviewer was the author. The duration of Li’s total interview, including the computer-based activities, was 55 minutes, but this study focuses only on the first 25 minutes that included work with the marbles box and the sample space. The protocol for these first two activities was as follows.

In the first activity, we present the marbles-scooping equipment, demonstrate its mechanism by letting the student scoop several times, and explain that scoops should contain exactly four marbles. We then ask the participant, “What do you think will happen when I scoop?” (or, “What might one get when one scoops?”). The question is intentionally ambiguous along at least three

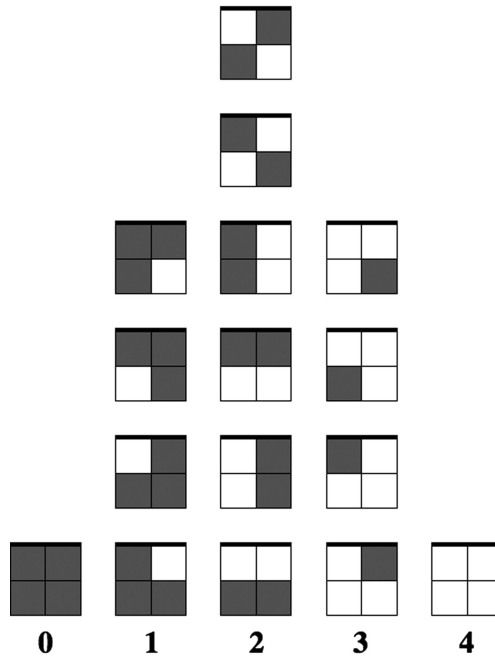
dimensions: (a) possible versus probable—a legitimate construction of the question is that it refers only to *what* we might get, not *how often* we will get it; (b) one sample versus numerous samples—we do not specify whether we would scoop only once or many times; and (c) combinations versus permutations—we do not specify whether we are attending only to the number of green and blue marbles in each scoop (the “five-objects view”) or to their spatial configuration as well (the “sixteen-objects view”). If participants interpret the question as referring to a single sample, we discuss the point with them, so as to elicit their reasoning, and then ask them to consider the alternative, many-scoop meaning (see also Konold, 1989). Once participants offer a response, we ask them to explain their reasoning.

In the second activity, we present participants the cards and crayons and ask them to color in “all the different scoops we could get.” If necessary, we clarify that we do not mean for them to conduct a real experiment in which they would use the cards to record actual scoops but rather to create the entire collection of scoops that one could possibly receive if one were indeed to conduct an experiment. If, while building the sample space, the participant creates any less than the total of 16 outcomes, we probe the participant to understand why and then prompt him/her to construct the entire sample space.

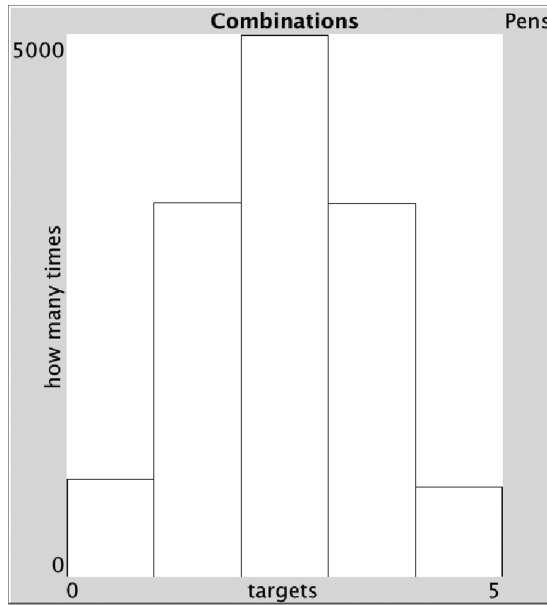
Finally, the interviewer guides the participant to assemble the sample space into a *combinations tower*, a histogram-shaped structure, here constituted of 16 discrete units set in five columns (see Figure 3a). Note the exterior contour of the combinations tower—it is the same as the contour of a 1:4:6:4:1 histogram (see Figure 3b). Indeed, outcome distributions in actual/simulated experiments with the marbles box are expected to converge toward the shape of the combinations tower, as more and more samples are drawn (Abrahamson, 2006b).

Note that we do not ever tell the students explicitly that the combinatorial-analysis procedure and, in particular, its product, the sample space assembled as the combinations tower, would possibly serve us in any way beyond showing us “what we can get.” That is, we do not explicitly communicate to participants any potential connections between the two activities—guessing an outcome distribution in a hypothetical experiment with a random generator and building its sample space.¹⁰ However, in an attempt to motivate students implicitly toward seeking connections between these activities, we manipulated the pragmatics of the dialogue (Grice, 1989; Schlegloff, 1996) as well as its socio-mathematical norms (Cobb, 2005; Cobb & Bauersfeld, 1995; Ernest, 1988), so that once participants have offered an intuitive inference they would feel inclined to seek a means of defending the inference. Also note that, due to the objectives of this particular study, we consciously “funnel” the students (Voigt, 1995) toward ultimately constructing the entire sample space and assembling it in the form of the combinations tower. At the end of the interview, however, we revisit and discuss with the participant all these “arbitrary” directives. As the data analysis will demonstrate, this somewhat unusual activity-sequence design created for the interviewer valuable opportunities to elicit and track milestones in students’ evolving reasoning (Abrahamson & White, 2008).

¹⁰The two activities can be further contrasted. In the first activity, participants are asked to cast a judgment about a specified property of an available object, but they are not told how to go about making this judgment. In the second activity, participants are guided to build and assemble an initially unavailable object, but they are not told why they are engaged in this activity or what they are to do with its product.



(a)



(b)

FIGURE 3 (a). The combinations tower—a sample space assembled in the form of a “bar chart.” (b). An actual outcome distribution in a computer-based simulated experiment of the marbles-box random generator.

Data Collection and Analysis

The Li interview is one of 28 that were all audio/videotaped for subsequent analysis. We worked in the traditions of collaborative microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1991; Siegler & Crowley, 1991) and grounded theory (Glaser & Strauss, 1967). Members of the research team familiarized themselves thoroughly with all 28 videotapes, and during meetings we each presented and discussed video excerpts and accompanying transcriptions that appeared to shed light on the emerging research questions, which we progressively honed. Observations that were generally vetted as pertinent to these research questions led to investigations for similar patterns in the data corpus; wherever we detected a pattern that we could not explain in terms of our knowledge of the literature, we considered the pattern temporarily as an ontological innovation and made it a focus of our subsequent analyses, readings, and discussions.¹¹

Thus, Li is but one of several students whose data we have been examining closely. In the following discussion of the Li case study, I will occasionally draw on results from analyses of the entire corpus of data.

An Activity Design Serving a Set of Emergent Research Questions

Having presented the materials and procedure used in the interview, I am now in a position to recontextualize and focus the research questions of this study, as follows:

- How will Li warrant his intuitive judgment of anticipated outcome frequencies?
- Specifically, will Li come to realize that the combinations tower can serve as a means of supporting his initial judgment? Namely, will he come to use the 16 elemental events to express intuited properties of the five aggregate events?
- If so, what personal, interpersonal, and technological resources might support Li in building this mathematical argument?
- What is the nature of any difficulties Li may encounter en route to connecting tacit and semiotic aspects of this argument?
- What is the nature and content of any further realizations that Li has yet to achieve so as to solidify his understanding of binomial distribution?

ANALYSIS OF A CASE STUDY: THE VICISSITUDES OF CONNECTING IMMEDIATE AND MEDIATED INFERENCES

This article reports on a study that investigated the nature and interactions of personal, material, and facilitation resources that enable students to productively engage their intuition in learning

¹¹The data analysis process has bootstrapped our research group into expanding our reading of cognitive science, sociocultural theory, cognitive- and cultural semiotics, pragmatics, philosophy, design-based research methodology, and critical theory. The caveat of making sense of the data impelled us to consider how these additional resources, which each introduced cogent perspectives, could possibly be integrated into the evolving model of guided tool-based learning. As one result of this collaborative process, we published on the potential efficacy of this theory integration (Abrahamson, 2008a; Abrahamson et al., 2008; Abrahamson & White, 2008). As a second result, I am developing a graduate seminar that uses the Seeing Chance data as a fulcrum for a survey of learning-sciences literature.

to perform mathematical procedures with understanding. In this section, I present an interpretive description of one case-study interview. The narrative density varies across the 25-minute episode analyzed, because I have chosen to zoom in on several moments that I view as pivotal for addressing the research questions.

An Initial Aggregate-Event-Based Intuitive Sense of Likelihood

I show Li the marbles box and ask him to guess what we will be doing with it. He says the box may be related to fractions, and specifically to the ratio of green marbles to blue marbles. He guesses that the ratio is about 60:40, respectively. I stir the contents of the box several times, so as to modify the observed distribution of color, and Li adjusts his guess to 50:50. I confirm that 50:50 is the correct ratio. I then introduce the scooper, and Li guesses that it is a utensil for holding marbles. I demonstrate the scooping action and, so doing, I happen to draw out a “1 green, 3 blue” sample (hence “1g, 3b” or just “1g3b,” i.e., an outcome or sample corresponding to compound events that are any of the four permutations on the combination “1 green and 3 blue marbles in any order”). Next, Li tries this action himself, and gets a 2g2b sample. The following conversation then ensues:

Dor: What might one get when one does that?

Li: Any . . . Up to four . . . What do you mean?

Dor: Yeah. Like that’s . . . kinda tell me more about that.

Li: Either four of one, none of the other; three of one and one of the other; two and two; one and three.

Dor: Do you have any . . . I donno . . . like, expectation as to what you might get?

Li: Knowing that there’s an equal amounts of marbles, the chance is probably that you’ll get two and two.

Li, similarly to the other 27 students we interviewed, predicts 2g2b as the most likely event. So doing, it appears, the students draw a population-to-sample inference.¹² Note, however, that Li never appears to attend to particular arrangements of marbles within the scooper—only to the numbers of green and blue marbles. Thus, in enumerating what one might get and stating what the chances are, Li is construing the experimental samples as aggregate-, not elemental events, albeit subsequent exchanges suggest that as yet Li is oblivious to this aggregate/elemental distinction. In passing, note also that Li has designated only four events (he omitted the “none and four” category). Let us look closer at the aforementioned exchange, because, as I explain later, it holds a key for understanding the remainder of this interview.

