The Double-Edged Sword of Constructivist Design

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Abstract

23 middle-school students participated in a randomized-control clinical-interview-based study examining psychological processes underlying effective utilization of artifacts in the context of a constructivist learning activity in classical probability. The interview activity centered on perceptually ambiguous artifacts that act as a random generator of binomial distribution, and can be used to ground several conflicting intuitions about random sampling and outcome distribution. We examined how eliciting students' intuition about likelihood affected their ability to understand the sample space. We found that students whose intuitions about the activity had been evoked initially struggled more than others in the activity due to conflicts within their intuitive framework; yet ultimately these students were more able to discern conceptual connections. The ambiguous artifacts grounded multiple conflicting intuitions to facilitate abducting mathematical procedures within the context of the student's own intuition. We examine the delicate role of the teacher in constructivist design, and the paradoxical nature of grounding conflicting ideas.

Background & Theoretical Framework

Our study is situated within the ongoing development of ProbLab, an experimental unit for probability (Abrahamson & Wilensky, 2002, 2005), and continues the work of the Seeing Chance project (Abrahamson, PI), which is itself situated within the Connected Probability project (Wilensky, 1993, 1994, 1997). Taken together, these projects comprise a body of design-based research spanning multiple years. Abrahamson's design philosophy draws on models of learning from the cognitive sciences, sociocultural theory, and cultural semiotics (Abrahamson, 2008). In particular, Abrahamson's design philosophy leverages students' intuitions regarding mathematical phenomena to build a constructivist approach to teaching that encourages conceptual re-invention. In this approach, students' introductions to normative mathematical procedures are grounded in their intuitions, and the mediating artifacts become internalized as enduring imagistic vehicles of concept-specific mathematical reasoning (Abrahamson & Cendak, 2006).

Central to Abrahamson's framework is designing perceptual ambiguous artifacts that mirror milestone conceptual tensions students must reconcile to gain a deep understanding of mathematical content. By analogy to Jastrow's (1899) famous duck-rabbit ambiguous figure (Figure 1), the 2-by-2 matrices in Figure 2 can be viewed either as "the same," in which the unordered ratio of green to blue is fore-grounded, or as "different," in which the particular configuration of green/blue marbles is attended to. As we will explain, these two views bear directly on students' understanding of the nature of combinatorial analysis and the notion of sample space. Namely, as random events, these matrices belong to the same event (n = 4; k = 2), yet it is essential to differentiate between them to compute the events' relative likelihood.

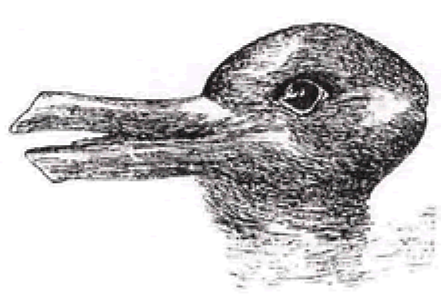
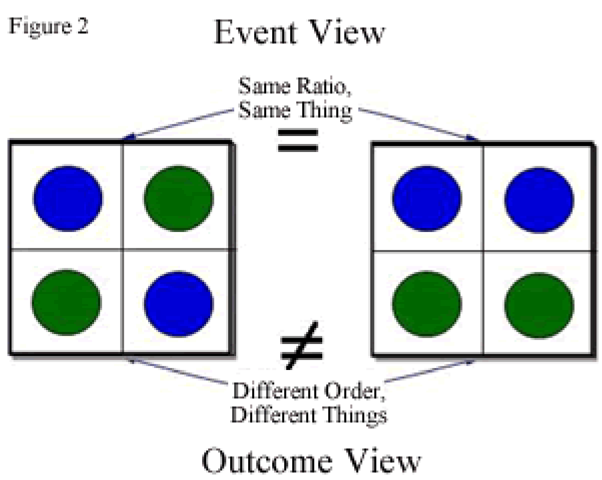


Figure 1 - Duck or Rabbit?

Figure 2



This study is a follow-up to prior Seeing Chance work, and it uses a portion of the standard Seeing Chance protocol with a few specific alterations, including our experimental groups. We will review the protocol and results of the original studies, discuss the artifacts in some additional depth, and then detail the specific deviations we made for our study.

Original Seeing Chance Interview

Students initially agreed to take part in a research study that they were told was concerned with learning mathematics. The interview begins with brief introductions, and then the student is presented with a transparent bin filled with green and blue marbles mixed in what appears to be equal proportion (figure 3). The interviewer asks the student to make a few guesses as to what we might do with it. After considering the marble bin and discussing a few guesses, the interviewer gives the student a utensil to examine, (figure 4). Again, the object's possible uses are discussed, and the student might note its resemblance to a spatula, with the odd exception of having four divots arranged in a square array. The student's term for this device (often "scooper") will be used for the duration of the interview. Next, the researcher invites the student to scoop marbles out of the bin, and after taking a few scoops and noting their contents, the interviewer asks the student to guess what you might get if you scooped this device into the bin, making sure to fill each of the four divots with a marble and letting all the rest fall out. The student then invariably makes the guess, “2 Green, 2 Blue.” But the interviewer presses for a rationale, and although students generally expresses certainty in the accuracy of the guess—it so compellingly “feels” right, they are unable to provide a more “mathematical” justification for this intuitive claim.

