UNDERSTANDING PROOF: TRACKING EXPERTS' DEVELOPING UNDERSTANDING OF AN UNFAMILIAR PROOF

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In this study, we adopt the notion of dense connection *in the understanding of mathematics, and trace the development of these connections over time as participants make sense of an unfamiliar proof. By representing participants' verbalized sensemaking with a network of ideas and resources that changes over time, we can investigate what features of a mathematical proof play more or less central roles in one's developing understanding of that proof. Preliminary results indicate that though all participants in the study were at a graduate level of study or above, different participants revealed different aspects of the proof (a formal definition, a specific example, or a specific property or component of the focal mathematical idea) to be central to their developing understanding.*

INTRODUCTION

One of the most important aspects of mathematical proof is the relationship between a reader and a proof as a disciplinary tool – that is, how does one use a proof to learn and make sense of the mathematical ideas contained within? In this project, we provide expert mathematicians (graduate students and university professors) with an unfamiliar mathematical proof, and ask them to think aloud as they make sense of it. We use these interviews to trace how experts construct their own understandings of the mathematical ideas contained within the proof, and identify which aspects of the proof serve as hubs or remain on the periphery of this developing understanding.

Unlike several studies of expert mathematical knowledge and expert mathematicians' proof practices, this study concentrates specifically on experts as they interact with an *unfamiliar mathematical idea*. We believe that such an approach may begin to address the discrepancies often cited between novice and expert practitioners of mathematics – namely, that novices rely on empirical and informal knowledge, whereas experts rely on coherent, formal definitions when thinking about mathematics (Vinner, 1991; Schoenfeld, 1985; Tall, 1991; Sfard, 1992; Dubinsky 1992). While certainly experts are able to describe their well-established mathematical understandings in such a way, this does not necessarily suggest that experts *learn* about new mathematics this way. As such, we believe that a deeper look into how individuals with a deep mathematical knowledge base *construct* such knowledge may yield different implications for secondary and tertiary mathematics education than expert/novice studies that focus on mathematical ideas that experts already understand well.

ANALYTIC FRAMEWORK

In both the formulation and the analysis of this study we relied heavily on the notion of *knowledge as dense connection* (Skemp, 1976; Papert, 1993), and were interested to investigate the extent to which expert knowledge, and particularly the development of this knowledge, can be described in the context of new and unknown mathematical content. The notion of mathematical knowledge as connected elements accounts for a number of aspects of expertise – for example, one of the identifying aspects of expertise is the ability one has to deconstruct and reconstruct mathematical knowledge in new and different ways (Tall, 2001); and it is certainly expected that different participants, with their varied experiences and backgrounds, may have different ways of "slicing up" the elements of the proof in order to construct their own understanding (Wilensky, 1991). As such, the coding system described below was developed using a bottom-up iterative process (Clement, 2000), though connections to existing literature were made when these relationships became apparent during development of the codes.

Our coding scheme consists of two levels – *ways of understanding* and *resources for understanding* – that closely mirror Sierpinska's (1994) distinction between acts of understanding and resources for understanding.

<u>Ways of understanding</u> include *questions*, *solutions*, and *explanations*, and align well with Duffin and Simpson's (2000) descriptions of *building*, *enacting*, and *having* understanding.

<u>Resources for understanding</u> include *parents*, *definitions*, *fragments*, and *instantiations* (examples provided by the proof itself, introduced by the reader, and so forth). Several resources for understanding can be identified within a question, solution, or explanation: for example, if a participant questions how two definitions presented within a proof are related to one another the statement would be coded as a question involving two definitions; if a participant makes sense of a definition by enacting it on an example provided within the proof, this would be coded as a solution involving a definition and an instantiation. The coding system is described in much more depth in Wilkerson and Wilensky (2008).