¹²Tversky and Kahneman’s “representativeness heuristic” was originally articulated to explain how people judge an outcome presented in the absence of the random generator that produced it. In our design, conversely, the participants are asked to guess an outcome in the presence of the random generator. Nevertheless, I submit that these two decision-making processes both involve object-to-sample inferences, because I assume that some mental imagery and simulation are involved in each of them (cf. Xu & Vashiti, 2008, in which both the population and the sample are displayed).

The Volatility of Tacit Knowledge Amid the Intrusion of Semiotic Media

When one first analyzes the aforementioned transcription, one might interpret the probes for *what one might get* and *what one expects to get* as alternative linguistic formulations of the interviewer's single intent that Li communicate his intuitive inferences with respect to properties of the random generator and specifically its expected outcomes. Indeed, from the perspective of mathematical theory, the interviewer's prompts might be regarded as commensurate, because knowing what we might get (the possible) anticipates knowing what we expect to get (the probable)—these knowings are of similar ontological status as phases in the analytic process. However, Li has neither psychological insight into the interviewer's intentions nor mathematical hindsight into probability theory—he is doing his best to respond sensibly to the prompts as they arrive. In so doing, I believe, Li draws on two very different types of resources. Namely, he applies: (a) analytical reasoning when he enumerates the possible outcomes sequentially, focusing on the structure of the scooper and the color types in the box; and (b) tacit judgment—a population-to-sample inference—when he selects 2g2b as the most likely event, focusing on the structure of the scooper vis-à-vis the color ratios in the box.

These analytic and tacit resources were both immediately available to Li, yet—as the subsequent transcriptions will demonstrate—Li cannot readily reconcile inferences emerging from these complementary resources. Namely, it is hardly obvious how an enumeration of all the possible aggregate events, which *prima facie* does not appear to privilege any particular one of them, might agree with a sense as to the most likely of these events. In fact, this brief transcription heralds the thematic challenge of reconciling inferences from these two ostensibly competing resources—a challenge Li is about to experience throughout the episode in question. In what follows, I will be interpreting this challenge as a pedagogically necessary perturbation to Li's tacit knowledge: by engaging in activities that utilize the cultural tools of combinatorial analysis, Li will soon experience cognitive conflict between inferences from “intuitive math” and “school math” (cf. Prediger, 2008). Li will initially resolve this conflict by revoking his intuitive inference, and only toward the end of the episode will the dyad negotiate a resolution by which Li can begin to regard his tacit and analytic resources as valuable and compatible. We now continue with a narrative of the interview.

I ask Li to clarify what he means by “chance,” and Li, after some thought, says that if we were to scoop all of the marbles out of the box and lay out these scoops before us, the “average” number of green and blue marbles in these samples would be 2g2b. In response, I pose for Li a variant on his hypothetical experiment—a *with*-replacement scenario, in which each four-marbles sample is returned to the box after it is drawn out—and Li says that the chances would be consistent at each scoop. Both of these responses are mathematically correct in and of themselves—if roughly worded—yet neither, at least when considered verbatim, appears to support Li's initial statement that “the chance is probably that you'll get two and two.” In fact, as the next excerpt demonstrates, even as he engages in producing what should constitute mathematical supports for his intuitive judgment that 2g2b has the greater chance, Li paradoxically loses faith in this very inference. I will interpret this apparent contradiction as the result of a reflexive process of distributed problem solving, in which Li's initially unfluent engagement of the available expressive media fails to sustain his intuitive inference and, instead, impels him toward an alternative conclusion that is incompatible with that intuitive inference. As such, Li's learning process is emblematic

of ontogenetic development in the social milieu, wherein individuals laboriously appropriate cultural tools that initially hamper their native instincts yet eventually enhance these capacities (see also Stavy & Strauss, 1982, on U-shaped growth).

I ask Li to explain his reasoning. Li offers an unsystematic combinatorial analysis, and then the following exchange ensues, in which he yet again miscounts the events:

Dor: So, when you say that “the chances that you’ll get two and two . . . ” What *are* the chances?

I mean, wh . . . wh . . . or . . .

Li: What their . . . I think there’re eight possibilities in total. No, there’re seven . . . possibilities.

Dor: Possibilities of what?

Li: Of the different colors you could get in this [scooper].

Dor: Ok.

Li: So, you scoop it, and you could have: no blues and four greens, one blue and three greens, two blues and two greens, three blues and one green, and four blues and no green. . . . And then you could have no greens and four blues, one green and three blues, and two greens and two blues, which is one of the . . . uhhm, and three greens and one blue, and four greens and no blues.

Subsequently, and although he had just enumerated 10 events (twice five, enumerated in opposite directions), Li concludes that there are seven possible events [sic]. Regardless of the miscount, Li then infers that the chance of getting 2g2b is one-in-seven. Thus, through attempting to support his initial intuition that 2g2b has the greater chance, Li arrives at a contradictory conclusion that the aggregate events are equiprobable. Clearly, Li does not see the property of internal order as pertinent to combinatorial analysis, and therefore he cannot construct an analytic counterpart to his intuitive inference. Still, the quick transition in Li’s expectation, from a varied distribution to a flat distribution, suggests that by engaging the process of generating the possible events, Li created new mental objects that, in turn, collectively offered him new inferences that were at odds with his initial inference. Li’s conclusion is reminiscent of LeCoutre (1992), who reported on cases in which participants claimed that by virtue of being random, all experimental outcomes are perforce equally likely (an *equiprobability bias*; see also Falk & Lann, 2008). However, Li never makes such claims explicit. In the next section, I will be offering an alternative explanation for the “loss of variance” in Li’s anticipated outcome distribution.

Li has been engaged in cognitively demanding analysis. Material media supporting him in this task were the marbles scooper as well as his fingers, but he was not supported by any media geared to enable a recording of his reasoning and thus a reviewing of his combinatorial analysis, and so he was prone to error, such as double counting 1g3b and 3b1g. We now skip slightly forward to a point in the combinatorial-analysis activity when into Li’s working environment are introduced the 2-by-2 cards—material objects that improve his organization of the analysis, resulting in a sample space of only five objects.

Implicit Imposition of Conceptual Categories: A Perennial Pedagogical Dilemma

I produce the cards and crayons, laying them on the desk, and ask Li to use these materials so as to show “what we could get when we scoop.” I point out that the 2-by-2 matrix in the cards has

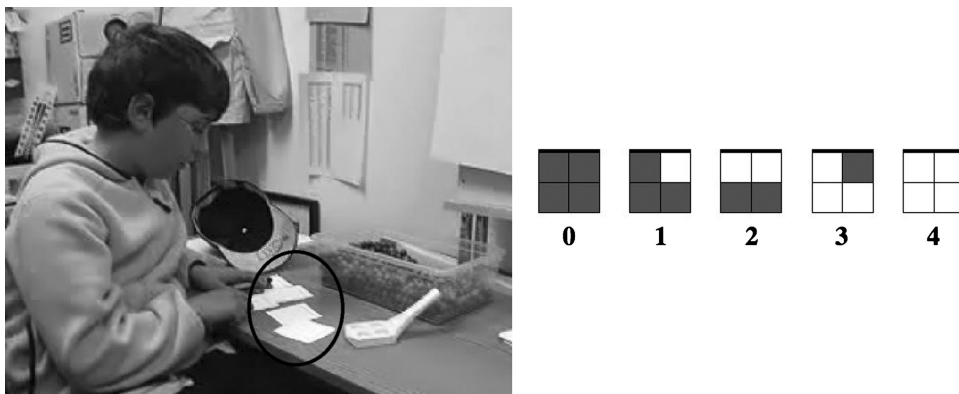


FIGURE 4 From left: Li completes his combinatorial analysis; reconstruction of Li's five-event sample space.

one thicker border that should be taken to indicate its upright orientation. Li takes a total of five cards from the stack, places them in front of him, and rapidly creates a sample space that includes cards with exactly 0, 1, 2, 3, and 4 green squares, respectively (the other squares are blue; see Figure 4). As he is completing this task, I re-indicate for Li the thicker border he apparently had not been attending to, and Li responds by rotating the cards such that they are each oriented with the thick line above. Li stares at the five cards for a few seconds and says, "I think that's it. Yeah" (see Figure 4).

The Reinvention of Permutations. Meaning to probe for whether Li indeed sees the collection of five cards he has produced as exhausting the possible outcomes, I initiate the following exchange, in which the property of order ("placement") first emerges in our conversation (cf. "colors") but is judged by Li as irrelevant to the task, as he perceives it. Concurrently, the objective of the combinatorial-analysis task is first enunciated as constituting an inquiry into the anticipated frequencies.

Dor: So, are you saying that if I . . . If I . . . If I took, uh-hm, samples now, like, will . . .

Li: The chance that you got two-and-two would be one-out-of-five.

Dor: Oh I see, but . . . Whatever I get, am I bound to get one of these [five] now, when I scoop?

Li: Yeah, I think so. Oh! With the *placement*, here [points to a card], no!

Dor: So what do you mean by "placement?"

Li: Like, *this* one [points to a green square in a 2-by-2 matrix] might be here [points to a different location in the same matrix], and then *that* one [pointing to the color currently at that location] would be there [first location].

Dor: Ok, do you want to count that as different?

Li: [in a *very* reluctant voice that American readers might recognize as the "whatever" tone] Uhhh . . . *Ok* . . . Should I write all of those combinations?

Dor: Yeah./4 seconds/Do you think . . . Do you think it's necessary to do that? Do you think it's relevant to this . . . issue?

- Li: Well, it depends on . . . what you want to find out.
- Dor: Well, what we wanted to find out was, uhhh—I'm just going back to the thing *you* said. . . —What are the *chances* of getting two green two blue.
- Li: Well, then it's *not* relevant.
- Dor: It's *not* relevant?! Ok . . . Why?
- Li: Because, if you get . . . [continues coloring in an additional 2g2b card, so that now there are 5+1 cards on the desk] Well, you can get two green and two blue in a couple different ways—like *that* [lifts the two cards that are different permutations on 2g2b], but it's still “two-green–two-blue.”

The interviewer misinterprets Li, as though by “it's still two-green-two-blue” he was referring to a consistency in his *expectation* of 2g2b, whereas Li actually meant to convey that the two cards show the same *combination*. (For the interviewer, “it” stood for the most likely event, whereas for Li, “it” stood for the count of green and blue squares in the card.) The conversation thus turns to issues of chance, even though Li had not really been entertaining thoughts about chance.