Figure 3 - Marble Bin

Figure 4 – The Scooper

Figure 5 – Cards and Crayons

Setting this unresolved question aside, the interview continues, proceeding into an activity where the interviewer asks the student to “show what you could possibly get” and provides some green and blue crayons and a stack of cards representing the empty scooper (figure 5). When introducing the cards, the interviewer indicates that the thicker edge-line on the card should be interpreted as corresponding to the handle of the scooper. As they begin to color all of the options, many students quickly realize that you could show the possibility of “2 Green, 2 Blue” in a number of ways, depending on the placement of the marbles in the scooper. They ask the interviewer if they should color in these different arrangements (outcomes defined as the ordered permutations) or just make a single card (the event defined as an unordered combination). When the interviewer turns the question back to the student, they feel certain that the different arrangements shouldn’t matter. The interviewer acknowledges this theory, but suggests that they draw all the arrangements anyway. The student then attempts to draw all the different arrangements (engaging in combinatorial analysis), eventually creating sixteen cards representing all the arrangements of “what you can get” when you scoop. The interviewer allows students to use their own methods to find all possible combinations, and to check if they've missed any, and only prompts if the student is unable to discover all possible arrangements.

The interviewer then asks the student to arrange these sixteen cards in a way that would convey “the most information possible” to someone just beginning the activity. The student organizes the cards on the table, and after some conversation with the researcher regarding the arrangement, the interviewer guides the student towards arranging the sixteen cards into a formation (henceforth the combinations tower) (figure 6). At this point, or during the subsequent discussion of the combinations tower, students recognize this formation as a warrant for their earlier intuition. They notice prominence of the tall middle column of cards representing the six different possible arrangements of “2 Green, 2 Blue,” and generally explain this in terms like "You get 2 Green 2 Blue most often because there's more ways to make it." Some students may also elaborate as to how this explains the need to attend to order in creating the combinations tower.

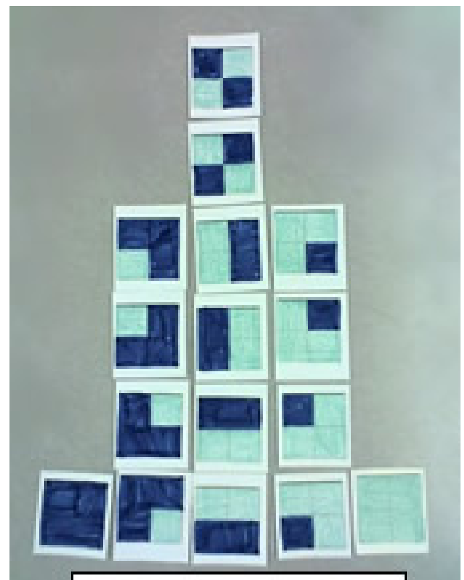


Figure 6 - Combinations Tower of 4-block cards

In initial Seeing Chance interviews, 27 out of 28 students had the "aha!" moment of recognizing the combinations tower as a justification for their guess of 2G2B, and many developed an even deeper understanding of the material in the subsequent debriefing interview and computer simulation activities included in the original Seeing Chance protocol. In our variant on the Seeing Chance activity, we adapted this interview portion only, and made several changes to the protocol. The preceding Seeing Chance studies clearly demonstrated that this activity could give students an intuitive guide to explore the traditionally un-intuitive concept of sample space, and our study hoped to examine possible weaknesses of this approach.

Materials and Conceptual Affordances

The 4-block

The 4-block mathematical object is central to the activity design, as it constitutes the primary ambiguous mathematical object. The 4-block is in the form of a 2 x 2 visual array. This object is embodied in: (a) the marble scooper, contextualized with respect to the marble bin; and (b) represented iconically on the card stock and then pluralized into the constructed sample space of the combinations tower. Abrahamson et al. (2008) maintain that the generative ambiguity of the 4-block is grounded in its capacity to be viewed either as event or outcome. Abrahamson et al. (2008) assumed that the event orientation of view was elicited as a result of the marble bin context in which students were made to consider the probable, predicting “what you would get” when you scoop. The elicitation of the outcome-view was attributed to the combinatorial analysis task in which the student is asked to consider the possible and “show what we can get” using the 4-block grid cards and blue and green crayons. The 4-block object's presence in both of these contexts allows students to retain their intuitions about probable events while exploring the sample space of specific outcomes.

The Marble Bin Context

The marble bin context refers to the portion of the interview where the student is asked to guess “what we would get” or “what would happen” if we were to dip the marble scooper into the bin, insuring that 4 marbles fill the divots. By asking the student to make a claim about “what we would get,” the activity becomes contextualized within what is probable. Consequently, the student’s assessment is constrained toward the functional relationship between the bin and scooper. This consideration of likelihood assumes the unpredictable nature (randomness) of the anticipated act of sampling from the bin, and in doing so casts the bin of green and blue marbles as a population from which the scooper extracts a sample of 4 marbles. The marble bin and scooper, formerly undefined with regard their perceived functionality, thus become jointly instrumentalized as constituting a random generator. So conceived, the student's intuitions generally align with the representativeness heuristic (Tversky & Kahneman, 1974), a cognitive mechanism by which the likelihood of a sample from a population is judged based upon the degree to which that sample is representative of the population. Thus, the representativeness heuristic acts as an intuitive means for determining the most likely scoop. Employing the representativeness heuristic, students are enabled to make the claim of “2 Green, 2 Blue” or “half-half” as the most likely scoop, insofar as it most resembles the marble bin.

What is crucial about the moment in which this claim is made is that in the very act of making the claim of “2 Green, 2 Blue”, the 4-block is cast as an event, with particular salient properties defining its form. It is the representativeness heuristic that selects the salient properties of the bin (population), in this case two-tone color (green or blue) and distribution (half-half), and blends these properties onto the scooper according to the affordances and constraints (Collins, Neville, & Bielaczyc, 2000; Gibson, 1977) of the device (four slots in a 2x2 array), and thus conditionalizes the student’s view of the 4-block in the event-view.