Research Questions

In keeping with the themes of the ICMI Study as outlined in the Discussion Document, we believe that this study (a) begins to address questions of individual differences in how one understands and makes sense of proofs, (b) identifies what aspects of proof (definition statements, examples, detailed description of processes and machinery) serve as central components of one's understanding, and (c) provides a language with which to investigate how learners interact with disciplinary materials in order to make sense of new and unfamiliar mathematical ideas. For this paper, our research questions include:

1) What aspects of a proof play a more central role in one's developing understanding of the mathematical ideas contained within?
2) What are the similarities and differences between different individuals and the proof elements that find more or less central to their understanding?

METHODS AND DATA

Participants

10 participants, including 8 professors (assistant, associate, and full) and 2 advanced graduate students from a variety of 4-year universities in the Midwest participated. Participants were identified primarily through university directory listings, and contacted via email to see if they would agree to be interviewed.

Protocol

Students and professors who wished to participate were given semi-structured clinical interviews using a think-aloud protocol (Ericsson & Simon 1993). Each was provided with the same mathematics research paper (Stanford, 1998; see below), selected for its accessibility in terms of topic and vocabulary. They were asked to read the paper aloud and try to understand it such that they would be able to teach it to a colleague. Interview data was videotaped, transcribed, and coded using the TAMSAnalyzer software (2008).

Proof

The research paper provided to participants (Stanford, 1998) concerns *links,* which can be thought of informally as arrangements of circles of rope that are entwined with one another, and the conditions under which those circles can be pulled apart. If a link has the property that when any single circle is removed from the arrangement, the rest can be pulled apart, that link is said to be *Brunnian*. If in a given two-dimensional representation of a given link, there are *n* distinct collections of over- and underpasses that, when switched, make the loops fall apart, the link is said to be *n-trivial.* The proof establishes a systematic relationship between the properties that make a link *Brunnian* and *n-trivial*, such that any Brunnian link can be described as *(n-1)-trivial*.

Analysis

For the construction of each experts' network, each participant's *resources for understanding* were converted into network nodes and *ways of understanding* into links between those nodes. For example, when a participant asks how two definitions (say, the definitions of trivial and of Brunnian) are related, this is reflected in the network by establishing a question link between trivial and Brunnian. If later the participant tries to find out how the two aforementioned definitions are related by manipulating the Borromean Rings as a specific example of a Brunnian link, this is reflected in the network by establishing a solution link between Brunnian, trivial, and the Borromean Rings.

After the network is built, it can be analyzed to determine which nodes, or mathematical resources presented in the proof, served a more central role in each experts' sensemaking. Network measures were computed using the statnet package (Handcock, et al, 2003) in the R statistical computing environment. The measure used in this report, betweenness[1], is a measure of the extent to which a given network node (or element of the proof) serves as a "bridge" between other nodes. In other words, a proof element with high betweenness is one that enables the participant to form many connections between other elements of the proof.

RESULTS

At the time of this proposal, data is in the process of being analyzed. Preliminary findings, however, suggest that while there is relative consistency in experts' *ways of understanding*, they use very different *resources for understanding* introduced by the proof while making sense of the ideas presented within. In other words, while experts seem to be relatively consistent in the number of questions, solutions, and explanations they discuss as they read through the proof, the specific aspects of the proof discussed within each of these questions, solutions, and explanations differ greatly. For some experts, specific instantiations of the mathematical object being explored serve a central role in building a densely-connected description of the proof; while for others a formal definition or several small components of the mathematical object serve this purpose.