- Dor: Ok. So you found . . . But, before, you said there's a one-in- . . . five chance of getting two green two blue, or something like that? Or, what? . . .
- Li: Yeah.
- Dor: So where's *that* from, that “one in five”? Because you said. . .
- Li: Well [moves aside the additional 2g2b card he had just created and lays his hands only on the five cards he had initially created], these are all the possibilities you can get, not counting where the marbles are, but. . .
- Dor: Ok, so based on *them*. . .
- Li: Yes
- Dor: you're saying there's a one-in-five chance. So . . . if I scoop, then . . . then, kind of . . . over lots and lots of scoops, about a fifth of the time I'll get . . . I'll get two-green–two-blue?
- Li: Yeah.
- Dor: Ok. We'll get back to that. But now, let's include the placements, alright?
- Li: Ok . . . So you want me to write all the rest of them out?
- Dor: Yeah, if you could please.
- Li: Yeah!

Li complies with my request that he expand the sample space so as to include permutations on the five combinations. Yet this compliance by no means indicates a change in Li's reasoning about the situation. Rather, having abandoned the initial intuitive sense that “the chance is probably that you'll get two and two” in favor of the analytically established sense that, “The chance that you got two-and-two would be one-out-of-five,” Li appears to interpret my request to expand the sample space as a prompt to perform a procedure that probably would not bear on the substance of the situation as he now sees it. Li is not alone in his sentiment—not one of the 28 participants in this study, at this particular phase in the interview protocol, appeared to perceive an attention to order as instrumental to warranting their initial sense of distribution. Many participants asked us flatly whether or not they should create the permutations and, when asked, in response, for their own thoughts on this issue, some participants replied with variants on, “*You're the teacher—you decide!*” By and large, even as they were expanding the sample space, participants still did

not perceive any practical utility in these additional objects, which they took to be redundant duplicates of the original set.

Ontological Imperialism: The Case of Imposing an Ordered Semiotic Tool on an Unordered View of the Sample Space. With his “one-in-five” estimation, Li is now treating as equally likely events that are not so. I have commented earlier that previous research on combinatorial reasoning has reported on a similar phenomenon, the equiprobability bias (see in Jones et al., 2007, p. 917). However, previous research has not tied this equiprobability bias to tensions between students’ intuitions and the semiotic tools made available for the students to express their intuitions. That is, I contend that Li’s initial, mathematically correct anticipation of a plurality of 2g2b, which he had determined on the basis of a perceptual judgment of the marbles population, evaporated the moment he was asked to enumerate the possible outcomes, as I now explain.

The school setting of the interview location, the disciplinary frame of the mathematical content, the pragmatics of the dialogue, and in particular the interviewer’s allusion to the sample space immediately after challenging the participant’s “2g2b” assertion, all gradually teamed to implicitly suggest to Li that he should determine all possible outcomes of the random generator if he is to support his sense of distribution (on the unuttered yet critical aspects of dialogue, see Grice, 1989; Schegloff, 1996). Thus, Li enumerates the possible outcomes—first only verbally/gesturally and then with the aid of the cards. However, Li does not yet know how to use combinatorial analysis as a *semiotic means of objectifying* (Radford, 2003) his presymbolic notion of likelihood—namely, Li does not yet know that the relative likelihoods of events are proportional to the number of unique instances of each event, and that one should therefore determine, tally, and compare all of these permutations, if one is to instantiate in mathematical inscription one’s intuitive sense of distribution. So, Li does not inscribe his initial sense of likelihood into the five cards, because he has no ready strategy for encoding that sense into the objects put at his disposal. Moreover, Li is not aware that he has lost information due to this incomplete passage from intuition to inscription. Thus, having created the five cards, Li construes them as though they encode his sense of distribution—even though that sense of distribution has in fact slipped away, unobjectified. The resulting absence of any inscribed trace of Li’s fragile sense of likelihood condemns that sense to perdition: When Li “reads” the sample space he has only just “written,” the cards bear no mnemonics to re-evoked and simulate the felt sense of likelihood. All Li sees is a set of perceptually similar cards (e.g., the 2g2b card does not appear privileged as compared to the 3g1b card). Thus, through the distorted reflection of a semiotic system, a variable distribution is flattened.

We seem to be facing a pedagogical dilemma that may well obtain beyond the particulars of this content and setting: Give students orderless semiotic means, for example, five eggcups that could each contain an orderless set of four marbles, and you cannot highlight for them the mathematically critical property of order; yet give students ordered semiotic means, such as these 2-by-2 cards, and the students lose the evanescent notion of likelihood. Indeed, Bamberger and diSessa (2003) alert us to the potential woes of *ontological imperialism*: by introducing to learners a semiotic system associated with a particular discipline, such as musical notation, we covertly impose upon the learners normative perceptual categories that do not necessarily carve the world at the same joints as the learners would, with potentially dire consequences for their construction

of personal connections toward the targeted content. The participants in the current study were temporary victims of this ontological imperialism—en route to construing a sample space as objectifying their pre-formulated sense of likelihoods, the participants are ill-equipped to use the provided media so as to express the likelihoods they had initially sensed and, then, looking back through the aperture of the expressive tools coerced upon them, the participants no longer see those likelihoods. However, as the continuing narrative of the interview will demonstrate, the participants' fleeting sense of the probable can be re-evoked by alluding once again to the random generator that had initially triggered that sense.

As we shall soon see, ontological imperialism may also surreptitiously create a chain of communication breakdowns between the “imperialists” and the “subjects.” Namely, when a student refers to a specific card as, e.g., 2g2b and says that it will occur most often, the student is seeing the card as meaning the 2g2b event. However, the interviewer, who *does* attend to the spatial organization of this card, construes it as meaning a particular permutation, for example, the 2g2b card with two green squares in the top row, and, consequently, the interviewer implicitly interprets the student's utterance, too, as referring to a permutation, not a combination. Moreover, the interviewer might assume that the student is using this permutation as *metonymically* referring to its entire 2g2b event class (Abrahamson, 2008a; Abrahamson et al., 2008). And yet, the student's repudiation of the other permutations as irrelevant to the analysis suggests otherwise. To wit, the student indeed *sees* the card as an orderless combination, just as—I contend—participants in the Tversky and Kahneman studies *saw* “HTHT” as 2H2T (in any order). That is, participants' biases associated with the “representative heuristic” are the result of non-normative categorization, not of non-normative inference.

The Tumultuous Alignment of Intuition and Inscription

Over the following 2.5 minutes, Li silently completes his construction of the sample space. The cards now lie on the desk in five groups, sorted by combination. Over another 3 minutes, Li and I discuss combinatorial-analysis strategies, and I guide him to create two additional cards that were missing from the 2g2b group (once he had found four of those, Li explains, he had assumed there could not be more—just as there could only be four cards in the 1g3b or 1b3g groups—so he stopped searching). The full sample space is now complete. I ask Li whether, having gone through this procedure, he has any new thoughts pertaining to his recent assertion that there is a one-in-five chance of getting 2g2b. Li responds in the negative—he still thinks that the chance of getting 2g2b is one-in-five.

The Reinvention of Classicist Probability. I guide Li to assemble the 16 cards into the combinations-tower form. When the tower is built, I ask Li whether he wishes to share any observations on this structure. Li comments on the step-like appearance of the five columns, albeit he initially miscounts the size of the steps (he sees them as steps of two). I ask Li how many cards there are in total, he says there are 15, I query his statement, and he re-counts the cards as 16 (he explains he had counted only 5 cards in the central column). It thus appears that Li has not been paying much attention to the particular distribution of the 16 cards in the sample space. However, in the following conversation, which lasts just over 3 minutes, Li will come to assert that the

entire collection of cards should be considered in determining the expected outcome distribution in hypothetical experiments with the marbles box. As the transcription that follows will make evident, though, Li is in transition: his discourse includes several lengthy periods of silence, his utterances are often hesitant, his speech and gestures are mismatched on several occasions (see Church & Goldin-Meadow, 1986), and he switches three times between interpretations of the sample space (marked in the transcription by underlined characters). Essentially, Li will relinquish the one-in-five expectation for the frequency of the 2g2b event in favor of re-acknowledging its expected plurality in the outcome distribution, and this re-acknowledgment will be quantitative (“six-out-of-sixteen”) and not only qualitative (“the chance is probably that you’ll get two and two”) as it had initially been. Following the transcription of this culminating episode, I will offer several complementary interpretations for how cognitive, technological, and interpersonal factors played into Li’s insight. (See <http://edrl.berkeley.edu/publications/journals/C&I/Abrahamson-C&I-Binomial-Li.mov> for a video clip of these final 3.5 minutes of the interview.)

Dor: And, again I’m . . . at the risk of boring you . . . The same question. You see this [gestures toward the entire sample space], and you say the chance of getting, uh, two . . . the chance of getting, uh, a . . . scooping something with two-green is one-out-of-five.

Li: /5 sec/Well, actually . . . /3 sec/yeah [one-out-of-five]!

Dor: Ok.

Li: /2 sec/Actually, /7 sec/it kinda seems like it would be six-out-of-sixteen.

Dor: Huh! Ok, so what . . . so . . . ‘One-out-of-five’ now went to ‘six-out-of-sixteen.’ Whathow . . .

Li: Well, it’s like . . .

Dor: That’s quite a difference!

Li: Yeah. It . . . /10 sec/Well, there are sixteen . . . /4 sec/Well, actually . . . /10 sec/No, it’s still—I think it still would be one-out-of-five.

Dor: Mm’hmm. So if I scoop, about a fifth of the time I’ll get a . . . something with two.

Li: Two of each.

Dor: Ok. So . . .

Li: ‘Cause like, these [indicates all the eleven cards above the bottom row of five cards] don’t really matter. [see Figures 5a and 5b]

Dor: In what sense?

Li: Well, if you’re looking to . . . /4 sec/Well, if the placement mattered [gestures back and forth between the scooper and the 11 cards], these *would* matter, but these [11 cards] are all the same thing. These [within the 1g3b column, indicates the three cards above the bottom card] are the same thing as this [points to the bottom card] except for the placement [repeats gesture pattern for the 2g2b column and then the 3g1b column]. So it’s these same original five [indicates bottom row] or, like, any one of these [indicates that any of the cards above the bottom row could replace its respective bottom card] . . . /3 sec/that matters.

Dor: /5 sec/So, I mean, this issue of “placement,” that seems to be what the . . . It’s not just, like, you and me deciding, “Let’s use placement” or “Let’s not use placement.” It’s, like, how does that relate to the situation in the world, like, the scooping?—Should we care about placement or not? And it seems like you’re saying . . . “not.”

Li: Uhhm, yeah.



