Reinventing Algebra Brick by Brick: Paradigmatic-Problematic Situations as Vehicles for Mathematizing and Didactizing

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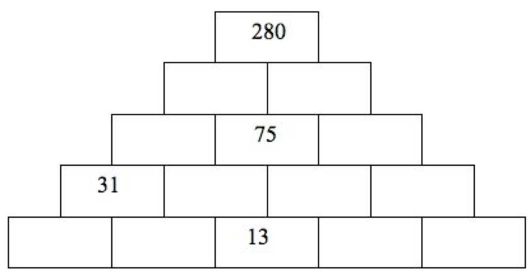
> "I believe problems of education are fundamentally the same all over the world, and so are the ideas for solving them. In all kinds of education our goal should be: more integrated subject matter, taught by teams rather than individuals, and in particular in teacher training, integration of subject matter with its didactics." (Hans Freudenthal, 1977, p. 371)

1. Introduction

This paper discusses an ongoing research project involving instructional design for mathematics teaching and teacher education. The project consists of an experimental graduate-level course for pre-service teachers. Central to the design of this course are "paradigmatic didactical–mathematical problematic situations" (hence pdmps). Pdmps are unique activities that serve as contexts for inquiry into mathematics and its didactics.[1]

In Figure 1 we present the pdmps that will be the focus of this paper. Readers are warmly encouraged to work on the problem prior to reading on.



Fill in the Brick Pyramid above according to the following rule: The number on each brick is the sum of the numbers on the two bricks it stands upon. How many solutions did you find? Could there be more? Explain.[2]

Figure 1. The Brick Pyramid: The central paradigmatic didactical–mathematical problematic situation in our study of experimental professional-development curricula.

It is one thing for the reader of this article to solve the Brick Pyramid problem individually and possibly reflect on this experience. Within a mathematics teacher-education context, the Brick

[1] Whereas in the Anglo-Saxon world, 'didactics' tends to be used, rather pejoratively, to indicate 'directive' or 'explicit' teaching, in continental Europe this adjective and its cognates have a positive connotation (e.g. *didactique, Didaktik, didactiek, didáctica*). Within the latter perspective, which we adopt herein, didactics is viewed as a scientific discipline that studies teaching–learning processes and designs theoretical models and instructional materials for improving these processes (Biehler et al. 1994; cf., Chevallard, 1999).

[2] The design of the Brick Pyramid problem was inspired by the *Wisk & Reken* textbook series. For discussions on the use of "brick walls" in German schools, see Selter (1997), Wittmann (1995), and Muller (2003).

Pyramid problem can take on additional dimensions that may be conducive to reflecting on pedagogical practice. Some of these dimensions can be studied by considering the following questions:

- What are some of the different ways this problem can be solved?
- How are those different ways similar and how are they different?
- What mathematical ideas/tools are implicated in the solution of this problem?
- How might you modify this problem so as to use it in a classroom setting?
- What challenges do you expect this problem may pose to prospective students?
- What forms of interaction may support these students in grappling with this problem and the algebraic ideas that are embedded in it as potential interactions?
- What mediating mathematical tools may further support the above process? How?
- What other problems or variations on this problem may serve as vehicles to expose students to those mathematical tools and ideas?

That one can meaningfully ask questions such as the above in the context of working on the Brick Pyramid makes this problem a pdmps. We hypothesized that this pdmps could serve as a powerful artifact for a cohort of future teachers, guided by an experienced instructor, to explore central questions about mathematics by integrating the subject matter with its didactics (Freudenthal, 1977) .[3]

More specifically, the Brick Pyramid problem allows prospective teachers to encounter central issues in the learning and teaching of algebra. These include: (1) symbol sense, informal sense-making, and formal algebra (Arcavi, 1994); (2) the relationship between language proficiency and algebraic learning (MacGregor & Price, 1999); (3) the role of realizable or imaginable situations in developing tools for building formulas and solving equations and systems of equations (van Reeuwijk, 1995; van Reeuwijk & Wijers, 1997); (4) the challenges in guiding students' transition from informal to formal algebra (Stacey & MacGregor, 2000; Swafford and Langrall, 2000; Nathan & Koellner, 2007); and (5) the complex interplay among algebraic thinking, algebraic generalization, and algebraic symbolization (Linchevski, 1995; van Ameron, 2003; Zazkis & Liljedahl, 2002; Radford, 2003, 2006).

[3] For a collection of pdmps such as the Brick Pyramid problem, the reader is referred to Zolkower & Uittenbogaard (in preparation), which includes annotated examples from implementations of particular pdmps in middle-school classrooms, teacher-preparation courses, and lesson study groups.

Our interest in pdmps such as the Brick Pyramid sparked from noticing that these artifacts engage classroom practitioners as well as researchers-in-training in productive inquiries around cognitive, social, technological, and other aspects of mathematics learning and teaching (cf., Brar, Galpern, & Abrahamson, 2006). Consequently, we designed and implemented a semester-long college course based on the study of pdmps and the subsequent try-out, by course participants, of adapted version of some of those pdmps in the form of teaching 'experiments' (Hiebert, Morris, & Glass, 2003). Our research aimed at investigating the effect of this course on strengthening participants' mathematical attitude (i.e., their disposition towards looking at the world with a mathematical eye) and developing in them a didactical attitude, that is, a disposition towards looking at mathematical problems from the point of view of learning and teaching.

This paper reports on the first implementation of our experimental course. The narrative of our findings is organized around a cohort's work with the Brick Pyramid problem: solution strategies and ideas exchanged during one of our weekly meetings, subsequent online discussions, planning towards an implementation of the problem with schoolchildren, and a reflective report by a course participant on his try-out of the problem with four 11th grade students. These data selections are brought forth to support our argument that the Brick Pyramid pdmps created unique opportunities for the participants in this experimental course to reflect on thematic issues germane to professional development. Namely, the discursive contexts that emerged through our joint work on this pdmps constituted vehicles for the co-production of a mathematical–didactical disposition in participating students.

2. Theoretical Perspectives

Freudenthal (1973, 1991) proposes we view mathematics as a verb, i.e. *mathematizing*. Mathematizing is a reflective human activity that consists of organizing or structuring subject matter from reality, including mathematics itself (Freudenthal, 1991). Mathematizing, "the process by which reality is trimmed to the mathematician's needs and preferences" (Freudenthal, 1991, p. 30), involves: (a) looking for essentials within and across situations, problems, procedures, algorithms, formulations, symbolizations, and axiomatic systems; (b) discovering common features, similarities, analogies, and isomorphisms; (c) exemplifying general ideas; (d) approaching problematic reality paradigmatically; (e) "The sudden emergence of fresh mental objects and operations;" (f) short-cutting or abbreviating initial strategies and symbolizations towards progressive schematizing, algorithmizing, symbolizing, and formalizing; and (g) reflecting on one's activities by considering the matter from different perspectives (ibid. pp. 35-36).

Within this perspective, known as *realistic mathematics education* (RME), the learning/teaching[4] of mathematics is conceived as the guided reinvention of mathematizing (Freudenthal, 1991). We guide students to reinvent mathematizing by engaging them in collaborative inquiries around problematic situations embedded in realistic or imaginable contexts (Freudenthal, 1991). As they organize those situations, students reconstruct their initial, situated material/mental activity by verbalizing, symbolizing, and diagramming the essential relationships found therein. Throughout this process, students reconstruct the meaning of families of situations that can be structured mathematically in similar ways along with the tools that might be used to that end. Consequently, key to RME is the design of situations that may serve as suitable contexts for progressive mathematizing.