The Combinatorial Analysis Context

Abrahamson and Cendak (2006) found that, when asked to draw the possibilities, all students asked whether they should draw the permutations. Confronted with the task of enumerating the possible, some felt satisfied drawing 5 cases representative of the 5 events: “4 green,” “3 green, 1 blue,” “2 green, 2 blue,” “1 green, 3 blue,” and “4 Blue.” (figure 7). When the interviewer asks the student to “show what we can get” using the 4-block card stock and green and blue crayons, this shifts the framing of the 4-block from the probable to the possible. Recall that the 4-block icon represented on the card stock exhibits a darker edge-line corresponding with the handle of the scooper. The card thus affords the creation of specific permutations (because rotation would create a significantly different card). Nevertheless, this affordance is often deemed inconsequential to students, as it is not a relevant consideration to the event view of probability.

Abrahamson et al. (2008) argue that requiring students to draw out the possible scoops constrains their conception of the activity with a paper medium that cannot house their intuitive sense of the relative proportional intensities of likelihood of scoops. Thus the 4-block card stock medium is simultaneously too perceptually rich, insofar as it affords the property of order deemed irrelevant by the student, and yet limited in its capacity as an expressive medium. Specifically, the card stock is limiting insofar as it falls short of being the semiotic means of objectification (Radford, 2003, 2008) needed by students to express their intuitive presymbolic notions of the relative frequency of scooping events. For example, students may want to be able to somehow indicate that the “2 Green, 2 Blue” event is special, since their intuitive judgment from the representativeness heuristic informs them that “2 Green, 2 Blue” is more representative, and therefore more likely, than “3 Green, 1 Blue.” Yet, the constraints of available media inherently limit their ability to express this notion. Abrahamson el al. (2008) note that the provision of the 4-block card stock results in ontological imperialism (Bamberger and diSessa, 2003) insofar as by affording the inscription of ordered configurations, which is vital for the activity sequence, the card imposes upon students a view of the marbles box that they never actually entertained. Thus, the inherent structural order of the 4-block constrains students’ ability to express their intuitions.

The Combinations Tower

The combinations tower is presented to the student as a suggested arrangement of the 16 possible outcomes created in the combinatorial analysis activity. As previously noted, students typically exhibit an “Ah-ha” moment upon seeing the combinations tower, claiming that it signifies what they were trying to express when they made their original claim of “2 Green, 2 Blue” as the most likely outcome (Abrahamson & Cendak, 2006). It is important here to tease apart the dual role of the combinations tower as both a visual-figure resonant with students’ proportional intuition of the relative likelihood of events, founded in the representativeness heuristic from the marble bin context, and as a handy tool to enumerate outcomes and thus build a mathematical warrant for their claim of “2 Green, 2 Blue.” Insofar as the combinations tower fills this dual-role, it serves as the previously unavailable semiotic means of objectification (Radford, 2003, 2008) for expressing in mathematically normative terms the pre-symbolic proportional judgment of relative frequencies of scoop events. That is, upon viewing the completed tower, “2 green, 2 blue” suddenly reveals itself as the tallest category. The student is thus able to instrumentalize the tower a means of justifying the earlier intuitive inference that “2 Green, 2 Blue” would be the mode, stating that since there are more (six) possible ways to get it than any of the other events, it is therefore the most likely scoop.

Abrahamson et al. (2008) discuss the cognitive mechanism that enables students to recognize in the plurality or tallest-ness of 2-green cards a warrant for their intuitive claim of “2 Green, 2 Blue.” That is, the question on the board is: How does a new object—the combinations tower—take on the semiotic function of warranting a claim made in the context of the marbles box and scooper? The student’s reasoning is purportedly thus: “The 2-green column is ‘more-tall’ than the 3-green column, therefore the 2-green is ‘more-often’ than the 3-green column (see the more of ‘A’ – more of ‘B’ heuristic, Stavy & Tirosh, 1996). Thus , the combinatorial analysis task, previously deemed a deviation from the context of the relative frequency of scoops, becomes retroactively justified as relevant to the question of the likelihood of scoops. That is, counting permutations becomes an inductively logical means of determining relative frequencies of combinations. Finally, Abrahamson and Wilensky (2005) concluded from student interviews that the constructed sample space itself could be construed as a “second-order” population, i.e., the sample space is seen as the sampling space, so that a plurality of 2-green cards implies that one is more likely to choose on of those cards.

Methods

Our study breaks from the precedent of all former Seeing Chance studies insofar as we have chosen to constrain the role of the researcher as educator and unfettered interlocutor. A previous Seeing Chance study (Abrahamson et al, 2008) found that the emergent variability inherent in authentic dialogue introduced pragmatic issues with profound effects upon students’ learning trajectories (for more pragmatics of discourse, see Grice, 1975). By requiring researchers to adhere to a rigid protocol, our study seeks to control for variability between subjects, thereby sensitizing our study as an instrument for measuring the effect of varying contexts of enactment of our ambiguous artifacts. The constraining of the researcher signifies a marked departure from the design-based research methodology (Design-based Research Collective, 2003; Cobb et al., 2003; Collins et al., 2004) in which a design is enacted in a naturalistic learning setting in order to monitor and maximize students' learning while refining the design and related theory.

Nonetheless, the nature of the one-to-one clinical interview requires that the researcher engage in fluent dialog with the student. Specifically, interviewers must adhere to the cooperative principle, and collaborate with the student's attempts to make sense of the activity (for more on the cooperative principle, see Grice, 1975). Discourse is framed as a negotiation between the learner's continual construction and articulation of meaning and the interviewer's attempt to reflect this meaning back to the learner. In order to hand over the construction of meaning to the student, the researcher proceeds through the activity introducing as little new vocabulary as possible, encouraging the student to coin new words when appropriate (i.e. when naming a novel object such as the "marble scooper"). Furthermore, the researcher must persistently guard against the assumption that the meaning of uttered words, both the researcher's own and the student's, are taken-as-shared (Cobb, Yackel, & Wood, 1990).