(a)

FIGURE 5 (a). Li sees the permutations as irrelevant to the task: “These don’t really matter” (gesture indicated by curved-arrow overlays). (b). A schematic representation of Li’s apparent perceptual construction of the combinations tower.

Dor: Ok. So your prediction is that if we scooped, say . . . I donno, 100 times, we’ll get about 20 of these, 20 of these, 20 of these, 20 of these, 20 of these? [each of the “20 of these” utterances is accompanied by a gesture, pen in hand, toward a column in the combinations tower, beginning with the right-most, 4g column, and moving to the left (see Figure 6)].

Li: /5 sec/Actually, no. I would . . . I’m going back to . . . there’s, out of all the possibilities you could get, six-out-of-sixteen are two-and-two, and these [indicates the 0g and the 4g cards] are only one-out-of-sixteen, so . . . Like, what I was saying—“one-out-of-five chance”—that would mean . . . /6 sec/‘Cause, [vehemently] you’ll get these [hand sweeps up and down the 2g2b column] more than these [holds up the single 4g card], ‘cause there’s six of these and there’s only one of these. So *that* [his own earlier “one-out-of-five chance” statement] would mean that you would get about 20 percent of . . . Uhh, you would

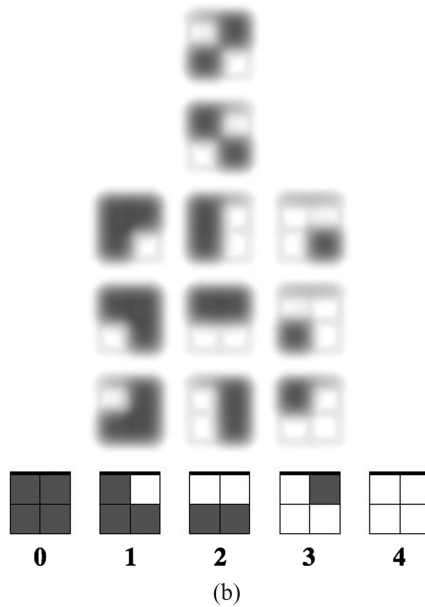


FIGURE 5 (Continued)

get 20 percent of the four-greens and four-blues . . . But now I'm realizing that's not true, because [indicates the vertical extension of the 2g2b column]. . .

Dor: Oh, I see, so you kind of . . . You built up this thing and suddenly you had a contradiction . . . where you say, on the one hand this [4g] is one-out-of-sixteen, but according to the "one-out-of-five," it should be one-out-of-fi . . . , like, 20 percent? And so, someth. . .

Li: Yeah, 'cause they're all . . . there's six possibilities of these [sweeps hand along the 2g2b column] and one of these [holding the 0g and 4g cards], and the chance of getting each one of these [indicates in the general direction of the entire sample space] is one-out-of-sixteen. So . . . but then, with *these* six [showing 2g2b column] it's *six*-out-of-sixteen, and these [holds up 0g card] are still only *one*-out-of-sixteen, and these [points to the 1g3b and 3g1b columns] are *four*-out-of-sixteen.

Dor: Uuhhh. So . . . so what made you change your mind, from one-out-of-five to six-out-of-sixteen?

Li: Well, when you said that there would be 20 percent, 20 percent, 20 percent, 20 percent, 20 . . . Then I realized that . . . that would be wrong.

Li's new realization is embryonic. My claim is certainly not that Li has achieved a stable understanding of the material at this point or that he has mastered the key calculations involved. Rather, I shall now focus on the aforementioned exchange, because, with Li's abrupt turns of mind, it helps reconstruct the microgenesis of his conceptual development.



20 of these...



20 of these...



20 of these...



And 20 of these.

FIGURE 6 “20 of these, 20 of these . . .”: Using gesture so as to link his speech utterances with particular stimuli in Li’s visual field, the interviewer orients Li’s view toward the columns of the combinations tower, thus cueing a particular visual construction of the tower that highlights the columns’ *variable* vertical extension. Yet, simultaneously, the interviewer contradicts the gestures by coding these vertical extensions verbally as indexing *equivalent* frequencies. Li soon responds to this rhetorical speech–gesture mismatch.

Order Matters: Intuition Reconciled With Inscription. Li’s initial deliberations in the aforementioned transcription might be summarized and heavily paraphrased as follows:

When I incorporate the entire sample space into my perceptual attention and construct the space as a set of vertically extending columns, I sense 2g2b as the most likely event. And yet I have no logical grounds to consider all these permutations as pertinent to the process—the eleven cards above the bottom row appear to be redundant duplicates of the actual five things I can get—and so I switch back to the idea that only five objects matter. But these five objects bear no sign that they are anything but equally likely, and so I reason that they each have a one-in-five chance of occurring.

Thus, guided by the interviewer’s prompts, Li initially vacillates between two vying mental constructions of the combinations tower but fails to achieve coherence: (a) a global view of the entire sample space is intuitively grounded in perceptual judgments of color ratios in the marbles box, yet a local view of the tower’s constitutive particulate elements brings into question the logical necessity of including the permutations; and (b) a view of the five cards in the bottom row appears to capture “what we can get,” yet the flat distribution is not intuitively grounded. The

interviewer dissolves this deadlock by juxtaposing perceptual and declarative aspects *across* these two views. Namely, the interviewer gesturally draws Li's visual attention to the vertical variability of the distribution whilst verbally constructing a flat distribution—"20 of *these*, 20 of *these*, . . .", and Li apparently finds this juxtaposition jarring enough so as to abandon the one-in-five view that incorporates five elemental events taken as aggregate events in favor of a six-out-of-sixteen view that incorporates five aggregations (sets) of elemental events. These five aggregations, which had been perceived as mostly superfluous icons, shifted in their semiotic status as their collective spatial property of height first took on the contextually pertinent signification of the sets' relative frequencies.

Finally, note a grammatical distinction, in the aforementioned transcription, between the first and second instances when Li articulates the viability of the 6/16 construction of the chance of the 2g2b event. In the first instance, Li says "it would be six-out-of-sixteen," where the singular pronoun "it" stands in for the noun phrase "the chance of scooping a 2g2b sample," a version of which appears in the interviewer's earlier interrogative statement. Critically, the five permutations above the bottom 2g2b card are not mentioned as integral elements of this 2g2b aggregate event but only as a property of the column associated with the event's expected frequency. In the second instance, however, Li states that "you'll get these [2g2b cards] more than these [4g card], 'cause there's six of these and there's only one of these." Here, the plural pronoun "these"—the linguistic patient of random selection encapsulated in the verb "get"—marks that Li has adjusted his ontology of the marbles-box experiment from a collection of five events *per se* to a collection of five event *classes* (aggregates, sets) that index his sense of distribution.¹³ Thus, Li's initial experience of judging likelihoods in the marbles box was "lived further in a new creation of meaning that takes account of, and also shifts, all the earlier meanings" (Gendlin, 1982, p. 166).¹⁴

After-Math: Overview of Li's Learning Trajectory in and Beyond the Interview

Over the course of the interview, Li shifted back and forth several times between two constructions of the artifacts—the marbles-box random generator and its combinations-tower distributed sample space—and each of these views, in turn, underlay the inference of either correct or incorrect mathematical propositions that Li asserted (see Table 1): (1) an aggregate-event-based view framed by the initial intuitive inference for the outcome distribution in the marbles-box

¹³It is this sense of an event class that underlies mathematical phrases such as "the chance of getting *any* of these" or "the chance of getting two green and two blue in *any* order"—phrases that may be challenging due to the ostensible assignment of a single chance value to a collection of objects (whose respective chances add up to the event's chance).

¹⁴Was Li's "6/16" mathematical proposition indeed grounded in his initial "2g2b" intuitive inference? That is, what role, if any, does the initial interaction with the marbles box play in participants' learning path? Could the initial interaction in fact be a superfluous activity such that we could cut to the chase and begin the design directly with the combinatorial analysis? To examine this question empirically, we ran an experiment in which 23 middle-school students were randomly assigned to three conditions that framed the nature of their initial interactions with the marbles box: (a) "leading question," in which essentially the current protocol was enacted; (b) "no question," in which participants were shown the marbles box but no additional problem or context were invoked; and (c) "distracter question," in which participants were asked to estimate how many without-replacement scoops are required to empty the box. We found that leading-question participants were more likely to question the necessity of permutations in the sample space as well as eventually interpret the sample space as meaning that 2g2b would be the most likely event (Mauks-Koepke, 2008; Mauks-Koepke, Buchanan, Relaford-Doyle, Sushkova, & Abrahamson, 2009).

TABLE 1
Mathematically Correct and Incorrect Propositions Associated With Aggregate-Event- and Elemental-Event-Based Views of the Artifacts, Listed by Their Order of Assertion in Li's Interview

| Orientation of View Toward the Sample Space | Mathematical Status of Proposition | |
|--|--|---|
| | Correct | Incorrect |
| Aggregate-Event-Based | A. Qualitative description of relative likelihood ("The chance is probably that you'll get 2 and 2") | B. Equiprobable bias ("The chance that you got 2-and-2 would be 1-out-of-5") |
| Elemental-Event-Based | D. Quantitative articulation of relative likelihoods ("It would be 6-out-of-16") | C. Permutations as redundant duplicates ("These [permutations] don't matter") |

hypothetical experiment—a view that was mathematically correct (Assertion A, Table 1) yet was initially objectified in the construction of only five discrete objects in the sample space (Assertion B); and (2) an elemental-event-based view that emerged only through the construction of the sample space and its assembly in the form of the combinations tower—initially Li was in a state of transition (Assertion C), but then he recognized that the expanded sample space could be instrumental in warranting his re-evoked view of the marbles box (Assertion D). In Table 1, the microgenetic arch from Assertion A (correct) through to Assertions B (incorrect), C (incorrect), and D (correct) is reminiscent of the U-shaped multi-year developmental trajectories discussed in Stavy and Strauss (1982). Also, whereas Assertion A was directed toward the immediate situation (the marbles box) and Assertions B and C were directed toward different constructions of the mediated situation (the cards), Assertion D synthesized the immediate and mediated situations (the cards, taken as semiotic means of objectifying the marbles box).

Clearly, Li's understanding of binomial distribution is at best embryonic and still requires much discursive interaction, such as in this clinical fashion, so as to achieve stability. Namely, I have characterized Li's move from Assertion A to Assertion D (see Table 1) as a form of *guided reinvention* (Freudenthal, 1986; Gravemeijer, 1994), such that Li is still to substantiate this discovery with further analyses of the sample space's internal properties and their implications for phenomenal aspects of the random generator. Indeed, data from the interviews with the other sixth-grade participants strongly suggests the initial instability of their Assertion D, as implicated in their great difficulty in explicitly reconciling ostensibly mutually exclusive inferences drawn from elemental-event- and aggregate-event-based views of individual cards (Abrahamson et al., 2008; Abrahamson & Cendak, 2006).