From the perspective of RME, becoming a mathematics teacher is viewed, too, as a guided reinvention process, namely, the guided reinvention of *didactizing* (Freudenthal, 1991; Goffree & Dolk, 1995; Goffree & Oonk, 1999). Ideally, as they participate in guided, collaborative activities of organizing indeterminate problematic mathematical teaching–learning situations by didactical means, prospective teachers acquire knowledge, skills, and dispositions for guiding their students' reinvention of mathematizing. Creating effective teacher-training programs that realize this vision is a complex task, which, in recent decades has been engendered much developmental research (Gravemeijer, 1998).

The research reported herein builds upon and contributes to a body of work that investigates the use of rich problems as contexts for learning to teach mathematics. Included in this growing domain are: realistic modeling problems (Verschaffel & de Corte, 1997); emergent modeling problems (Gravemeijer et al., 2000; van den Heuvel-Panhuizen, 2003); problems that yield multiple solutions (Silver et al., 2005; Zolkower, 2007); model-eliciting tasks (Lesh et al., 2008); substantial learning environments (Wittmann, 1995, 2002); open-ended problems (Cifarelli & Cai, 2005), spiral tasks (Fried & Amit, 2005), and example-generating problems (Watson & Mason, 2005).

Our experimental graduate-level course for prospective mathematics teachers is entirely organized and driven by problems rather than by learning-sciences or pedagogical, math-methods

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[4] In Dutch, *leren* means both to learn and to teach.

concepts. Paraphrasing Turkle and Papert's (1991) proverbial call to "put logic on tap, not on top," we place mathematics instruction theories on tap rather than on top. That is, our pdmps serve as scenarios for the targeted mathematical–didactical ideas to emerge out of the guided engagement of participants in the designed activities.

Our two central research hypotheses concern two complementary settings: (1) the study of pdmps during the university-based course; and (2) the enactment, via teaching experiments, of modified versions of pdmps in classrooms wherein some of our course participants were fulfilling their student-teacher practicum. We hypothesized that: (a) the selected pdmps would provoke, among course participants, experiences of constructive challenge around important ideas concerning the content and didactics of number-and-operation, geometry-and-measurement, algebra, and probability-and-statistics strands; (b) modified versions of these same pdmps would elicit meaningful mathematical activity among middle/high-school students; and (c) the presentation and discussion of these pdmps-centered classroom teaching experiments would allow participants to reflect upon central tensions and dilemmas in initiating their prospective students into mathematizing.

3. Design: The Brick Pyramid as Guided Activity Fostering Proto-Algebraic Reasoning

In the activity sequence that we have been using, students are first shown a picture of bricks configured in the shape of a triangle. The bricks' numerical contents are bound to each other by the following rule: In each brick triad, the number within the top brick is the sum of the numbers in the two contiguous bricks directly below it. The task is to solve the puzzle-like problem by filling in the missing numbers. When the bottom-row numbers are given, the activity is close-ended, in that there is only one way to complete the triad. Thus, in the case of the *entire* pyramid, when all the bottom-row numbers are givens, the solution procedure consists simply of successive adding upward. That said, the bottom-row numbers could be chosen so as to suggest interesting patterns (odd and even, consecutive numbers, consecutive odd numbers, consecutive even numbers, and so on). However, when only the top number is given, the filling-in activity becomes open-ended in that there is more than a single solution to the entire pyramid structure. Furthermore, entirely blank pyramids may be presented as "own productions" (Streefland, 1990) in that students are invited to fill in with numbers of their own choosing. Finally, the pyramid we used for this study discloses the top number and several more, thus creating a system of constraints that emerges as determining a single solution for each additional input inserted, resulting in a structure with surprising properties. (We trust that the gentle reader who has diligently solved the problem on p. 1, may have been intrigued by a certain phenomenon concerning the number 24.)

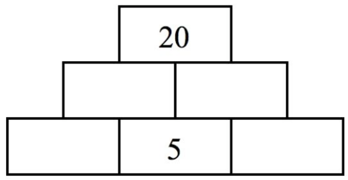


Figure 2. A Brick Pyramid problem supporting the reinvention of structured analysis

A Brick Pyramid such as the one in Figure 2 could be used, in the lower grades, for exploring the idea of compensation in subtraction (e.g. 20 = 10 + 10 = 11 + 9). In higher grades, this pyramid may be studied with the goal of bringing to the fore the utility of systematic problem solving as a means of finding all possible solutions and feeling confident convinced that possible solutions have been exhausted. In this case, the 11 possible solutions (using positive integers and 0) are easily determined by noticing that the acceptable numbers for the left brick on the second row (or, symmetrically, the right brick) are within the 5 to 15 range, inclusively. Interestingly enough, if the "5" in the original problem statement were simply moved one brick to the left (or right), the number of solutions that include integers and zero would be reduced to 8. This phenomenon can be explored at different levels of mathematizing, from using informal algebraic reasoning to operating with formal algebraic expressions. Furthermore, within a mathematics teacher-education context, such discussion allows for a change of perspective from (prospective) teacher to instructional designer.[5] Note that the Brick Pyramid is in fact a family of problems. Designing these problems—either as a teacher preparing instructional material or as a student engaged in extension activities—is possibly a richer activity than solving a single problem.

By adding more "floors" to the pyramid design and choosing carefully the initial numbers as well as their locations, the problems may become more challenging. Our Brick Pyramid problem (Figure 1) yields seven unique solutions, working with positive integers and 0. Although

[5] From an instructional design point of view, one may consider questions such as: How do we create brick pyramid problems? How do we adapt them for students of different ages? How do we design problems with a certain number of possible solutions, e.g. more than 5 but no more than 10? How do we design pyramids that elicit explorations about odd and even numbers, figurate numbers, etc.? What if we change the addition rule, which links two adjacent bricks to the one above them, to a multiplication rule? What mathematizing potential would this newly designed brick pyramid hold?

these can be found algebraically, as shown in Figure 3 below, the problem can also be approached with non-algebraic and proto-algebraic methods. Working with positive integers and 0, one possible algebraic treatment begins by assigning to the bottom-leftmost brick the variable *x* and then stepping upwards, sideways, and downwards, abiding with the addition rule for each triad, until all the bricks have been filled. This process indicates that the range of possible values for *x* is 0 to 6 (see the expression "6-*x*" for the bottom-rightmost brick), thereby giving 7 solutions. Interestingly enough, the rightmost brick on the second row is the only one where *x* "cancels out," resulting in the constant value of 24—a puzzling phenomenon that may merit an investigation of its own (i.e. Why 24? How is this number, 24, related to the four given constraints—280, 75, 31, and 13—and their respective locations on the pyramid?).

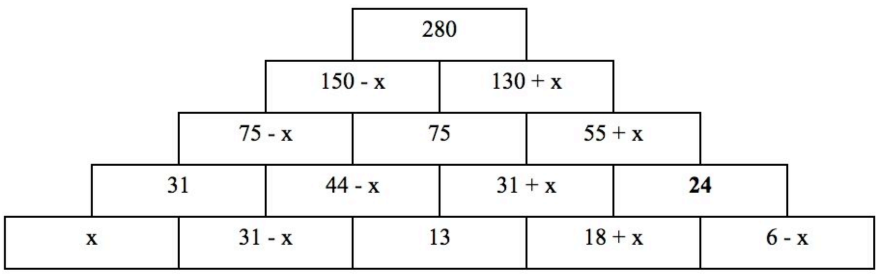


Figure 3. Algebraic treatment of the Brick Pyramid as a space shaped by numerical constraints

In sum, Brick Pyramid problems can serve as vehicles for exploring, at various levels in a continuum of progressive formalization, important algebraic ideas such as given and unknown, constant and variable, equations and systems of equations, and so on.