The introductory portion of the activity, in which students scoop a few times to familiarize themselves with the marble bin and scooper, has been removed. It was only when we analyzed the data from the previous study that we found that what we thought was a minor activity may have introduced an artifact in the design. Due to the inherent unpredictability of probability generators, in this case the bin of marbles with the scooper, it sometimes happens that after having sampled a few times students draw false conclusions based on this small sample. For example, one student from the previous study scooped "3 Blue, 1 Green" three of four times, thereby introducing a bias into his subsequent conception of the relative frequency of events. When asked to guess what one might get upon scooping, the student seemingly failed to employ the representative heuristic leading to the expected claim of "2 Green, 2 Blue," instead adjusting his expectation of subsequent scoops based upon this short term empirical evidence (Abrahamson et al., 2008). Since this portion of the interview may introduce unwanted variation within and between treatment groups' samples, we decided to do away with it.

Participants

Twenty-three grades 7 and 8 students from a public urban middle school (32.2% on free/reduced lunch; 66.2% minority students) voluntarily participated in the study. Students were selected from Pre-Algebra (4 students), Algebra (7 students), and Honors Algebra (12 students) classes. All participants were attributed an achievement level (‘High,’ ‘Middle,’ and ‘Low’) determined by the students’ current mathematics teachers on the basis of their performance on assessments. All students had been exposed to the study of probability in the context of a unit integrating probability with the concept of fractions. In this unit, both empirical and theoretical approaches to determining probability were covered, most often through the use of spinners and dice. Participants were randomly distributed into 3 treatment groups, balancing for gender, grade level, and achievement levels. No screening was conducted.

Treatment Groups

As the rationale for the design of the Seeing Chance activity hinges upon the assumption that the elicitation of the student’s intuitive claim of “2 Green, 2 Blue” serves as the pivotal moment in conditioning the student’s view of the 4-Block to the event orientation, we have chosen to manipulate the provision of context during the marble bin context, insofar as it directly precedes and ostensibly elicits the claim. In order to determine whether a student recruits the event orientation of view towards the 4-block, we examine each student’s response to the combinatorial analysis task, predicting that a student holding the event-view would view the 4-block as unordered, whereas a student without the event-view would conceive of the 4-block as ordered. Thus we hypothesize that students who does not hold the event-view will go Direct to Permutations (DP), engaging in drawing the possibilities by drawing ordered permutations of the 4-block. We hypothesize that students who recruit the event-view will question whether or not they should attend to order in drawing all possibilities, thus failing to go Direct to Permutations. Thus, a student is coded DP if he or she interprets the combinatorial analysis context such that he or she attempts to draw all the permutations possible, never questioning the relevance of order. Alternatively, if a student initially questions the relevance of order, the student is not coded DP.

Students will be placed into three treatment groups: leading question (LQ), no question (NQ), and distracter question (DQ). Each group will engage in the activity, with the experimental manipulation replacing the marble bin context orienting activity. While all three groups will be introduced to the marble bin and scooper, only the LQ group will be asked the standard Seeing Chance question designed to elicit the event-based orientation towards the 4-Block: “If I dip [the scooper] in here, what do you think will happen? What is the best bet?” The NQ group will not be asked any question, and will instead proceed directly from the introduction of the bin and scooper to the combinatorial analysis task. The DQ group will be asked an alternative question designed to divert the student from taking the event-orientation toward the 4-block: “If I were to scoop and make sure that I got 4 marbles every time, how many scoops would it take to empty the bin?”

The names of the three groups have been chosen to portray the expected performance of each group with regard to their ability to recruit the event-view of the 4-block. The LQ group is considered advantaged, the NQ group neutral, and the DQ group disadvantaged. It is important to note that the researchers are conscious that this ascription of “advantage” or “disadvantage” indicates our biased perspective as researchers, whereas students are understood to encounter the activity with fresh eyes, holding no such bias. Furthermore, although the research team maintains predictions regarding the influence of the treatment upon students’ subsequent performance in the activity, the interview protocol clearly defines the researcher’s role by providing criteria for proceeding through the interview, complete with responses to probable questions posed by the student. As the nature of the clinical interview does not afford the opportunity for double-blind administration of treatments, we have deemed it best to be fully aware of our bias, so as to guard against it.

Predictions

Leading Question Group

As the provision of context in the LQ group mirrors that of previous Seeing Chance studies, we expect this group to respond similarly to students in those studies (Abrahamson & Cendak, 2006). Namely, we expect that our LQ students will utilize the representative heuristic to make the claim of “2 Green, 2 Blue,” thereby manifesting the event-orientation towards the 4-block. Thus, we predict that LQ students, having recruited the event-view, will question the relevance of order and thus not be coded DP.

Distracter Question Group

The DQ group is designed to distract students from being able to recruit the event-view by engaging students in an estimation task in which they considered how many times one would have to scoop to empty the marble bin. We expect that the DQ context will serve to orient their view of the marble bin as a material quantity, effectively de-emphasizing the relevance of the colors of the marbles, which in turn will obfuscate the property of ratio of blue and green marbles within the bin. Thus, by disabling their inclination to attend to the half-half ratio in the bin, and casting the scooper's role as that of "bin-emptying device," we expect that students will be unlikely to enlist the event orientation of view on the 4-block. As previous Seeing Chance studies posited the event-orientation of the 4-block as causing student's difficulties conceptualizing the possibilities of "what you get when you scoop" in terms of ordered outcomes, we expect that DQ students will not experience this confusion and will conceive of possible scoops in terms of ordered permutations. Thus, we predict that DQ students, having failed to recruit the event-view, will be coded DP.