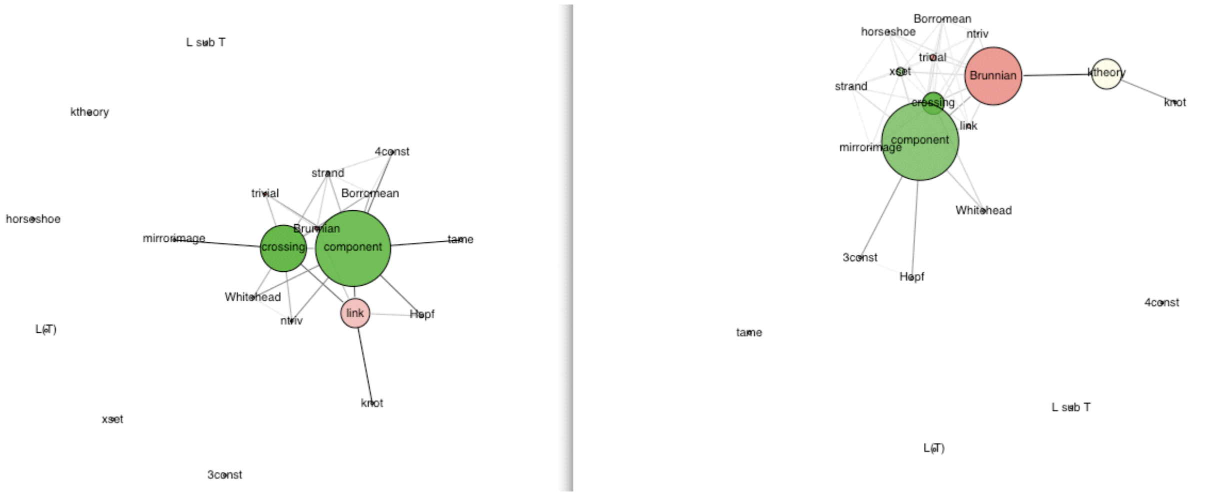

Figure 1. Network representations of two participants' coded interviews

In the graphic above, the network produced from each participant's entire coded interview is featured. The darkness of links between any two elements represents the frequency with which those elements were mentioned together.

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[1] Betweenness $C_B = \sum_{i \neq j \neq v} \frac{g_{ivj}}{g_{ij}}$, where g_{ivj} is the shortest path between i and j through v, and g_{ij} is the shortest path between i and j.

The color of elements indicates which code each element belongs to; the most visible here are *fragments*, that is, smaller pieces of the main idea to be proved (green), formal *definitions* (red), *parents* or background knowledge introduced by the participant (white). The graphs indicate that Joe's network is more dense, but that Ana more frequently linked the same elements together. Furthermore, definitions, background knowledge, and pieces of the larger proof all played a much more important role for Joe's developing understanding, while Ana made sense of the proof mostly in terms of its smaller pieces only.

(average betweenness[2])	Joe	Ana	Mark
Fragments	.28	.05	~0
Parents	~0	.09	~0
Definitions	.02	.09	~0
Examples	.43	.05	~0
Constructions	.25	.71	1

Table 1: Betweenness of different types of proof elements for Joe, Ana, and Mark

In addition to exploring how different individuals might utilize different components of a proof in order to make sense of it and the mathematical ideas contained within, it is interesting to consider what elements are important for all participants, regardless of their "proof style". The table above shows that while Joe, Ana, and Mark relied on different types of proof elements to very different degrees (Joe heavily relied on fragments and examples; Ana had a more distributed focus and used her background knowledge more), constructions – that is, examples that were constructed by the participant on-the-fly to illustrate, test, or otherwise investigate the claims laid forth in the proof – served as an important bridging element for all three participants.

CONCLUSION

In order to access the aspects of expertise that might best inform educational practice; it is important to recognize that the *mechanism* by which experts come to know mathematics should be investigated in addition to the structure of that knowledge they already have. In this paper, we outline a method for representing experts' active sensemaking while reading a proof, and some analytical tools for evaluating what parts of a proof serve central roles in individuals' developing understanding of that proof and the ideas associated

[2] Betweenness was averaged across all elements of each type, and across total betweenness of each element for each individual.

with it. Although our results are still in the preliminary stages, we believe that we are able to capture patterns in experts' developing understandings that might reflect different ways of coming to understand a proof, as well as other patterns that hold constant across participants.

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