In the remainder of the Li interview, where he worked with computer-based simulations of the marbles-box experiment, Li learned to interpret actual empirical outcome distributions, which he saw within five dynamically incrementing histogram columns, as randomized proportional "stretches" of the combinations tower (see Abrahamson, 2006b; Abrahamson & Cendak, 2006). Yet Li has still to develop his current understanding so as to accommodate cases of experiments with binomial random generators in which the probabilities are other than .5 (see Abrahamson, 2009a, for activities designed to support this extension). Finally, over the course of his mathematical development, Li should learn to conduct these forms of reasoning using alternative approaches to combinatorial analysis, such as tree diagrams, as well as a range of semiotic tools that ultimately include symbol-based algebraic strategies such as the binomial function.

Li was one of 27-out-of-28 participants who offered mathematically correct insights into the relation between the combinations tower and the expected outcome distribution in the marbles-box experiment, thus grounding the normative view of the sample space in their intuitions of likelihood. The *prima facie* surprising nature of the participants' insight as well as its pivotal role in their conceptual development toward a desirable learning outcome have made this insight a focus of our investigation into the interface of tacit and mathematical reasoning, as I now elaborate.

DISCUSSION: GUIDED MATHEMATICAL INSIGHT AT THE INTERSECTION OF COGNITIVE-SCIENCE AND SOCIOCULTURAL THEORY—A PANOPLY OF COMPLEMENTARY INTERPRETIVE PERSPECTIVES

Several cognitive mechanisms and interpersonal strategies may be implicated as factoring into Li's adoption of the normative view of the sample space. The following list supplements earlier discussions of the work of diSessa, Gelman, and their respective collaborators so as to demonstrate the potential utility of integrating cognitive-science and sociocultural theory into a comprehensive model of mathematical learning (a "dialectic" view, see diSessa, 2008b). Namely, I shall demonstrate that each of the perspectives, below, illuminates the data, yet neither of them on its own furnishes a viable explanation for how Li grounded the mathematical view of the sample space in his tacit judgment pertaining to quantitative properties of the marbles-box hypothetical experiment. Following the survey, I shall therefore return to the construct of semiotic leap, which I introduced earlier in this manuscript, so as to evaluate whether this proposed construct coherently addresses the questions that rose from the analysis of Li's behavior in light of the apparent shortcomings of the theoretical models surveyed in what follows.

Note that by pointing to the inadequacies of the surveyed frameworks to furnish complete explanations for Li's behavior, I am by no means critiquing these frameworks, because they were not necessarily conceptualized to address research problems relevant to phenomena of this particular nature. Rather, I have selected major perspectives that each provide a piece in what I believe is a larger puzzle.

Intuitive Rules

Stavy and Tirosh (1996) argue that students use a set of simple, at times fallible, heuristics to draw inferences under conditions of limited information (cf. Van Dooren, De Bock, Weyers, & Verschaffel, 2004). Notably the "More A—More B" intuitive rule guides students to compare two objects along some accessible dimension as a means of inferring their relation with respect to some other, inaccessible dimension, regardless of whether or how these two dimensions are truly related. Thus, in comparing the 2g2b and 4g events, Li sees that $2g2b > 4g$ along the accessible dimension of "number of permutations" (or "height") in their respective combinations-tower columns ($6 > 1$), and so he opportunistically infers that $2g2b > 4g$ along the inaccessible target

dimension “frequency of occurrence.” This interpretation, however, leaves out Li’s motivation and tacit criterion for evaluating the veracity of his inference.

A Sample Space as Second-Order Random Generator

A sample space with equiprobable elemental events, such as in the case of the marbles-scooping experiment used in this study ($p = .5$), can be perceived as constituting an open “urn” onto itself. That is, selecting randomly from the 16 cards lying on the desktop is commensurate with sampling from the marbles box, for example, both activities have a $6/16$ chance of drawing a 2g2b combination. Moreover, the material, iconic, and discrete make up of the sample space strongly suggest this affordance as a sampling device (see T. Martin & Schwartz, 2005, for another case where new affordances emerged for semiotic artifacts taken as objects). Finally, note in the opening of the aforementioned culminating transcription that the interviewer explicitly refers to the combinations tower when asking Li for his expectation of outcomes in the marbles-box experiment. Whereas the interviewer sees the combinations tower as modeling conceptually relevant properties of the source population of marbles (i.e., as a semiotic artifact), Li has not yet constructed this relation normatively and therefore he interprets the interviewer’s gesture toward the combinations tower as going no further than that set of 16 cards (non-semiotic objects per se). Thus, when Li says, “[Y]ou’ll get these [2g2b] more than these [4g], ‘cause there’s six of these and there’s only one of these,” he may be inferring on the basis of a perceptual judgment of distribution broadly similar to his strategy for the marbles box.

Whereas this suggested view of the sample space as a *sampling* space is logically appealing, still it does not explain how Li is tying between the source domain (the marbles box) and its iconic model (the combinations tower). Namely, we are left with the question of whether and how Li construes inferences drawn from the combinations tower as bearing on properties of the marbles box.

Conceptual Blending

Fauconnier and Turner (2002) explain human problem solving as based on conjuring into the mental problem space imagistic cognitive material and then manipulating these images so that they become aligned and merged, as a means of arriving at a coherent imagistic solution (where images are multimodal and not only visual, cf., Goldin, 1987). Hutchins (2005) points to the roles of material objects in anchoring these conceptual blends. I submit that Li blended into the columns of the combinations tower—which constituted the material semiotic tools made available for his mathematical argumentation—his preformulated image of relative likelihoods. This blending was facilitated by Li’s guided attention toward the five columns’ variable vertical thrusts (1, 4, 6, 4, 1), a construal of the sample space that he could then align with his felt sense of greater and lesser chances associated with the five possible events. Subsequent to establishing the blend, Li runs the blend by treating each column as a collection of discrete objects on which he applies enumeration, comparison, and calculation. This explanation still leaves open the question of Li’s initial motivation for blending his tacit inference into the available semiotic medium.

Semiotic Means of Objectification

Radford (2003; 2006a; 2006b), building on Soviet Activity Theory, yet expanding it, argues for a broader cultural-semiotics conceptualization of the phenomenon of mathematical reasoning as a *praxis cogitans*. I borrow from this framework the powerful idea that an individual's mathematical learning involves appropriation of cultural artifacts, which are embedded in a system of practice, as a means of objectifying personal presymbolic notions regarding quantities, relations, and operations—a “linguistically impossible-to-articulate mode of reflecting which, in its turn can but *reveal itself* through action” (2006a, p. 13, original italics). Through these acts of objectification, Radford argues, the learner builds meaning for the personal notions as well as for the artifacts (for related work, see Bartolini Bussi & Boni, 2003; Bartolini Bussi & Mariotti, 1999; Hutchins & Palen, 1997; Lemke, 1998, 2002; Radford, 2008).

Ever since Li was asked to evaluate the outcome distribution of the marbles-box experiment—and moreover, once his evaluation had been challenged—he had been groping for a semiotic means of objectifying his intuitive sense of distribution. The sample space in its totality of 16 equiprobable outcomes did not initially appear to directly *articulate* his preformulated intuition of likelihood, which had been tacitly couched in a 5- and not 16-event construal of the experiment, but the combinations tower did ultimately *substantiate* this intuition as a discursive tender and, reciprocally, lent meaning to the mathematical artifacts (i.e., to the sample space “product” as well as, retroactively, the combinatorial-analysis procedure-tool).

Radford's framework, with its discourse-based view on personal meaning making, helps situate Li's cognitive operations within the broader material and interpersonal contexts of the interview. However, a purely cultural-semiotics perspective cannot treat innate ontogenetic perceptual capacities, such as those evident in population-to-sample inferences, because these are presumably available prior to an infant's immediate immersion in the acculturation process (Xu & Vashti, 2008).¹⁵

Critical Features of Ambiguous Objects

Li's deliberations are grounded in an alternating view of the combinations tower as either, on the one hand, a collection of five objects (the bottom row) with 11 additional objects above them, or, on the other hand, five integral columns. The combinations tower is thus an ambiguous figure whose meaning is contingent on its perceptual construction (its gestalt). Tsai and Kolbet (1985) found that interpretations of ambiguous figures can be manipulated by drawing participants' visual attention to particular regions in the figures that inhabit critical features for disambiguating the figures one way or the other. This, I believe, is precisely what the interviewer does, if implicitly, when, using language and gesture, he draws Li's attention to particular parts of the combinations tower or frames Li's viewing of the tower. As in the cognitive-psychology experiments, where participants swayed between “duck” and “rabbit” sightings of the classical Jastrow drawing, so Li

¹⁵Evolutionary sociocultural theorists might argue in return that, on a phylogenetic scale, innate capacities can nevertheless be attributed to the survival of gene carriers whose chance innate capacities were best adapted to socially emergent needs and hence were naturally selected and further honed over subsequent generations, recursively.

oscillated between two semantic attractors that were associated with particular structural elements of the combinations tower.

The experimental design of this cognitivist treatment is instrumental in highlighting the role of external manipulation in individuals' construction of visual percepts. Next, I discuss related anthropological and learning-sciences treatments, which draw in issues of content, practice, and discourse.

Professional Vision, Disciplined Perception

Goodwin (1994) has described discursive devices that experts use in order to indoctrinate novices into disciplinary practice. In particular, the expert highlights some particular aspect of a domain of scrutiny and codes it with respect to a professional task (see also Goodwin & Goodwin, 1996). Stevens and Hall (1998), who focused on the mathematical discipline, used a similar framework to monitor the conceptual development of a student by analyzing the discursive practices of a tutor–tutee dyad. Notably, the tutor acculturated the tutee to particular *orientations of view* conducive of successful negotiation of meaning for symbolical representations, such as a graph associated with a linear function.

In like vein, the interviewer in our data oriented Li's view toward the combinations tower in a way that enhanced the tension between Li's extant inference for the marbles-box experiment and the visible collection of 16 elemental events. Specifically, the interviewer's gestures-within toward the combinations tower, accompanied by the speech utterance "20 of these, 20 of these, . . . ," highlighted the columns' variable vertical trajectories yet coded them verbally as counterfactually demarcating a flat distribution, thus suggesting to Li a possible conflict in his own reasoning.