No Question Group

The NQ group is situated between the LQ group and DQ group insofar as students in the NQ group are neither provided context designed to orient (LQ) nor distract (DQ) the student's recruitment of the event-view of the 4-block. In this way, we also expect that our NQ students will perform, on average, somewhere between the LQ students and the DQ students. Given the lack of guidance from the researcher, NQ students' orientation of view on the 4-block will be largely contingent upon the extra-experimental factors relating to each particular subject: prior knowledge, ability to infer intended learning goals based on the affordances and constraints of available media, pragmatic awareness, etc. Thus we predict that the NQ will be a mixed bag, with some coded DP and others not, but this treatment group may act as a sort of control for gauging the additional factors affecting students understanding of the activity.

Results & Analysis

Data Collection

Data were collected using an audio-visual recording device positioned such that all utterances made by the student and interviewer, gestures performed by the student, and manipulations of the learning tools by the student were captured. In addition, the interviewer took notes during and immediately after the interview. Data analysis involved both written summaries of each interview and coding of student behaviors. Video data were coded using a rubric developed by the research team, with each student's interview cross-checked by at least two members of the research team. Inconsistencies in coding were brought before the research team for review and were all resolved.

Quantitative Results

The 4-block

Our results indicate that all LQ students were not *DP*, while DQ students were, by in large *DP*, with NQ students falling roughly between the two groups (see table 1). As our data would not be suitable for analysis by a chi-squared test, because the expected values in the table are all below 10, Fisher's exact test was used, as it is traditionally used in the analysis of categorical data where sample sizes are small. Fisher's exact test, both two-tailed and one-tailed, yielded a p=0.015 , thus strongly suggesting an effect of the treatment group on the *DP* behavioral specification.

	Does the student go direct to permutations (*DP*)?		
	Yes	No	
LQ	0	7	7
NQ	4	4	8
DQ	6	2	8
	10	13	23

Table 1

As predicted, LQ did not go Direct to Permutations (*DP*), DQ students generally did go *DP*, with NQ students falling between both groups. These findings suggest that the provision of context influenced students' view of the 4-block, with LQ students taking the event-view, causing them to question the relevance of the property of order with regard to the task of enumerating the possible 4-blocks. Some of the *NQ* and *DQ* students immediately recognized the marble bin as affording the possibilities of a random generator and framed the discussion this way themselves, effectively altering the interview protocol to align with *LQ*. These students were designated self-primers (SP) and are grouped with *LQ* data in Table 2. In order to be designated *SP*, a student had to discuss, without prompting, their understanding of "what you can get", thus answering the unasked question from the LQ protocol. It is worthwhile to note that the only two DQ students to be *not DP* were high achieving and were coded as self-primers (*SP*), suggesting that they may have taken the event-view prior to the combinatorial analysis task and therefore belong in the LQ row, thus fortifying the '7' (albeit they officially earned the *SP* designation later in the interview). In sum, our findings strongly support the anecdotal evidence from previous Seeing Chance studies that provision of the marble bin context impacted students' recruitment of the event-view of the 4-block. This evidence further supports our hypothesis that students' orientation of view toward the 4-block is extremely sensitive to the context of enactment of the 4-block.

The Combinations Tower

As expected, all LQ students claimed "2 Green, 2 Blue" as the most likely scoop, thus meeting our criteria for judging their recruitment of the event-view. Four non-LQ students received the designation of Self-Primer (*SP*), two NQ and two DQ. We pooled these students into one group labeled LQ+SP, and compared them to the pooled group labeled NQ+DQ, representing the remainder who had not met the *SP* criteria.

Does the student instrumentalize the combinations tower? (*CTI*)?

	Yes	No	
LQ + SP	9	2	11
NQ + DQ	3	9	12
	12	11	23

Table 2

Both pooled groups were then compared using the *CTI* specification (see table 2), effectively testing whether the recruitment of the event-view was associated with students' ability to instrumentalize the tower as either an index for their intuitive frequency or towards grounding the combinatorial analysis procedure. In order to meet the criteria for *CTI*, a student had to make an assertion about the larger number of possibilities in the 2G2B column, and relate this to the likelihood of choosing scooping two green and two blue marbles in a scoop. Again, as the expected values in the table were all below 10, we chose to use Fisher's exact test. Fisher's exact test found the one-tailed $p < 0.01$ and the two-tailed $p=0.012$. By either measure, our results strongly suggest an association between pooled groups and the *CTI* specification.

As predicted, a student's ability to make a specific claim signifying the recruitment of the event-view of the 4-block was found to be a key factor for predicting whether that student would be able to use the combinations tower towards connecting the relevance of the number of outcomes within each event-column to the likelihood of that event. On the one hand, these results are not surprising, as students who had not held the event-view might not have conceived of the activity as pertaining to the question of likelihood whatsoever. However, of the nine NQ+DQ students who were *not CTI*, four (two NQ and two DQ) made passing references in which they noted the activity of reminding them of 'chance' or 'probability.' This suggests that while the students' consideration of likelihood is essential for recruiting the event-view, the consideration of likelihood did not itself constitute the event-view of the 4-block. Overall, these findings suggest that a student's ability to instrumentalize the combination tower as a tool for coordinating judgments regarding the relative frequency of events is highly contingent upon the context of enactment.

A Note Regarding the Significance of Achievement Level

One of the profound findings of our study was that of the twelve students earning the *CTI* designation, eleven of twelve were high achieving (the remaining student was middle achieving). While these results aren't too surprising given the skewed distribution of high-achieving students volunteering for our study (15 of our 23 students were high achieving), we also found that the four of four *SP* students were high achieving and were classified as *CTI*. Furthermore, of the 7 LQ students, the only 2 to not earn the designation of *CTI* were a low achieving student and a middle achieving student. Juxtaposing these findings with those of Abrahamson and Cendak (2006) indicate the crucial role of supportive feedback for mediating low and middle achieving students towards learning gains.