Following a Methods section, below, we present data elicited within the context of our experimental graduate-level teacher-education course, wherein the Brick Pyramid was treated as a pdmps.

4. Methods

Participants

Four graduate students—the total number of students enrolled in this experimental course a UC Graduate School of Education—participated in the study: Justin, Emily, and Nora—three students all in their first year in a Masters program, and Zoran—a student in his second year in a doctoral program.[6] The first author—the designer of this course—assisted the second author,

[6] All student names are pseudonyms.

remotely, in facilitating this course. Occasionally, the first author participated through video-conferencing.

Materials

Materials for this study included regular stationery employed in “hands-on” classrooms, such as crayons and scissors, as well as a computer-supported discussion platform (“bSpace”).

Procedure

A uniqueness of pdmps-based instruction is the apparent spontaneity of the lesson flow, characterized by ad hoc supportive reactions to students’ ideas, and the resourcefulness of the instructor, who brings to bear an intimate familiarity with the pdmps as a context for guiding students’ reinvention of mathematizing and didactizing. As in any typical graduate-level course, the course syllabus of the pdmps seminar specifies a list of weekly readings that, in turn, inform classroom discussions, which begin online, prior to each lesson, and are then summarized in situ. However, these readings are treated not as learning objectives onto their own. Instead, these serve as resources for illuminating and deepening students’ inquiry into the mathematical–didactical matters encountered in and through engaging in studying pdmps.

Class meetings lasted three hours. Typically, lessons begin with a “warm up” pdmps and continue with the main pdmps. For the main, students share their emergent solution procedures, often even as these ideas are emerging, which they become accustomed to present on the blackboard, and share their mathematical struggles and emergent questions about the didactizing of these pdmps—what doing so would require and entail. The instructor guides this procedure by probing with challenging questions, drawing in the literature resources, and generally maintaining a collaborative ecology. Cultivating such an ecology is essential for students to shift from a “solution mode” to a “solving mode,” whereby they recognize that sharing their thought *processes*, not just their thought *products*, is the very matter this course is made of.

That said, the inclination of a group of students to collaborate on the solution of mathematical problems, we find, is greatly contingent on micro-cultural practices cultivated by the teacher. Whereas some students may initially resist what feels to them as premature sharing, most of them ultimately realize that any such discomfit trades off with an authentic excitement of engaging in team work and the “therapeutic” virtue of recognizing that their classmates err as much as they do, as they explore solution procedures. This "therapeutic" virtue, in turn, may help these future teachers accept the fallibility of their own mathematical reasoning such that they would anticipate and accept such fallibility in their future students. That said, the emphasis placed by our pdmps approach to thinking aloud together should not be interpreted as marking our belief that mathematicians do or should always necessarily work likewise—it would be a naïve interpretation of the Vygotskyan legacy to ignore the crucial role of internal thought processes and to therefore coerce the sharing of ideas while these are still in the making, so much as to run the risk of precluding genuine reflection.

The focus lesson took place during the 5th week of instruction of a Fall semester. Three of the participants—Masters students—were new to the program. Their main course burden at the time, in addition to our course, consisted of a heavy-load introductory survey course that covered the learning-sciences literature. During the 5th week, course participants were just beginning to become involved in school placement observations and other fieldwork related assignments. In the classroom, we all sat around a single large desk, and stationery resources were pooled in the middle.

Data Collected

In order to evaluate the extent to and manner in which pdmps-based activities supported the development of participants' mathematical–didactical insights, we collected the following data: (a) video-taping of course sessions; (b) the instructor-as-researcher's field notes (c) participants' written work; (d) classroom artifacts resulting from student teachers' try-outs of selected pdmps; (e) records of on-line discussion boards; (f) final papers, which consisted of narrative accounts of a teaching experiment wherein course participants tried out a selected pdmps with students in their field-placement classroom; and (g) exit survey/reflection on the course.

Methods for Data Analysis

Text. As a situated practice that consists of making, operating with, and thinking about inscriptions, mathematizing is realized in and through language. This social semiotic perspective calls for attending explicitly to the language of mathematics (its register and multiple associated genres, e.g., word problems, explanations, argumentations, formal proofs). It follows from this perspective that spoken interaction and, in particular, whole-group discussions, can serve as a teacher-guided interpersonal gateway for enlarging the potential of what learners can mean using the spoken and written language of mathematics (Halliday, 1993; Zolkower & Shreyar, 2007; Shreyar, Zolkower, & Perez, in press). Furthermore, this social semiotic approach calls for the centrality of diagrams as thinking devices, level-raising models, or semiotic mediation tools (Lemke, 1998; Gravemeijer et al, 2003; van den Heuvel-Panhuizen, 2003; Radford, 2003; Bakker and Hoffmann, 2005).

The above ideas framed our analysis and interpretation of the two main kinds of data we generated in our research-design project: speech and artifacts.

Our rationale, acting as researchers analyzing the data corpus, much reflected our rationale as the designers and teachers of the pdmps and, reflexively, our rationale for how the course participants should facilitate these very problems in their own respective secondary-school mathematics classrooms. Namely, we sought to track the progressive mathematization of the Brick Pyramid, and in particular the emergence of proto-algebraic discourse and action. We applied this general rationale both to our graduate-level-classroom text and to the texts elicited by one participant, Justin, selected for this paper due to his choice to work with the Brick Pyramid in his school placement setting.

5. Results and Discussion: Reinventing Algebra by Thinking Aloud Together About the Brick Pyramid and Beyond

5.1 Overview

This section presents empirical results from our pdmps course. In order to communicate a sense of the entire trajectory this course takes—from classroom work through planning an implementation, conducting the implementation, and reflecting on it verbally and in a course papers—we have chosen to discuss only one pdmps, the Pyramid Brick problem, which we have discussed earlier in this paper. The "meat" of this section is a description of the lesson when this pdmps was first introduced. In addition, we present excerpts from Justin's preparatory work toward taking the pdmps to a public-school 11th-grade classroom as well as his narrative of conducting this teaching experiment.

5.2 Annotated Description of the Classroom Activity

In this section we analyze and discuss selected excerpts from a whole-group conversation about the Brick Pyramid problem (see Appendix A for the complete transcription). This text, made up of 432 turns (changes of speaker), consists of a series of exchanges, guided by the course instructor, whereby participants engaged in thinking aloud together as they attempted to collaboratively solve the problem at hand (Zolkower and Shreyar, 1997; Shreyar, Zolkower, and Pérez, in press).

Viewed as a curriculum micro-genre (Christie, 2002), the text was parsed into the following five episodes (see Appendix A for the entire transcription):

I. (1-38) Opening: Framing the Brick Pyramid problem
II. (39-176) Solving the problem: Thinking aloud together, scribbling, speaking, diagramming, gesturing
III. (177-230) Comparing and contrasting approaches and, at the same time, moving forward to other 'more algebraic' approaches
IV. (231-356) Reflecting back on the experience as reinventing algebra while moving forwards towards algebraizing the situation
V. (357-432) Closure: Projecting—a shift from pmps to p*d*mps—by considering the problem as a potential classroom activity while also discussing it from the point of view of instructional design

Following, we describe the overt activities in this text, i.e., what a non-omniscient "fly-on-the-wall" observer, who is engaged in the solution process, would witness. We begin with episode II, when students began to think aloud together. In the interest of brevity and clarity, we shall use the following coding system to refer to each of the fifteen cells in the Brick Pyramid (see Figure 4, below).