These frameworks help us understand how Li learns *what* to look at, *how* to look at it, and what to *associate* with or *ascribe* to phenomena, in order to participate successfully in a socially meaningful system of practice. Still, I contend, we must address the symbol-grounding question and, in particular, the cognitive substance of Li's a priori tacit sense and how it becomes elaborated as a perceptual construction of an available artifact.

Instrumental Genesis

Vérillon and Rabardel (1995), exponents of the French school of mathematical didactics, differentiate between an artifact, any human-made object, and an instrument, an artifact that has been "instrumentalized," that is, put toward a particular objective. In our study, students were never told explicitly that the combinatorial-analysis procedure they were engaging might bear on the issue of expected outcome distribution. I submit that only once Li perceived the combinations tower as a means to support his initial intuition did he instrumentalize the sample space as a useful mathematical tool and—retroactively—combinatorial analysis as a useful procedure. This study thus sheds new light on the pedagogical principle of guided re-invention. Namely, the study demonstrates that students can re-invent a procedure-*as-instrument* even when they are explicitly taught to perform the procedure mechanically.

Instrumental genesis is a powerful framework for our purposes, because it patently ties the respective work of Piaget and Vygotsky. Namely, the framework explains individual learning

in the social context as the development of utilization schemas that are internalizations or “instrumentations” of activities in goal-oriented situations. This French didactics framework, however, does not address the question of how the utilization schemas emanate from tacit knowledge, that is, how they emerge from the application of enabling constraints in perceptual activity.

Reification and the Commognitive Framework

Any investigative analysis of mathematical behavior at the cusp of intuition and objectification suggests the relevance of research on *reification*. Sfard (1991) has defined reification as “. . . an ontological shift—a sudden ability to see something familiar in a totally new light . . . an instantaneous quantum leap: a process solidifies into object, into a static structure” (pp. 19–20), e.g., “ $a + b$,” once a multi-symbol directive to perform the addition operation conjoining two unknown values, is re-framed as a single integral entity that can be referred to and operated on. According to reification theory, learners, loosely mirroring historical evolution of mathematical knowledge, develop understandings of content by maturing from process to product, albeit the process–product duality remains available as a productive dialectic agent of subsequent reasoning and discovery.

Getting back to the case of Li, in order to apply the construct of reification we must identify the process that becomes a product. Yet, whereas the combinations tower is the summative product of the combinatorial-analysis process that Li enacted himself, Li’s insight was in objectifying not this analytic process involving the construction of 16 cards but the presymbolic sense of population-to-sample inference invoked by the marbles-box scenario. Moreover, the activity sequence and dialogue strongly suggest that Li’s initial intuition recruited cognitive mechanisms epistemically incompatible with the combinatorial-analytic process. Therefore, Li’s is not a case study of reification in the sense that Sfard and collaborators have used this term.

That said, Li’s data are elucidated by considering a pedagogical tension discussed by Sfard and Linchevski (1994), who implicate premature introduction of mathematical artifacts that reify processes still under routinization as liable to jeopardize students’ grounded passage from solid understandings toward the target conceptual structure. For example, students introduced to functions prior to consolidating algebraic reasoning experience challenges in conceptualizing functions as reifying their emerging algebraic reasoning. Implications for instructional design are nuanced. Indeed, Li’s case study may be viewed as a proof-of-concept that initially “irrelevant” artifacts can in fact be individualized as useful consolidations of earlier knowledge (Sfard, 2002, 2007). This apparent ambiguity in the literature regarding the potential of artifact-based mathematics instruction is unraveled by illuminating discursive as well as meta-discursive contingencies of students’ productive interaction with symbolic artifacts, as I now elaborate.

Sfard (2007) introduced the *commognitive* (communicative + cognitive) framework, which draws from the cognitive sciences and sociocultural theory and builds on a career of scrutinizing teacher–student interactions. According to this framework, mathematical reasoning is a form of internal discourse. Mathematical learning is the expansion of this discourse so as to accommodate new symbolic artifacts as well as new meta-discursive norms that guide the perception and use of such artifacts (cf. “register” in Duval, 2006). These artifacts and norms are individualized (internalized), thus enabling a learner to participate and gradually assume agency in the mathematical

discursive community and ultimately perhaps impact the evolution of the field and the education of new initiates, and so on.

Looking specifically at students' attempts to use unfamiliar symbolic tools within the context of goal-oriented curricular tasks, Sfard (2002) has identified two critical inquiry activities: (a) *intimations* are attempts to intuitively construe the new artifact, for example, a graph of a statistical distribution, in light of familiar visual or interactive cues; and (b) *implementations* are the exploratory applications of these intimations, including an evaluation of their utility to the task at hand. Indeed, in terms of Sfard's framework, one might say that Li responded to "visual" as well as "metalevel intimations" induced by the formatted sample space, which he was guided to construct, by applying the sample space as a "source" toward the "target" of his presymbolic notion of the anticipated distribution in the marbles-box experiment. This particular "attended focus" on the sample space brought about a "decision" that the sample space affords a warrant for the earlier inference. That is, Li acknowledged that "the application of the attending procedure would produce the same decision as the one dictated by the intuition" (Sfard, 2002, p. 341). Note, however, how—as in the process-product duality of the reification phenomenon—the original intuition treated in the commognitive description is evoked by attending to a symbolic tool in question, not to a situation encountered previously, as in the case of Li who is trying to make sense of the combinations tower in light of his prior encounter with the marbles box. In particular, the commognitive description is not geared to treat the cognitive mechanism by which imagistic aspects of an intuition can be invoked by one artifact then carried over and grounded in another.¹⁶

¹⁶I wish to emphasize the potential contributions of Sfard's magnum opus to my own research program. The Seeing Chance interview can be viewed as an asymmetrical negotiation, in which the interviewer creates an opportunity for the student to consider the relevance of the analytic procedure as complementary to, and enhancing of, his naive inference that was based on perceptual judgment. Pivotal to the completion of this negotiation was that the student be able to view the thematic mathematical object—the 2-by-2 matrix—as one of 16 (equiprobable) events and not only as one of five "things you can get," and that he understand the implications of alternating between these views. Combinatorial analysis and the discussion around the products of this analysis—the sample space, first loosely grouped on the desk and then assembled into the orderly combinations tower structure—constituted the context in which the student was expected to expand his repertoire of views toward the events so as to accord with the interviewer's. However, seeing an event as one of 16 rather than as one of 5 is complexly contingent on understanding why one might wish to adopt a new view of the object. That is, relinquishing or modifying a world view begs a willingness to participate in a discursive practice that entails an adoption of what initially appears to be an arbitrary construal of material substance. In this sense, combinatorial analysis, as opposed to direct perceptual judgment, can be viewed as a

new [and incommensurate] discourse . . . governed by meta-rules different from those according to which the student has been acting so far, . . . [thus entailing] a situation in which communication is hindered by the fact that different discussants are acting according to different meta-rules (and thus possibly using the same words in differing ways). . . . Only too often, [discursive] commognitive conflicts are mistaken for factual disagreements, that is, as a clash between two sentences only one of which can be correct. . . . Without other people's example, children may have no incentive for changing their discursive ways. From the children's point of view, the discourse in which they are fluent does not seem to have any particular weaknesses as a tool for making sense of the world around them. (Sfard, 2007, pp. 574–575)

Sfard concludes that a student's willingness to engage in other people's discourse is complexly related to the student's sense of identity. Sfard thus paves methodological avenues for investigating relations between identity and learning, relations that are becoming increasingly central to the study of diversity in mathematics education.

Peircean Generative Abduction

This study is ultimately an exploration of the missing link between informal perceptual intuition and formal mathematical reasoning. Investigations of intuition and reasoning naturally hark back to the Attic philosophers, and so we need not look too far afield for a legacy of expository texts on either of these putative facets of human thought.¹⁷ In particular, a neo-Classical analysis of Li's utterances as disclosing a concatenation of logical operations helps us sort between thoughts that are logical imperatives (deductive), thoughts that build on patterns that have emerged in the phenomena under scrutiny (inductive), and yet other thoughts that appear to draw on resources not immediately apparent in the situation. The latter type of thoughts have been whimsically named *abductive*—the neologism attempts to capture the sideways or stealthlike introduction of an idea into the logical thought process (Peirce, 1931–1958). Abductive reasoning is typically triggered by an apprehension of a situation that appears surprising and illogical. Specifically, an abduction is the explorative conjecture of some would-be rule that, if only true, would reestablish order into the baffling situation. Thus, I am using the term “abduction” and its cognates to refer to a logical operation based on noticing relations among properties of (artifacts generated as part of an inquiry process into) a phenomenon and interpreting these observations as a basis for hypothesizing explanations for this phenomenon.

By introducing the notion of abduction as a sister operation to deduction and induction, Peirce both captured the unique nature of these inventive leaps of logical faith and legitimized their endorsement and scrutiny in the analysis of human reasoning. Indeed, some cognitive scientists (Thagard, 1981), semioticians (Shank, 1987, 1998), philosophers of science (Fischer, 2001; Midtgarden, 2005), and mathematics-education researchers (Norton, 2008; Rivera, 2008) have been pointing to the crucial role of abductive reasoning in the original invention and guided re-invention of scientific and mathematical knowledge.

Li's notion that entire sets of elemental events within the sample space—and not just five unique events at the base of the combinations tower—should be taken as bearing on the question of relative frequency could be explained as an abductive thought process. Namely, Li would be reasoning that,

If only it were true that, “The more unique events in a set, the more frequently are its events sampled,” then my initial intuitive inference, coming from the marbles box, that 2g2b is the most likely thing we can get, could still hold, even though engaging in combinatorial analysis has led me to assume otherwise.