Qualitative Results

The following case studies tell the story of 4 students from our study. They were chosen in order to highlight the sensitivity of the 4-block and combinations tower to their respective contexts of enactment. Commentary is embedded in each case study. Pseudonyms are used to protect student identities.

Case Study 1: Leading Question Supports Complete Understanding

Joe is an 8th grader who was characterized by his math teacher as high-achieving. After a few minutes of working with the marble bin and scooper, he is asked what he thinks he would get if he were to scoop. He responds that "after a while, it should even out to 2 green, 2 blue." The interviewer then asks what he would say if he had to make a bet on a single scoop. Joe's response indicates his sense of the entire distribution: "If you feel crazy [you could bet] all green, all blue…but to be safe, I'd say two green two blue is the most probable." To explain his claim he cites the equal number of blue and green marbles in the marble bin. Joe is then asked to use the crayons and cards to show every possible scoop. He initially draws five cards, one from each number-of-green group. When asked if those five cards represent everything that he could possibly get, he replies, "Well actually, this is in general speaking. You could get, instead of two blues here [gestures at blue squares in the 2g2b card], two blues here [points to two other squares]." He asks whether he is supposed to draw all the possible arrangements, but immediately answers his own question when he remembers the thick black line: "If the line is there, that means I have to do placement." Joe does not yet see the relevance of placement to the overall activity, but is able to use a feature of the medium to successfully create the complete sample space. This ability to attend to the affordances and constraints of available media and judge them according to the pragmatics of discourse is a key skill of high-achieving students.

Once the sixteen cards are completed, Joe is asked to arrange them in a way that he thinks would be most informative. His arrangement indicates that he is thinking of the cards in terms of ratio of blue to green. He creates four distinct groups, and describes them as "all of something [4 green or 4 blue]…or 3-1 for green, or 3-1 ratio blue, or even." This ability to recognize outcomes as members of groups defined by ratio will help Joe when he is subsequently faced with the combinations tower. Once the tower is built, Joe is asked if it tells him anything. He jokingly answers that it looks like a person with no arms doing the splits. The interviewer asks a second time, this time making reference to the marble bin and the scooping activity. It is at this point that Joe, having been asked the leading question, has an advantage over those students who were not asked to make a claim about likelihoods. The interviewer's reference to the scooping activity reminds Joe of his claim that two-green two-blue is the most likely event, and he is able to connect the combinations tower to this prior work with probability. He quickly answers, "You're more likely to get these ones [gestures to the 2:2 column]. It's safer to bet halfway, in the middle, like I said." Joe is thus able to connect the combinations tower to the earlier activity, and use it to justify his claim that 2g2b is the most likely outcome. He uses the tower to determine that two 2g2b has a 6/16 chance, and reads off the corresponding probabilities for the other groups. Joe's understanding does not end there. He goes on to point out the distinction between plurality and majority, an issue that often causes students confusion (Abrahamson & Wilensky, 2007). He notices that there are more cards with a 3:1 ratio (that is, a total of 8 - 3g1b pooled with 1g3b) than there are cards in the 2g2b group (6), and elaborates on his earlier claim of what bet he would make: "It depends on how you're betting. If you're betting to get a three-to-one ratio, it's better to bet this [gestures to the two 3:1 columns], but if you're betting exactly, like three-to-one ratio in blue's favor, you should just say 'half'." Finally, to test his understanding further, the interviewer picks up two cards, one from the two-green two-blue group and one from three-green one-blue, and asks if one is more likely than the other. Joe initially responds that the card from the two-green two-blue group is more likely, but immediately corrects himself: "They're equally likely if you're just talking about those two. They're all equally likely, if you're being specific about placement."

Case 2: An NQ student Does the Best with the Context She's Provided

Rebecca is an 8th grader characterized by her teachers as high achieving. She recognizes the marble bin as possibly used for counting marbles and comments that the scooper was possibly a marble holder, though it looks like an egg-fryer. When asked to color in "what we can possibly get," Rebecca creates 7 cards, one of each event-type, with three cards representing the event of 2g2b (providing one card in the vertical arrangement, one in the horizontal arrangement, and one card for crossed 2g2b). After having created these cards, she pauses to ask if she should go on. The interviewer asks whether she thinks she has shown “everything we can possibly get,” and she answers, “Well, not every combination, but … we could do more if we switched them around [she gestures to one of the cards she has drawn, twisting her hand over it].” Asked whether it (the order) matter, she indicates that it does matter, and soon proceeds to draw more ordered outcomes. Rebecca’s behavior is typical of NQ students, insofar as she exhibits both characteristics of the event and outcome-views. Judging from her first 7 cards, she classified 4-blocks as types, with ratio playing a key roll. At the same time, she originally considered the 3 arrangements (vertical, horizontal, and crossed) of the 2g2b event as distinct. Namely, Rebecca’s view on the 4-block begins as a mixture of classifications typically used by LQ, and even though she indicates that she should color in more, she seems unsure.

Rebecca takes a long time to build the sample space, and asks several times whether she is done. Her difficulty populating the sample space was likely due to the fact that she did not organize the cards by ratio while she made them, but continually stacked cards in a column two cards wide with no apparent order to her stack. When asked to arrange all 16 outcomes to give the most possible information, she arranged the cards into a 4x4 array with the 3:1 cases making up the bottom two rows, followed by the parallel cases of 2g2b next, with the top row having the 4g and 4g on the left, and the crossed cases of 2g2b on the upper right. She explained that this arrangement showed the categories of the different types. She also mentioned that this arrangement tells the person entering the room that this is everything we can get. When she saw the tower, she initially saw it as a rearrangement of what she did before. When asked about whether it told her anything about scooping, she originally drew a blank. Given time to contemplate the tower, she eventually noted that it told her that the 4g and 4b are least likely, because they are only 2 out of 16. She said that was all she could say. When pressed to look further, she noted that the 3:1 ratio cards will happen half of the time, because they are 8 out of 16.