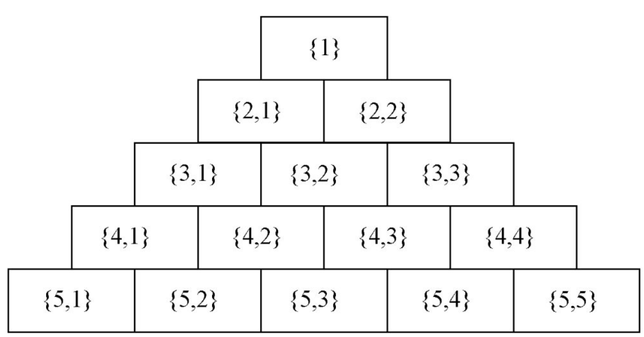
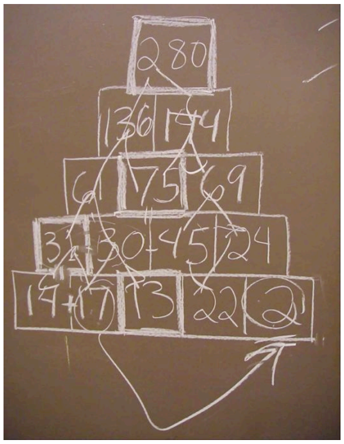


Figure 4. Notation system used in the transcription: In each ordered pair, the first numeral represents the row number, and the second is the cell number within that row

II. Solving the problem (39-176). Emily sets off the discussion by going up to the board and presenting her solution procedure (see Figure 5).

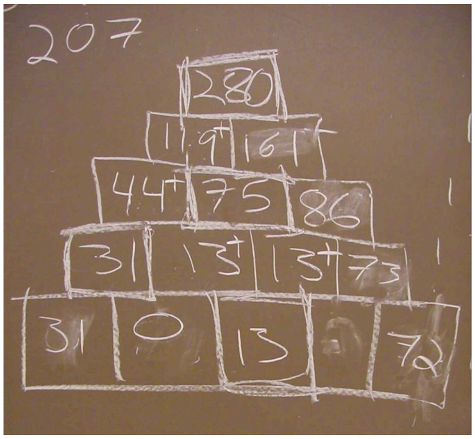


[62] ….“What I decided to do was, pick a value, put it somewhere in here [*indicates bottom row*], build off of that. And I figured I’d pick a value that was under one of these given, permanent numbers. So I put a number here that’s less than 31. [*indicates {5,2}*] Any number I wanted. [*enters 17 into {5,2}*] And I went with 17 because… I don’t know why….”

Figure 5. Emily: Uses numbers and moves from the bottom up

In order to begin familiarizing herself with the problem, Emily inserts a value, 17, which she selects somewhat arbitrarily, into an empty cell in the bottom row {5,2}, and then works that value so as to fill the entire pyramid according to the addition rule. As it turns out, assigning that particular value to that cell–variable is not permissible in this system. Yet, due to an arithmetic error, “45 – 13 = 22,” this violation appears to go unnoticed.

Next, Justin replaces Emily at the board and presents his solution (see Figure 6). Justin, possibly building off Emily’s work, is already more systematic than her, in that he orients toward searching for the range of the solutions, and so his choices of input numbers are governed by an attempt to determine one limit of this range. As he solves, he articulates emergent insights. Namely, Justin begins to realize that the pyramid can be viewed as a system, rather than as a collection of local overlapping three-cell triadic structures: because each sum is constituted by addends below it, recursively, there appears to be some overall systematic set of relations governing the distribution of addends. However, Justin does not as yet specify the nature of this system of relations.

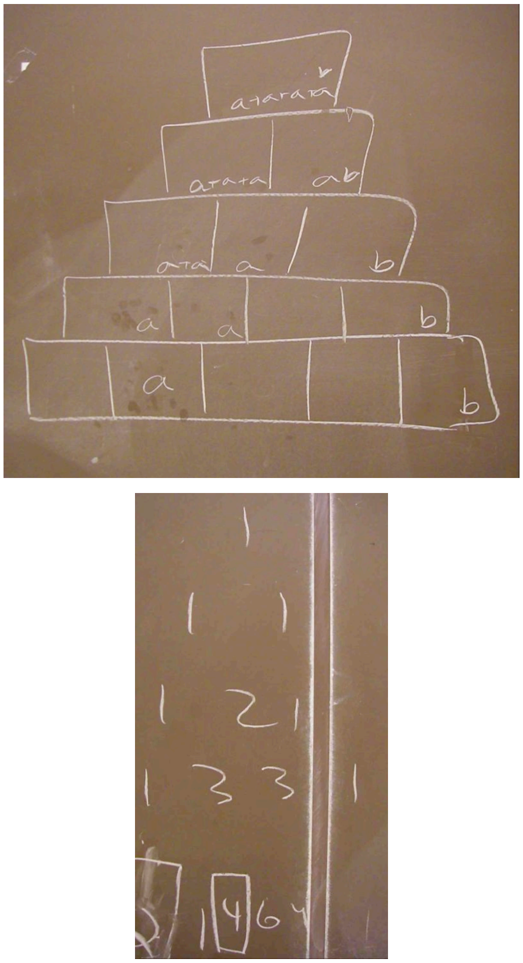


[71] "OK, so, 75 here, 31 here, 13 here, 280 here, oops, right. [*enters values as he utters them*] And so, <what> I did was, I looked at each box and thought, what are the limits bounding the number in each box? [*emphasizes boxes with given values*] So, I guess I kinda like started up at the top, and I was like, well these have to be 75 or more, [*indicates {2,1}, {2,2}*] right, and then…." Nora points to the 'less than' constraints in Justin's approach [152]. Dor highlights the double-using issue, which affects the way one distributes the values throughout the pyramid [159, 165]

Figure 6. Justin: Uses "numbers +" or "greater than" constraints and moves from the top down

Nora, commenting on Justin's work, adds that the in order to determine the range of values, one would need to find the other limit, too. Dor (the instructor and second author) comments on embodied constructions of addition that, he believes, may be implicitly biasing the solution procedure: (a) addition as "adding up," i.e., a privileging of up-ward-adding construction of the problem, at the expense of attending to the equally important downward constraints; (b) addition as "using up" resources—once a value is used in one sum, e.g., in conjunction with the cell to the right, it must be used again for the cell on the left, and this re-use might be violating the grounding multi-modal dynamical images that tacitly underlie the sense of addition. Emily responds that, indeed, the sum of the numbers on the bottom row is not equal to the number up top, so that her implicit model of the situation seems to have shifted.

Nora, thinking aloud, explores how—given the repeated use of addends—a sum that is written fairly high in the pyramid could be viewed as constituted by the addends below it. She recognizes that the relation is not a direct distribution of the sum into equal parts, but initially she cannot explain just how this distribution works or could be represented so as to support this inquiry. A significant move forward in the group's collective inquiry is when Nora uses algebraic symbols so as to explain how a numerical value on the bottom "ripples" up (see Figure 7). She distinguishes between values in the center of the bottom row (*a*) and those on the extremities (*b*), in terms of their contributions to the upward accumulating sums.