Subsequent to this abduction, Li pursues an *inductive* line of reasoning by which he compares other columns in the combinations tower so as to evaluate whether their relative heights, too, resonate with his intuitive expectation of relative outcome frequencies in experiments with the

¹⁷The Vygotskiiian legacy, and in particular the ethnographical work of Alexander Luria in the Ural, has demonstrated that “naïve” logical reasoners do not necessarily reason syllogistically as “scientific” reasoners do. Therefore, I never judge middle-school students’ reasoning by the extent to which they subscribe to the ineluctable deductivity of the formal syllogistic format, because I view this format as a cultural tool in which the participant may well be unfluent. Indeed, whereas the content of the interlocutors’ turn taking in the transcribed excerpts of this interview might be construed as constructing a syllogistic sequence of statements, these structures are at most suggested and never explicit.

marbles-box random generator. Thus, the new rule that Li had hypothesized may become available for performing *deductive* inference within this context and perhaps beyond.¹⁸

Mathematics-education researchers working in a Peircean framework (e.g., Bakker, 2007; Bakker & Hoffmann, 2005; Norton, 2008) have contended that Peirce's constructs offer a solution to the "learning paradox" discussed by Bereiter (1985), namely the question of how students construct meaning for signs if the notions denoted by these signs were not *a priori* available to them (see also Duval, 2006, on the "cognitive paradox"). The Peircean conjecture that new insights—abductions, hypostatic abstractions, collateral knowledge—can emerge from creative perception of diagrammatic information constitutes a response to this paradox and re-casts the question of meaning making as semiotic activity within discursive interactions.

Semiotic Leap

Indeed, I view Li's inference as essentially abductive and as constituting the basis for him to further build that which Peirce calls "collateral knowledge" of the semiotic artifacts and thus of the targeted content. And yet, I searched for a construct that might better capture the necessarily semiotic nature of students' insight-based learning, as they interact with the cultural tools introduced into their learning environment; a construct that would capture the epistemological gulf to be forded between tacit inference and formal reasoning. Hence, I have coined *semiotic leap* as a means of describing an individual's discourse-based pragmatic appropriation of an expressive object or form whose perceptual construction resonates with the original tacit inference. First, the leap occurs. Second, the logical operation of abduction determines an explicit rule that sanctions the semiotic leap, and this rule is the very stuff of sense making and deep content learning.

With the construct of semiotic leap, I thus intend to describe cognitive actions in which learners, who are solving a problematic mathematical situation, identify and use objects they find in their environment as semiotic tools for objectifying intuitive inferences pertaining to the situation, even before they fully understand the mathematical relations between the object and the situation, that is, before they have made sense of the mathematical object. Semiotic leaps are further characterized by emergent, pre-articulated reasoning that draws greatly on perceptual cues from objects yet may bootstrap a learner toward building sense for the target mathematical content. The mechanism enabling semiotic leaps is conceptual blending (Fauconnier & Turner, 2002) of mutually attracting multimodal percepts experienced in immediate (source situations) and mediated (represented, modeled) situations, by which a learner first comes to inhabit the semiotic space between the signifier and the signified.

I theorize such blending as instigated by a combination of cognitive and sociocultural resources. That is, the learner's heuristic inferences (Stavy & Tirosh, 1996) with regard to an unfamiliar mathematical artifact are triggered by the facilitator's perceptual highlighting (Goodwin, 1994; Stevens & Hall, 1998) of the artifact's critical features (Tsal & Kolbet, 1985), such that the artifact comes to constitute a material anchor for the conceptual blend (Hutchins, 2005). Initially,

¹⁸A Peircean analysis of Li's behavior as exemplifying abductive reasoning is further elaborated elsewhere (Abrahamson, 2009b). To read further on the history, roles, and mechanisms of C. S. Peirce's "generative abduction" and diagrammatic "hypostatic abstraction" in mathematical [re]-discovery, see Bakker and Hoffmann (2005), Heffer (2006), and Radford (2008).

the learner may experience the blend as a fortuitous discursive device—an available medium she or he has appropriated to build a warrant for a local mathematical argument. However, once the blend is evaluated by the learner (and sanctioned by the facilitator) as an acceptable mathematical support for the initially intuitive inference, the participant instrumentalizes this artifact (Vérillon & Rabardel, 1995), that is, appropriates the object as a tool for accomplishing similar mathematical tasks. Still, the learner's reasoning that underlay the semiotic leap might be abductive and skeletal, but now the learner may be in a cognitive position and affective disposition to explore the object more thoroughly, inductively, and determine how its elements relate to the intuitive inference. Through dialogic reflection, the learner thus begins to synthesize knowledge emanating from intuitive faculties and analytic inference (Schön, 1981).¹⁹

SUMMARY

I have presented an analysis of one student's learning process, which I view as an individual's conceptual micro-development within the context of a concerted effort involving a researcher-as-tutor, a variety of media, and an activity sequence that included a problem, potential solution paths, and a protocol-based set of prompts. In order to make sense of Li's behavior in this complex setting, I drew on theoretical models and constructs from cognitive sciences, sociocultural literature, and educational semiotics, which treat issues of intuition, artifacts, teaching, perception, reasoning, learning, and the relations among these. This integrated perspective illuminated the interview as a scenario where a student's mediated interaction with material objects both elicited his conceptually relevant presymbolic intuitions for a problem-based phenomenon and guided him to coordinate these intuitions with the standard mathematical tools and procedures related to the disciplinary handling of this class of situations. Leaning heavily on distributed-cognition and cultural-semiotics research as well as on conceptual-blending theory, I suggested that Li's coordination of his tacit inference and the mathematical artifact was a semiotic action engendered by the pragmatics of the problem-solving context. Namely, I submit that Li, who initially made an intuitive inference regarding the properties of an object yet then had his inference challenged by the interviewer, was motivated to blend presymbolic, embodied elements of his naïve notion into resonant structural aspects of an available, substantive expressive tool as a means of objectifying—and thus warranting—his intuitive inference.

The study has thus offered an example of an instructional sequence in which critical aspects of students' mathematical learning appear to transpire in the imagistic realm prior both to the manipulation of symbolic procedures and prior to the performance of logical reasoning. Namely, *Li anchored an image, which had been evoked by a problematized situation, into the material mathematical model of that situation.* Only then did he appropriate the mathematical view of the situation, which required adopting a new basic object and articulating how it related to the tacit basic object evoked by the problemaized situation. This process, I believe, constitutes the core conceptual learning required for a deep understanding of the targeted mathematical notion. Furthermore, although Li had in practice enacted the mathematical procedure and built the model

¹⁹See Abrahamson and Wilensky (2007) on a design-oriented formulation of principles for fostering content learning as the reconciliation of vying perceptual constructions of mathematical objects.

prior to anchoring the conceptual blend, it was only once he had made sense of the model *imagistically* that he could, retroactively and transitively, make sense of the procedure.

For the most part, the theoretical models cited and discussed in this article each focuses either on individuals' intuitive and heuristical resources or on the social context—a context rife with pre-instituted artifacts, semiotic systems, and implicit goals, beliefs, and norms of discourse—that the individuals grow to partake in and, ultimately, embody. Yet each of these models *alone* did not appear to constitute a perfect match for the data. Rather, I maintain, Li's learning process can be fully understood only through the complementary perspectives of cognitive sciences and sociocultural theory and, in particular, a *dialectical* perspective (diSessa, 2008b) that accounts for the cognitive structures intimating students' uptake of cultural tools. More generally, I submit, an integrated model would enable us to develop deeper understandings of students' experiences as they engage in activities designed to create opportunities for them to learn mathematical concepts.

Toward these ends, I introduced an ontological innovation—the hypothetical construct of semiotic leap—so as to characterize instances when learners, who are attempting to support an unsubstantially warranted intuitive assertion, are impelled to appropriate an available mathematical artifact in their environment as an argumentation means, even before they have completed constructing a systematic understanding of the mathematical properties of this artifact and its implications. Thus, semiotic leaps are student-initiated appropriations of perceptually available or *imagistically* simulated mathematical structures, and these leaps may potentially serve as powerful mechanisms of learning.

CONCLUSIONS

Students can construct an understanding for the mathematical procedures they learn to use as problem-solving tools, even when they initially do not understand the rationale or objective of these cultural artifacts. This claim *per se* has been made before (e.g., Sfard, 2002). Yet, as I have attempted to demonstrate, further work is required in order to explain both the source of students' knowledge basis that ultimately enables such sense-making and the mechanisms by which these personal sources become invested in the cultural artifacts. A fuller description of such learning-through-using processes requires an integrated theoretical perspective encompassing both the cognitive sciences and sociocultural theory. That is, analyses of artifact-mediated learning as apprenticeship (Lave & Wenger, 1991), participation in discursive activities (Cobb & Bauersfeld, 1995; Sfard & McClain, 2002), or acculturation into reflective praxis (Radford, 2006a) should be complemented with attention to individuals' tacit knowledge that informs their intuitive inferences (Gelman & Williams, 1998; Polanyi, 1958; Xu & Vashti, 2008; Zhu & Gigerenzer, 2006) and in particular individuals' struggle to align these inferences with the cultural formulations they are encouraged to engage and utilize (Bamberger & diSessa, 2003). That is, individual learning transpires at the nexus of complex bottom-up and top-down dialectical processes (Clancey, 2008; diSessa, 1993, 2008b), and the learning sciences would greatly avail by adopting perspectives and methodologies geared to take on this dialectic complexity. The case study presented in this article demonstrated the nature of this dialectic complexity and offered one possible route to addressing the challenges it poses for educational researchers interested in developing theory of learning and articulating its implications for design, instruction, and professional development.

A premise of this project is that procedural and conceptual mathematical knowledge should be intertwined (see Dewey, 1916/1944, on philosophical motivations for deep conceptual understanding; see Freudenthal, 1968, on the intellectual roots of what became the Realistic Mathematics Education pedagogical framework). Elsewhere (Abrahamson, 2009b), I have borrowed Vygotsky's distinction between "sense" and "meaning" and have emphasized that the effort is to enable students to build inner sense of mathematical procedures (to "connect," Wilensky, 1997) not only to recognize and master the meaning (i.e., the cultural function), of these tools. However, this distinction between sense and meaning enables me to articulate that this study has demonstrated the possible contingency of sense-making on meaning-making: dictated processes may be completely transparent yet make sense as solution procedures only once their end-product is evaluated as meaningful. In turn, to build meaning for a semiotic artifact, I have argued in this article, is to come to see the artifact such that its apprehension resonates with intuitive inferences pertaining to a situation that the artifact is said to model. I have named this moment of apprehension a semiotic leap, because it is then that the student first constructs cognitive ties between ineffable, tacit cognitive operations and explicit, mathematical semiotic objects, thus fording the fundamental epistemological gap between the tacit and the semiotic, the phenomenological and the inscribed. Inherent to the idea of a leap is that it denotes progress toward a desirable goal (in the sense of a leap of faith rather than a *salto mortale*). But having leapt, the learner must still bridge the epistemological gap between the tacit and the formal. I have implicated abductive reasoning as the logical operation that triggers the incremental thickening of the requisite collateral knowledge between the learner and the new mathematical notion.