Though at first if may appear odd that she didn't mention 2g2b as happening more than the other columns, it is clear from her claims that she views the property of uncolored-ratio as relevant, rather than the colored-ratio. Rachel was coded *CTI*, as she related the number of ways to get a particular event to the likelihood of that event. However, unlike Joe, Rebecca had no epiphany upon seeing the tower. Whereas she was able, after some time, to instrumentalize the combinations tower towards linking the number of outcomes to the relative likelihood of an event, her difficulty suggests that she never conceived of the 4-block in the event-view, and thus lacked *a priori* intuitive sense of the relative frequency of each event.

Case 3: A DQ student with a flair for the aesthetic

Bobby is an 8th grader who has been characterized by his teacher as low achieving. When asked to comment on the marble bin, Bobby guesses that we might use the marbles for counting or separating by color and comments on how they remind him of his bouncy-ball collection. When presented with the scooper, he comments on how it reminds him of a game, and goes on to note that it holds marbles. Upon being asked what we might be doing with the scooper he says, "Picking up the balls and see if, kinda like a probability thing. If you scoop it in there and shake them off (gestures scooping and shaking off extra marbles) and then how many ever times you do it you record how many are in which and then probably put them there (gestures to the side of the table) and it will most likely have to be, it will most definitely be even at the end... I'm assuming there is an even number of both colors." At first glance, it might appear that Bobby has earned the designation of *SP*, since he spoke of drawing out samples, "probability," and getting an "even" result in the end. Having rigorously scrutinized his case after repeated viewings, however, we find that he never actually makes a claim specifying the relative frequency of a particular scooping event. Instead he appears to have made the claim that if you scoop a lot of times, record the number of green and blue in each scoop, and place the marbles you scooped out to the side, you will eventually get a distribution of half-half in this side pile of marbles. Thus, Bobby did not earn the *SP* classification.

When asked the distracter question, Bobby carefully considers the bin, shaking the marble bin; he guesses that it would take 120 scoops to empty the bin. Upon being asked to color in all the possibilities, he goes directly to permutations (*DP)*, and uses the language “Green, Blue, Blue, Green” in conjunction with a succession of pointing gestures towards the 4-block card to denote the placement of each marble. Bobby’s attention to the placement of each marble in the scooper suggests that he doesn’t view 4-blocks in terms of color-ratio, rather he attends to the ordered placement of marbles in the scooper. Describing his strategy for finding all the possible, he says, “I just think of every pattern and make sure to do the opposite.” When asked to arrange the cards in such a way as to convey the most information, he first groups the 3g1b and 1b3g in a 2x2 square, noting that he chose to put the majority color on the outside. This suggests his view of the 4-blocks as aesthetic objects, where the order affords different aesthetic possibilities. For Bobby, the collection of 4-blocks appears to furnish no sense of relative likelihoods.

Upon seeing the combinations tower constructed before him, he notes how it looks like a rocket ship. The researcher points out how the columns are similar to the groups he constructed, and gestures to the marble bin, asking Bobby if “that grouping tells us anything about this scenario.” He notes how the blue are more on one side of the tower, and the green on the other. Asked if the combinations tower “tells you anything about scooping,” he replies that “these are all the combinations that you can get. If someone saw this (the combinations tower), they would probably look at it and say, ‘This is what I’ll get if I scoop that thing (the scooper) into the bin of blue and green marbles.” Thus even though Bobby at one point conceived of the activity as possibly pertaining to probability, his mathematically impoverished view of the 4-block seems to have hindered his ability to see the combinations tower as relevant to the relative frequency of events. The fact that Bobby made a passing connection to previous experiences regarding probability, and yet failed to recognize this as particularly salient to the activity, suggests that the provision of context is especially important for low-achieving students. Furthermore, while Bobby’s view of the 4-block as ordered contributes to his ability to easily populate the sample space, this view was not associated meaningfully with likelihood. Lacking an event-view that would furnish an intuitive sense of likelihood severely limits Bobby’s ability to engage the combinations tower as an ambiguous object generating cognitive conflict.

Case 4: A High Achieving DQ-SP Kid with All the Answers

Isaac is a high achieving 8^{th} grader. Upon being introduced to the marble bin, he originally says that the marbles remind him of chocolate covered candies, and goes on to surmise that we might be guessing how many beads are in the bin. He also mentions that we could figure out the volume or surface area of the marbles. He sees the scooper as a spoon, and wonders whether there is a game we are going to play with it. Asked the question of how many scoops it would take to empty the bin, he guesses that it would take 60-70 scoops. When Isaac first populates the sample space, he initially makes the 5 event cards. At this point he pauses, and notes that "this could take a while," but rather than attempting to draw more cards, he begins to rotate each card, asking whether mentioning these rotations satisfies the interviewer's demand to "show" the possible. Asked whether the different arrangements would be important he says "not really." Given his pause and the fact that he stated the order to be irrelevant, Isaac was classified as 'not *DP*.' The interviewer urges him to draw the rest, which he does efficiently, several times mentioning that the different arrangements could be represented just as easily by rotating a card. All this suggests that Isaac may have already recruited an event-view of the 4-block. His strong inclination to attend to color-ratio likely aids him in attending to the proportion of green to blue marbles in the bin, if he has not already. Thus by viewing the 4-block as an unordered ratio, Isaac builds a solid foundation from which he can compare the 4-block ratio with the marble bin ratio, which may retroactively trigger the representativeness heuristic, further developing his event-view of the 4-block.