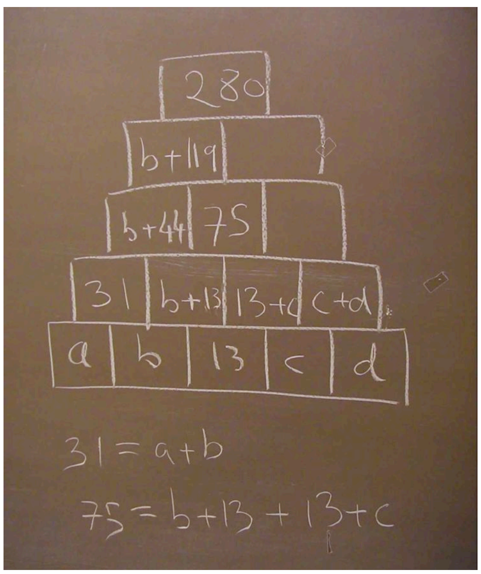
[175]: “So if we did it in this square [*indicates {5,4}*], we would end up, maybe we would only have to do, like, a fourth of 72, or something. Because then, by the time it gets up to the top, then you’ll have the full 72 that you want. But then I don’t know if that takes into consideration… I don’t know exactly how this ** but I think <it would fit it> somehow if you did some fraction of 72, then, you know, it’s gonna be multiplied here, there’s gonna be… two of them up here, and then that’ll count as one, and that’ll count as two, so that’s three. [*indicates boxes above {5,4}*] If we had some number here, then we’d have that number, then twice that number, then three times that number, and then…”
[185]: “If we had some number here [*enters “a” into {5,2}*], then… [*draws next row*] OK, well, kind of ignoring the numbers that are already in there, once you go up here, so, well, it’s like what you were saying, we feel like OK, if the *a* is here then it’s already taken care of, we don’t need to worry about it again, but actually it’s gonna – you have to count it again in this box and in this box. [*enters a into {4,1}, {4,2}*] And then, when you come to the next level….”

Figure 7. Nora: Enter *a*, then *b*; link to Pascal’s Triangle

Up to this point, the course participants have each made unique contributions to the collaborative problem-solving process. One can discern a progression, from Emily’s first hesitant exploration of a single solution, through Justin’s analysis of the pyramid as an emerging system of constraints, to Nora’s introduction of algebraic symbols in an attempt to spell out the spread of upward addend contributions of a single number on the bottom row, depending on whether it is in a central or extreme brick, and how two such “deltas” (the spreading contribution tributaries) mingle in an addend confluence. The course instructor insistence that the work be done up at the board had two related results. On the one hand, students were initially diffident to share half-baked ideas. On the other hand, these ideas, like Nora’s *a* and *b* addends, appear to have spread and mingled upwards, receiving at each level the input and reformulation of additional minds on tap.

The narrative now continues. We have parsed the discussion at this point, because we sense a slight shift in the collaborative process: having lain all the cards on the table—that is, having placed on the blackboard their various approaches to the problem—the students now turn to offer comments, modifications, and further insights that appear to go *across* the contributions.

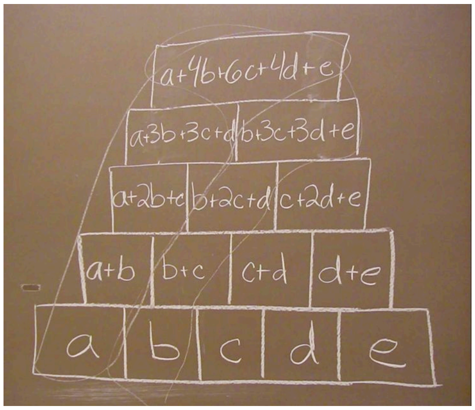
III. Comparing and contrasting (177-230). In this turn, which ensues directly after, Zoran elaborates on the work of Emily, Justin, Nora, and Dor. Zoran, too, recognizes the inadequacy of trial-and-error techniques to cope with the load of arithmetical constraints imposed by the pyramid system. He acknowledges the value of the algebraic system introduced by Nora, to support this inquiry. Zoran introduces additional variables (*a* through *d*), in order to articulate general relations among the fifteen cells of this particular pyramid, see Figure 8).



Zoran [177] continues discussing the distribution issue, talking in terms of ‘sets’: “With this we have this set here, and this set here… this set is only going up to 31”…; concludes that it’s too much work to try doing all of this “purely arithmetically,” thereby calling for an algebraic approach

Figure 8. Zoran’s Brick Pyramid: Enter, *a*, *b*, *c*, and *d*.

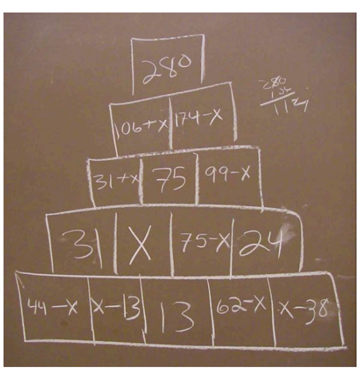
Emily, possibly building on Nora and Zoran’s suggestion, expands the number of symbols to five (*a* through *e*) and uses these ‘variables’ to demonstrate attributes of the pyramid family, regardless of the given numbers. In particular, Emily develops the idea that Pascal’s Triangle can be viewed as lodged upside-down in the pyramid (Figure 9).



[220] “Say you just have your numbers down here: a, b, c, d, e. [*enters letters as she utters them into bottom row, in that order*] OK, there’s one of each of those, I’m just saying they’re… distinct, or they could be <one>, I don’t know, um, they’re variables. So, to get here, you do a+b, and this is b+c…” [227] “…. this is sort of becoming the, uh, Pascal’s Triangle in reverse. And this one would be b+2c+d. [*enters value into {3,2}*] This would be c+2d+e. [*enters value into {3,3}*] And up here, I need to start making the bricks bigger. And now, we have a+3b+3c+d, I think. [*enters sum into {2,1}*] Someone correct me if I’m wrong. And then b+3c+3d+e. [*enters sum into {2,2}*] And for the last one… Make it a little bigger…! [*expands first box borders*] OK. a+4b+6c+4d+e. [*enters sum into {1}*] And then, <see> you have 1, 4, 6, 4, 1, just like this guy. [*indicates Pascal’s Triangle fragment on the board*] And this is just like coefficients.”

Figure 9. Emily again: Enter *a, b, c, d, e* in the bottom row; more explicit link to Pascal’s Triangle

IV. Reflecting (231-356). The instructor, apparently feeling this to be an auspicious moment to enrich the conversation by situating it within state-of the-art learning sciences debates, introduces the following constructs as they apply to the joint mathematizing effort under way: “sprouting” of algebraic notation from speech and gesture (Radford, 2003); “distributed cognition” (Hutchins, 1995); “cognitive artifacts” that amplify reasoning (Norman, 1991); “working memory” (Baddeley & Hitch, 1974); and “hypostatic abstraction” (Peirce, 1931 – 1958). This mini lecture may or may not have any direct impact on the collaborative solution process. In any case, Nora—possibly also inspired by Emily and Zoran’s attempts, then suggests working with a single variable—x—and expresses all of the numerical values using this variable (see Figure 10).



[279] “So I picked this box right here. Just because it was touching so many other things, I figured I could get a lot of information out of it. [*places x in {4, 2}*] So this one is 31+x, and then this is 106+x, and then this is 75-x, and this is, let’s see, x-13, and then….”