An auspicious mathematical terrain for the investigation of semiotic leaps, because there the leap must bridge a more persistent epistemological gap—is that of *intensive quantities* (Piaget, 1952; J. L. Schwartz, 1988). Intensive quantities, dimension-less magnitudes like ratios, cannot be meaningfully added to each other the way that *extensive quantities*, such as distance or mass, can. From a phenomenological perspective, I submit, some intensive quantities— $a : b$ qualia such as slope, density, or chance—are privileged domains for which humans have evolved enabling constraints that underlie their tacit inferences yet thus complicate the educational program of learning to describe these quantities using the mathematical semiotic system. That is, on the one hand, "[I]ntuition is generally seen as a primary phenomenon which may be described but which is not reducible to more elementary components" (Fischbein, 1987, p. ix). Yet, on the other hand, the requisite mathematical description of a phenomenon inhering an intensive quantity demands reducing the phenomenon to its elementary components—the a and b in the $a : b$ compound—and then recomposing this compound (Abrahamson, 2009b). The case study analyzed in this article demonstrated how a middle-school student could bring to bear tacit mechanisms for seeing chance in a box of marbles, yet when this irreducible qualia of chance was to be redescribed as an intensive quantity—that is, as the $a:b$ relation between the favorable events and the entire sample space—this student encountered great difficulty. Namely, the student struggled to coordinate his tacit synoptic inference with the analytic mathematical artifact he had constructed. The coordination was accomplished only once the student was guided to determine a way of perceiving the artifact such that it evoked the same inference as did the situational embodiment that the artifact was purportedly representing. This resonance between the multimodal images evoked by the intuitive perception of the situation and the disciplined perception of its mathematical representation, I believe, is the moment of meaning making and hence of sense making, of connecting. The abductive logical process then takes over so as to rationalize the newly hypothesized embodied knowledge.

IMPLICATIONS FOR INSTRUCTION

Essential to the learning process supporting students' sense making is that the activity sequence depart from problematized situations involving phenomenal embodiments of the curricular unit's targeted mathematical notions. It is these phenomenal embodiments that first evoke the tacit processes of inference making. However, the resulting pedagogical dilemma is that students do not necessarily know how to articulate their inferences using conventional semiotic devices. This study suggests the potential of designing for semiotic leaps and then orchestrating them.

Designing for Semiotic Leaps

An epistemological commitment of the proposed integrated theoretical perspective, as well as its concomitant heuristic design framework, is that some mathematical concepts can be learned through a process of initially "meaningless" tool use that nevertheless is crafted so as to lead up to students' abduction of the rules underlying the tool's function. One effective strategy for facilitating students' grounded appropriation of cultural tools is to design mathematical situations that tap students' intuitive schemes and then facilitate activities through which students come to see the tools as semiotic means for warranting their preformulated inferences, even before these processes are inscribed symbolically. Such guided learning can ultimately be as meaningful for students as other types of discovery-based processes practiced in constructivist curricula, because the students experience personal invention of the procedure-*as-instrument* even in the midst of learning to perform this procedure per the instructor's directions. Moreover, such a learning-through-using process appears to be well aligned with the accomplishment of curricular objectives—for example, those framed by national standards—namely, developing grounded fluency with mathematical tools.

Orchestrating Semiotic Leaps

Different situational embodiments of one and the same mathematical concept may elicit different intuitive responses (diSessa & Wagner, 2005), because humans are ecologically adapted to respond to situations, not to mathematical reifications of these situations (Gigerenzer, 1998). Ultimately, teachers should help students understand how *all* these various intuitive responses they experience vis-à-vis different situational embodiments of a target concept relate to that concept. Yet run-of-the-mill educational programs require curricular materials, for example, student textbooks and teacher guide-books, and creating these, in turn, necessitates design decisions with respect to the specific nature of the experiences that teachers are to foster through a structured activity sequence. Thus, amid the repertory of relevant concept-specific intuitions familiar from the literature, constructivist designers of mathematical curriculum necessarily make optimization selections in targeting intuitive responses best conducive to deep learning.

This article reported on a study in which the researcher attempted to help a set of individual students make sense of a mathematical concept by eliciting their mathematically correct, conceptually relevant intuitive response and then ushering the students to bring that response to bear in interpreting a mathematical artifact that supports and enhances that intuition. Along

the way, untargeted responses, too, were elicited, and so the researcher-cum-tutor had to make moment-to-moment decisions as to whether these intuitions should be nurtured and applied or rather only acknowledged and detoured. A teacher, like a creative conductor of a teaming jazz band, first orchestrates the performance of an improvisatory pedagogical score—be it a semi-structured clinical interview, or, later, a lesson activity sequence—and then, through cueing students’ desirable intuitions and quieting undesirable ones, marshals the emergence, convergence, and merging of the ensemble of tacit intuitions and learned procedures into resonant, dynamical stability.

Orchestrating Semiotic Leaps Toward the Binomial

Specifically for the topic of basic probability, this study constitutes empirical support for the conjecture that students’ intuitive expectation of likelihoods in experiments with random generators is qualitatively in accordance with mathematics and, moreover, that this intuition can ground the mathematical procedure of combinatorial analysis by which the expanded sample space is generated and investigated. Thus, given appropriate design, the intuitive sense of likelihood, which Tversky and Kahneman (1974) decry as a bias-prone heuristic, is in fact a useful cognitive resource. Specifically, empirical findings of erroneous inferences regarding the comparison of equiprobable samples on the basis of their assumed likelihoods demonstrate not faulty inference making, I wager. Instead, these inferences are due to non-normative categorization of the samples as orderless aggregate events rather than as ordered elemental events. Given appropriate design, however, wherein students’ initial inferences *agree* with mathematical theory, the intuitive sense of likelihood can and should be embraced in the teaching and learning of the binomial.

I thus join other researchers of probability learning (e.g., see in Shaughnessy, 2003) in a quest for designs that work *with* students’ intuition rather than quell them. In fact, the current study suggests pedagogical advantages of setting up conditions that encourage students to draw on intuitive judgments, just as long as those initial inferences are aligned with mathematical theory. The particular design employed in this study appears to create opportunities for students to effectively approach the targeted content. In particular, the marbles-box activity enables students to “list” the experiment’s sample space as the assembly of all possible iconic configurations of the 2-by-2 scoop, and the resulting imagistic resonance between the random generator and its sample space eschews the cognitive burden of interpreting symbolic notation and thus facilitates a honing of the central issues that must be addressed. Moreover, the tangibility and mobility of the sample-space cards readily enable reconfiguration of the space into formats that accentuate its parsing into event aggregates and thus render the space better conducive for the emergence and performance of comparisons vital for evoking the targeted inferences.²⁰

²⁰I have been repeatedly asked by leading scholars why the four-marbles scoop is structured in 2-by-2 form rather than 4-by-1, given that the normative mathematical listing of a sample space is as linear strings, for example, “HHTH,” and that we may wish to scaffold students toward that normative form. My prosaic reply is that the 2-by-2 configuration is a heritage of the square samples used in our computer-based statistics activity for networked classrooms (S.A.M.P.L.E.R., Statistics As Multi-Participant Learning-Environment Resource, Abrahamson & Wilensky, 2004b; Abrahamson & Wilensky, 2007), and we wished to create a uniform format across our curricular material, *ProbLab* (Abrahamson, Janusz et al., 2006; Abrahamson & Wilensky, 2002). Auspiciously, though, when square samples are rotated as a form of combinatorial expansion, they produce a set of four permutations as compared to only two that would be produced by rotating a card

In order to enable students to sustain and leverage these apparently useful inferences throughout a structured activity sequence, further conditions may be imposed on the activity so as to initially arrest students' impulse to experiment with an available random device, at least until the sample space has been constructed, analyzed, and coordinated with the inferences. This sequencing would defer to a later stage in the design competing inferences that come from explorative data analysis and potentially interfere with the crucial population-to-sample inference. That is, students are "liable" to base expectations not on the global features of the mechanism but upon a set of available experimental outcomes resulting from several introductory trial draws from the random device. In like vein, the study demonstrates that, on the basis of their intuitive expectations alone, students can go quite a way toward understanding core notions of classicist probability, even before actual experiments are conducted. From that perspective, the intuitive and mathematical expectations could be regarded as triangulations, which subsequent random experiments validate.

The model of learning-through-using that I have put forth, if valid, would support the educational design heuristic by which didactical mathematical situations should tap students' intuitive knowledge—that is, learning materials should be designed with an eye on the enabling constraints of perception and reasoning (Gelman & Williams, 1998; Gigerenzer, 1998; Gigerenzer & Brighton, 2009; Smith et al., 1993)—so as to accommodate students' intuitive schemata, even before they learn to perform the mathematical counterparts of these automatic processes; even before the students come to appreciate how mathematical procedures enhance and elaborate intuitive reasoning so as to meet the demands of participating in the social complex.

ACKNOWLEDGMENTS

This article builds on several conference papers (Abrahamson, 2008a, 2008b; Abrahamson et al., 2008; Abrahamson & Cendak, 2006). The *Seeing Chance* project was supported by a National Academy of Education/Spencer Postdoctoral Fellowship. I am grateful to the students who volunteered to participate in this study and to the school principal and staff for opening their doors to this collaboration. A big thanks to Jeanne Bamberger, Allan Collins, Andy diSessa, Maria Droujkova, Alan Schoenfeld, Tobin White, and Betina Zolkower for commenting on earlier drafts of this article—shortcomings of this article reflect my failure to heed their advice. Thank you Cliff Konold for helping me be more precise and consistent with my mathematical terminology. Special thanks to members of the Center for Connected Learning and Computer-Based Modeling (Uri Wilensky, Director) at Northwestern University who assisted in implementing my design, and especially Paulo Blikstein, for his help in engineering and producing my design of the marbles scooper. At UC Berkeley, thank you to members of the Embodied Design Research Laboratory in the Graduate School of Education, including participants in the Undergraduate Research Apprenticeship Program, and to students of the graduate course Learning Chance. I wish to extend my gratitude to the prior and current editors in chief and executive editors

representing a linear scoop. Notwithstanding, a comparison of the learning affordances of square and linear scoops would make for an intriguing study. In any case, a combinations-tower format utilizing linearly configured compound events has been adopted by the Model Chance project and thus integrated into an experimental version of the TinkerPlots computer-based learning environment (Cliff Konold, personal communication, January 22, 2007).

of *Cognition and Instruction* as well as Susan Jurow and several anonymous reviewers for their highly effective comments. Finally, I thank Michael Monroe for “Simple Life” and “Wintersong,” pensive musical art that inspired much of the aforementioned search for coherence.

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