After creating all 16 ordered outcomes, Isaac arranges the cards into 3 rectangular groups by ratio (4:0, 3:1, or 2:2). He volunteers that when you scoop, you will notice that one of these three things happens. Gesturing to the 2:2 group he says,

> This would be perfectly proportional to what you think would happen. Like if there are half blue and half green (he looks at the bin), then this (gestures to the group of 2:2 cards) would represent your best odds, I mean your average, probably . . . well actually obviously…. It would represent your average scoop… your average result that you would have two of one color and two of the other because there are half and half (in the

> bin). And here (gestures to the 3:1 group of 8 cards) will be a little less likely (looks to bin), I think, because based on the averages, but this may happen just as often as this (gestures to 2:2 group) except that when you average it out it will come down to this (gestures to 2:2).

It is worthwhile here to note that while Isaac originally considers the number of ordered outcomes in the 3:1 group as relevant to the likelihood, his strong intuition from the marble bin leads him to believe that 2g2b is both majority and plurality, whereas in fact 3:1 is the majority and 2g2b is the plurality event. Gesturing to the 2:2 column, he says, “This is the most important figure here. And this (gesturing to the 4:0) would be very unlikely, I guess." It is this comment that led us to code Isaac as *SC*, as he clearly makes a claim specifying the relative likelihood of various events.

Upon seeing the combinations tower, Isaac indicates the middle column as the most likely. He then goes on to indicate that the 3g1b is less likely to happen on the “predominantly green side” of the combinations tower, with 4g0b being even less likely, and visa versa for the other side, thus being coded as *CTI*. While Isaac still falsely confuses the plurality (2g2b) with the majority (3:1), judging 2g2b to be more likely than getting 3:1, this confusion is common (Abrahamson & Wilensky, 2007), The fact that this confusion is so pervasive supports the argument that students taking the event-view indeed employ the representativeness heuristic, insofar as they tend to view 4-blocks as color-ratioed objects (rather than purely ratioed), thus accounting for their insistence of 2g2b as most likely despite contrary evidence suggesting that there are more ways to get a 3:1 ratio.

The double-edged sword of constructivist design

Analysis

The idea of using ambiguous artifacts was initially chosen to invite students’ intuitions into unfamiliar conceptual territory, where they might guide and ground exploration. Cognitive reactions to this approach are still somewhat generalizable, as most rich mathematical material evokes multiple, ambiguous perspectives. It is clear that incorporating students' intuitions about probability in the event-view guided their exploration of the sample space. Without an intuitive guide, students were unable to utilize the combinations tower as an indicator of the frequency of the various events. Eliciting these intuitions comes with certain entailments. One entailment, the notion that 2G2B is the most likely outcome, and the bi-directional gradient of likelihood ending in 4G or 4B, acts as a guide during the sample space exploration, as the student keeps in mind the ways in which those outcome classes are "special". But the notion that specific permutations of an outcome are irrelevant is also an entailment of this intuition, and this conflicts with the activities required to explore binomial distribution. Some students who were not *DP*, in fact, struggled with the idea of permutations through to the end of the activity, and never seemed quite able to accomplish what Abrahamson et al. (2008) called the "suspension of pertinence" (compare to "suspension of disbelief") necessary to wholly engage in the learning activity. Additionally, some students were aware that these two views were separate, and struggled with mapping ideas between the two (Abrahamson, D. 2008 is a case study of one of these students). This is especially unfortunate, as these students often made very intelligent explorations of the subject material, and yet still left with a feeling of *epistemological anxiety* (Wilensky, 1997), and the notion that they hadn't really figured it out.

Conclusions

The use of perceptually ambiguous artifacts in constructivist designs for learning introduces a "double-edged sword." On the one hand, this ambiguity imbues designed artifacts with a richness that can be extremely fruitful insofar as the perceptual ambiguity affords a generative cognitive conflict. Hidden within the richness of such artifacts, mathematically complementary ideas can be elegantly juxtaposed, creating a generative conceptual tension inextricably tied up in the various views afforded by the ambiguity. As evidenced both in the quantitative results of our study and in the case studies of Joe and Isaac, this ambiguity was crucial for the *LQ* and *SP* to have considerable insight into notoriously difficult mathematics. On the other hand, our study has demonstrated that designs incorporating ambiguous artifacts are extremely sensitive to their contexts of enactment. Students' navigation of perceptual ambiguity must be carefully orchestrated in order to ensure that students' engagement in cognitive conflict is productive. We cannot hold the illusion that well-designed learning tools can be left to students with the expectation that they will discover their utility, somehow bringing about insight by force of shrewd design. It is apparent that an educator's deft guidance at pivotal moments in a learning activity provides the orienting context that serves to enable learners to engage the ambiguous artifacts as material catalysts for calling-up, developing, and ultimately honing cognitive conflict.

Insofar as Abrahamson & Cendak (2006) demonstrated the efficacy of the Seeing Chance design for learning in the clinical interview setting, much work remains to be done. While the volatility of employing ambiguous artifacts in constructivist pedagogy has been demonstrated as manageable within the one-to-one, tutor-like implementation of the Seeing Chance activity design, our study emphasizes the considerable challenge for scaling up such constructivist designs for whole-class implementation. Insofar as constructivist designs are sensitive to contexts of enactment, they demand mechanisms through which context can be reliably provided as needed. Indeed, while a 9-block version of a related Problab activity has been implemented in a classroom (Abrahamson, Janusz, & Wilensky, 2006), the issue of the distribution of teacher attention during whole-class implementation was not a focus of that study, and thus requires further research.

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