Figure 10. Nora again: and then there was just x

What, then, makes the Brick Pyramid indeed 'paradigmatic?' In his homage commentary on the life of Hans Freudenthal, Goffree (1993) likens paradigmatic mathematical problems to scientific benchmark discoveries following which the scientists' extant paradigm shifts and the world is forever seen in a new way. Of course, our course participants are mathematically informed adults, according to normative standards, such that the algebraic machinery is quite at their finger tips, ready for application. The question is whether and when this application is triggered. Traditional instruction for the most part calls for algebraic solution procedures simply by token of the chapter title, e.g., "Algebra Word Problem," not unlike the proverbial Euro-bargain-tourist who knows he's in Venice because it is Monday. On the contrary, pdmps invariably present a scenario that calls for mathematizing and didactizing without furnishing a specific toolbox let alone a particular tool. Implicit in this principle is the belief that pedagogical approaches to mathematics instruction that deny students the opportunity to search for appropriate tools to structure or organize problematic situations have limited effects in developing in them a genuine disposition towards mathematizing. In other words, the strong claim we make here is that to the extent that students are deprived of opportunities to discover/uncover the mechanisms of a mathematical tool, such as algebra, they will never be fully "connected" to that tool (see Wilensky, 1997, on the consequent "epistemological anxiety"). Thus, although our students were not "learning algebra" in the traditional sense of the word, they were guided to reinvent algebra, in the RME sense of that phrase.

This emergence of algebra, we argue, was authentic:

a) Algebra was not dictated as a preferred solution method but arrived at bottom-up from the problem, stumbled upon as if haphazardly yet progressively recognized as a suitable approach, just as in real-life problems that do not come with instruction manuals;
b) Specifically, algebra was identified as addressing a computational challenge, i.e., the cognitive overload of constraints imposed between the elements of the problem;
c) Algebra offered unique affordances, i.e., the articulation of general situations and therefore the pursuit of general solutions, i.e., the reinvention of a variable which functions as a placeholder for all possible values in a given location on the Brick Pyramid;
d) Thus, algebraic *reasoning* impelled algebraic *symbolization*—the symbols were certainly enabling capstones in the solution process but they crystallized through the

need to inscribe the set of articulated constraints—again, unlike traditional curricula that invariably introduce the application of mathematical objects prior to establishing their grounding in common sense;

e) even once it was suggested, algebra was still debated, modified, and evaluated, i.e., witness how Zoran, Nora, and Emily progressed from *a* and *b*... to *a*, *b*, *c*, *d*, and *e*… to simply *x*;

f) Algebra was thus honed in response to the group's emerging appreciation of the potential of the pdmps as a didactical–mathematical artifact. That is, the debate over whether all fifteen cells or just the empty cells should be expressed algebraically was contingent on perceptions of the *purpose* of these algebraizations vis-à-vis the level of mathematization engaged as well as its orientation ("horizontal" or "vertical"); and

g) The pdmps created sustained engagement, afforded multiple solution procedures, was conducive to the generation of diagram that could be elaborated with arrows and other spontaneous symbols, enabled students to make their reasoning explicit to and for others, and fostered a shared experience that would be repeatedly cited in the future (i.e. a memorable experience, that is perhaps the type of paradigmatic experience that is the condition for durable learning).

Participants in the course engaged in collaborative problem solving focused on presenting, discussing, and elaborating multiple approaches to the Brick Pyramid. The Brick Pyramid acted as a shared, index-able artifact that enabled individuals to make manifest their thinking to others (and to reason with others in and through those very inscriptions).

As a result, the problem solving process became distributed among the cohort, both explicitly and implicitly. Explicitly, students: (a) explained their reasoning using speech, inscription, gesture, etc.; (b) received feedback on the mathematical validity of their assumptions and productions; and assigned each other tasks, often based on an emerging division of labor that made manifest, radicalized, and capitalized on a tacitly assumed diversity of expertise within the cohort yet (the "mathematician," the "intuitionist," the "designer," etc.). Technically, these explicit discursive acts were instantiated in the form of each student, in turn, making salient to other the properties and relations they were first noticing among the emerging values in the grid. Implicitly, these public discursive actions triggered in the active recipients' idiosyncratic responses that had not necessarily been realized by the speakers yet could then be brought to bear back into the thinking-aloud-together plane.

The data collected made salient a process of building meaning for a semiotic artifact. The process was monotonically incremental yet demarcated by pivotal moments of insight (aha! moments), when the mathematical machinery of notions, inscriptions, and strategies appeared to rise an echelon. The discursive norms that were highly tolerable to half-baked ideas, imposed generative constraints on students' algebraic thinking in *statu nascendi*. Namely, students were impelled to objectify their emerging notions in the form of articulated properties of the semiotic artifact, e.g., grounding their observations in structural elements of the Pyramid, relations among elements and values, and related mathematical constructs.

We propose that the Brick Pyramid served as a polysemic semiotic artifact, in that each member of the cohort, at different moments, could build different sense for it as a whole or for singular elements of the Pyramid, which in turn allowed it to be constructed in manifold ways. It is thus that the Pyramid drove, facilitated, coordinated, and anchored the distributed problem solving. Namely, different personal framings of the object were co-present in the thinking-aloud-together jointly constructed conversation text, creating opportunities for individuals to juxtapose different notions. We witnessed a collage of notions cohere into a patterned mosaic that captured the math-didactical value of the Brick Pyramid as a paradigmatic instantiation of an algebraic system.

On paper or on the blackboard, the Pyramid per se appears as just a grid with numbers. Yet the Pyramid is not just a "problem-solving tool." It is more like a "partner in cognition" (Salomon, Perkins, & Globerson, 1991) with a potential residual effect: it is more like an "object to think with" (ascribed to Seymour Papert), a thinking device (Lotman, 1988). That is, what is ultimately learned is not about pyramids but about algebra. By mathematically structuring or organizing the Pyramid, the students re-invent algebraizing (Freudenthal, 1991). Lyrically speaking, the Pyramid—once a curious inscribed structure on a worksheet—becomes colorful, animated, mobilized, and enmeshed in multiple intersecting dimensions of mathematical thinking thereby functioning as a "model for" thinking about the very meaning and purpose of algebra as a situated human activity.

V. Projecting (357-432)—from mathematizing to didactizing. At this point, the instructor was satisfied that adequate progress had been made in the context of working on the Brick

Pyramid problem. It was now time to move the discussion toward considering the prospects of trying out the problem with school-children, in accord with the course requirements. Therefore, the instructor posed the following question: "So, what do you think might happen if you take this problem to middle school?" In response, Justin commented:

> [358] Well, you'd definitely want diversity in solutions. So that's something you'd, I would focus on…Not, you know, when presenting the problem… Don't make it seem like there's one particular way. So you can get the birth of algebra…
>
> [360] Because then some students will have more algebraic thinking incorporated in their answers as opposed to the others, and you can have more ways of thinking to choose from, to try to get to that… if that's what...
>
> [368] I guess if you ask them to come up with multiple solutions, then they would necessarily be able to see how the different solutions relate, or something. So, if we did Emily's way, like Emily picked 17, then let's say, OK, well this is the way I can find solutions, just picking a number and filling it in, right? But even if they did, so let's say they did 17 and on the next one they say, OK, let's do 10. Then, as they're filling it out, they might see a pattern, oh, every time I move to this next block, I'm just doing… I don't know what it would be, 31-17 to get the next one or 31-10, or 31 whatever. Maybe they can see... kind of see the patterns and maybe that would help them move to the more generalized…Thinking of it as general patterns, general rules, instead of just-
>
> … [429] I like how this… creates the need for algebra by extending arithmetic. And so, I mean… yeah, showing a kid *a* plus *a* equals 2*a* is just kind of cramming it down their throats, whereas, like, you could get to that later, but first, I mean, maybe not exactly presented that way, but, um, having and creating a need for their motivation for having symbols is really important, I think. And I think this does a good job of it.…

We have reproduced Justin's words verbatim here, because—as we are about to witness—these words augur quite accurately key aspects of Justin's actual implementation of the Brick Pyramid problem in an 11^{th}-grade classroom where he was placed as part of his teacher training. Specifically, we draw from the above Justin's attention to diverse solution procedures, which he perceives as: (a) a condition for eliciting the generative interplay of thinking aloud together that begets algebra as an authentic guided reinvention; (b) inviting more students into the joint text building in that each student could grab onto someone else's idea as an entry point to further his/her thinking; and (c) conducive of reflection as a means of reconciling the different views co-present in the discussion. In addition, and possibly inspired by the course readings for that day (van Ameron, 2003; van der Kooij, 2001) Justin speaks to the virtue of supporting pre-symbolical algebraic reasoning.[7]

5.3 Justin Prepares to Conduct His Field Work: Implementing the Pyramid PDMPS in School

As the above narrative of the Brick Pyramid may have conveyed (see also Appendix A), our course participants evaluated this problem as a potentially powerful pdmps. Justin, in particular, had opined that this problem could enable him to conduct an effective algebra lesson in his school placement. The instructor encouraged him to pursue this plan, and several weeks later Justin implemented the Brick Pyramid with four 11th-grade students in a local public high-school.

In the final paper he submitted in partial fulfillment of our course requirements, entitled "Embers of Reinvention: Almost Reinventing Algebra," (p. 4) Justin reports that he presented the problem to the school students in the following manner:

1) Find a solution to the puzzle, that is, fill in the pyramid with numbers.
2) Describe your thinking in writing. What was your strategy? Did you notice any patterns? Any other features of the puzzle that you found interesting?
3) What do you think is the best way to complete the Brick Pyramid? Why? Make sure you indicate clearly where the starting point is.
4) Find all possible ways to fill in the Brick Pyramid

As he further reported (p. 6), Justin anticipated that students would possibly need more support in making sense of the problem, and so he prepared in advance several "problematizing" questions, as follows:

> I understood that students may have trouble making the transition from arithmetic to algebra, and might need guidance during the reinvention process. For this reason I arrived at a list of problematizing questions that I thought would be useful to guide students toward the use of algebra. I pre-formulated questions that might encourage them to extend their solution into new representations, for example: What is your strategy? Will it work every time? How do you know? Can you generalize your strategy? I also prepared additional problems in case there was extra time, for example: "Can you find an algebraic solution for the top brick of an empty pyramid?"

As the title of his term paper discloses, Justin's try-out was hardly as rich as the corresponding college experience. And yet, Justin himself, his college classmates, as well as the instructor were

[7] Appendix B presents the contents of an online discussion that followed the Brick Pyramid lesson. We shall not analyze this discussion here, yet the reader is invited to peruse the text so as to appreciate students' reflections on their experiences, the integration the readings, and the thoughts ahead toward an implementation (in particular Justin).

all impressed that he had been able to operate so well under the circumstance. Justin's implementation is the focus of the next sub-section.

5.4. A Secondary-School Classroom Try Out of the Pyramid Problem

In this section we discuss one narrative account of a try-out of the Brick Pyramid problem with four grade 11 students. The account was submitted by Justin, one of the course participants, as his final paper (see Appendix C for selected excerpts).

Justin's report is tinged with a certain sense of disappointment. It transpires from his reflective narrative that he had hoped that those four 11th-graders had been able to recognize the relevance of using their (already acquired) algebraic skills and tools in order to find all possible solutions to the Brick Pyramid. Or, in the event that they were not able to do spontaneously, Justin seemingly wished to have had the opportunity to guide these students along the very same pathway of reinvention he and his classmates had followed, during the course session, as they thought aloud together about how to solve that problem. Notwithstanding the above, Justin reports that the students were truly intrigued by the problem, produced different solution approaches, and engaged in spontaneous collaborative work. Granted, it would have been gratifying for him to be able to report here on a crowning success, e.g., that the 11th-grade cohort was guided to collectively reinvent algebra and/or that they were all able to explain the recurrence of 24 in all seven solutions found.

Various logistical constraints militated in unison against this achievement. A time constraint (10 minutes to teach what had been planned as a 45 minute lesson [sic!]) as well as Justin's own need for closure, which led him to "spoon-feed" algebra as the time was running out, defeated the purpose of this short lesson. One could explain this phenomenon as the (rather typical) breaking down of the didactic contract as an effect of the two internal constraints: time and the teacher's own beliefs about mathematics, learning, and teaching (Brousseau, 1997). Yet, we submit that Justin's accomplishment lies in his didactical–mathematical reasoning, as illustrated in his report, much more than it is or could be in guiding the mathematizing efforts of the 11th-grade students he worked with. At the time, Justin was a teacher in training with less than one semester of class hours as an observer, and as such, we do not expect him to expertly tackle the manifold aspects of teaching on his very first attempt, especially with this demanding type of curricular material. Moreover, his reasoning is a testimony of precisely the effect we had wished to impact on participants through this course, and the tryout was only a partial evaluation of this effect.

What we find most impressive is the range of educational issues that Justin addresses in his paper: he treats algebra as a system (mathematics); reflects on his practice as a teacher (didactical–mathematics), in light of cumulative understanding of realistic mathematics education as well as other relevant research literature (scholarly input); comments on the experimental unit (teacher as instructional designer); and learns about his practice through analyzing video tapes and lesson transcripts (teacher as researcher). Finally, we read in Justin's reflective narrative an attempt by a prospective teacher at creating a cohesive view of mathematics instruction that includes teaching, designing, reflecting, and researching. That vision, namely that the mathematics teacher should be an informed reflective practitioner who can feel at home in teaching, designing, and analyzing teaching/learning episodes, is spelled out in our course syllabus.

5.5 Coda

It is an annual tradition at the second author's Graduate School of Education to celebrate our work with a Research Day. The pdmps course participants put up a poster show that attracted many attendees. Justin's Brick Pyramid problem, in particular, created quite a crowd, with some colleagues sitting down at available desks to solve the problem. Spontaneously, the conversation took this pdmps to levels quite unexpected of high-school students. It was remarkable to view Justin carry the vertical mathematization of this deceivingly simple grid to those heights.

6. Conclusion and Closing Remarks

We have found that the Brick Pyramid is a unique vehicle for eliciting the reinvention of algebra among people, like the participants in our course, who are well-versed in using the symbolic machinery of formal algebra. This is so in great part because the problem hides its algebraic meaning. That the call for algebraizing is implicit allows for re-creating algebraic thinking and doing so from scratch, as opposed to inviting the application of algebraic tools. This property of the Brick Pyramid, which is one it shares with other such problems (e.g. arithmagons, cf. Wittmann, van den Kooij, 2001), makes of the problem an invaluable tool for mathematizing and didactizing, in that prospective teachers can easily put themselves in the place of their students, thereby un-learning the algebra they already know. When guided by an instructor who can gear the conversation such that algebra is jointly brought into being in and through the here-and-now exchanges, this imaginary travel to the past is likely to prepare prospective teachers for their future role as guiding their own students' reinvention of algebraizing.

In designing and promoting this course, we are conscious of valuing depth of experiencing over breadth in coverage. When a graduate course is built around a methods-for-mathematics-teaching textbook with its complementary readings, it seems rather certain that one would get through the syllabus. However, when mathematical content, pedagogical subject matter, and assigned readings are positioned as ancillary to the 'actual' learning, one may not be as certain to cover it all (and perhaps even more troublesome, the course is unlikely to be instructor-proof so as to enable standardization). Yet, as a tradeoff, there is the perceived opportunity that participants would have rich experiences of learning "what really matters."

And what is it that matters, really? Learning to teach mathematics involves reflective back-and-forth movements between classroom practice and instructional theory; mathematical content and didactical form; observing one's learning processes and observing those of others (Freudenthal, 1991). To the extent that our pdmps engage future teachers and mathematics education researchers in working on and thinking through mathematics problems, mathematics learning problems, and problems of learning how to teach mathematics, this course is itself paradigmatic of a teacher education practice which, by grounding college-classroom discussion in actual school-classroom experiences, takes on the challenge of making the learning sciences matter.

7. Limitations

The pioneering character of this study and, in particular, the relatively small number of participants, lack of a control group, and exclusively qualitative nature of the analyses, do not enable us as yet to substantiate, in the form of broadly accepted generalizable claims, our strong personal conviction in the promise of this RME-inspired pdmps course to engender didactical-mathematical literacy in prospective teachers. For example, we cannot guarantee that equally productive collaborative problem solving would be the experience of any college cohort that used our syllabus. Similarly, we cannot reject the plausible critique that Justin would have done just as well in his classroom tryout without the benefit of his experience in our course. Finally, at this point we cannot isolate unique contributions of the classroom discussion, the online discussions, the problem solving, and the reading. Rather, we have attempted to persuade the reader in the efficacy of our approach by sounding deeply shared convictions of many mathematics-education researchers. That said, in the United States and elsewhere the mathematics education research field is witnessing a surge interest in RME work, and empirical studies that lend validity to our convictions are increasingly available (e.g., Shreyar, Zolkower, & Pérez, in press).

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Appendix A: Excerpts from the Brick Pyramid Classroom Discussion

Emily

[47] I guess what I was thinking is after you pick one **number** <that> pretty much tells you what ** You can be creative a little bit more, but it all starts falling into place after one. So we have… the bottom **row**. [*draws the bottom row*] Right, there's 5 **boxes** in the bottom, and then… [*continues drawing the pyramid frame*]. Three, then… [*to herself*]….

[62] So, that was **given**. 13 in the middle, <right up here>, 75, *** and on the top, 280. [*enters values as she utters them, emphasizing boxes*]. All right. So those **values** are **unchangeable** because they are **printed in there**. What I decided to do was, pick a value, put it somewhere in here [*indicates bottom row*], build off of that. And I figured I'd pick a value that was under one of these **given, permanent numbers**, so I put a number here that's less than 31. [*indicates {5,2}*] Any number I wanted. [*enters 17 into {5,2}*] And I went with 17, because – I don't know why. So then in order to get up to 31 I **had to find** the other value, which when added to 17, would equal 31. Which is 14. [*enters 14 into {5,1}*] And, now, more, more values can be placed in because I know that the value of this **has to be** the **sum** of 17 and 13. [*indicates {4,2}*] So that's 30. [*enters 30 into {4,2}*] And now I can figure out what this box is. [*indicates {3,1}*] I **have to sum** these two [*indicates {4,1}, {4,2}*], which would be 61. [*enters 61 into {3,1}*] And I see that 75 is given, I **have to get up to** 75, and I have 30 here, so the value here [*indicates empty box under 75*] **has to be** the rest of the way up to 75, which is 45. [*enters 45 into {4,3}*] And… now I **had to**… sur, er, sum these [*indicates {3,1}, {3,2}*] to get this **value** [*indicates box above {2,1}*], because this **is going to be given** - given this, given this, given this [*indicates boxes lined up underneath {2,1}*]. All of them are building up, so this was 136. [*enters 136 into {2,1}*] And… in order to get to 280, we **have to add** 144. [*enters 144 into {2,2}*] So, that's **a given** [*indicates 280 in {1}*], so I **can't break from** that, so I have to put 144 there. Now I **have to get to** 144 using 75, so I subtract 75 from 144 and got 69. [*enters 69 into {3,3}*] And I **had to get to** 69 from 45 and something else [*indicates {4,4}*]… which is 24. [*enters 24 into {4,4}*] And now there's, uh, I **have to get to** 45 using 13, that's **unchangeable**, so… uh, 45 minus 13 is 22 ([*sic*], incorrect)]. [*enters 22 into {5,4}*] And 24 **has to be there** because of all the other **numbers**, and 22 **has to be** there, so 22 plus 2 equals 24. [*enters 2 into {5,5}*] And that's what I did.

Justin

[71] OK, so, 75 here, 31 here, 13 here, 280 here, oops, right. [*enters values as he utters them*] And so, <what> I did was, I looked at each **box** and thought, what are the **limits** bounding the number in each box? [*emphasizes boxes with given values*] So, I guess I kinda like started up at the top, and I was like, well these **have to be 75 or more**, [*indicates {2,1}, {2,2}*] right, and then I went down here and this one **has to be more than 31** [*indicates {3,1}*], oh, so then this one **has to be actually more than 31 and 75 together** [*indicates {2,1}*], and I eventually went down to the bottom, right, and was like, this one **has to be at least 13** [*enters "13+" into {4,2}*], and I used that notation, **13 or above**. So is this one. [*enters 13+ into {4,3}*] **Has to be.** This one now, **has to be over 44**. [*enters 44+ into {3,1}*]

[79] All right, so… um. Exactly. Right, OK. So, **this one has to be 13 or above**, **this one has to be 13 or above**. That makes it… that makes **this one, constrains this one to 44 or above**. And

then, **this one being 119 or above**. [*enters 119+ into {2,1}*] And then, let's see, so this one has to be at least 13 or above. [*enters 13+ {3,3}*] This **has to be 88 or above** … right? [*enters 88+ into {2,2}*] And now, the sum of these two [*indicates 119+ and 88+*] is one, two… two, uh… Oh my gosh, ****.

[85] 207? Is that right? 207. [*writes 207 on board, next to the pyramid*] OK. So, the difference then is, uh, 73, right? So, 73 is [*writes 73 on board, under the pyramid*], has to be added to one of these two [*indicates {2,1}, {2,2}*] in order to make this 280. So I was just like, all right, I'll put 73 down here [*enters 73 into {5,5}*]. And make all the rest zero. But, pretty much, that made me, that, that let me know that all of these different boxes **could have 73 distributed between them** [indicates some of the empty boxes on the pyramid]. But then * I was <starting> to figure out, **if you put 73 here** [*enters 73 into box under {5,2}*], that gets added to both of these squares [*indicates {4,1}, {4,2}*]. So, so I just put it there [*indicates 73 {5,5}*], and I made all these 0 [*enters 0 into remaining empty boxes and overwrites 73 in {5,2}*], and I was like, all right.

Nora
[152] What I was thinking when you started it, is, cause you were saying you had the constraints, but… you only had the **greater-than constraints**. But then there also, if you went from the top to the bottom then you would also have **less-than constraints**, and I don't know if that would actually help you, but… it seems that at the beginning it would at least give you more information about what each… actually…

Dor
[157] So it seems that, I think, what you, what you, uh, what we neglected in this approach is the fact, it's something you commented on in passing, which is why I was trying to point it out. At one point you said, "Well, if I put 72 there, it will add up to both here and there." Remember when you said that?...

[159] In the bottom of the line, second to the left. And I think that aspect of the structure was neglected. Because we take it, I think just implicitly, we think of them as all kind of – every **row**… it's as if there's the bottom row, and then there, it just builds up from there, but… We say that, but we don't actually attend to the fact that **each bottom row of numbers is going both to here and there**. And I think that what – the reason that happens is… that… might sound fanciful to you, but implicitly we have this kind of image, that we're taking the number from the bottom, and, like, we've added these two, and they're there. And it's like we've already taken this stuff, we've already used it and put it there. How – it kind of seems to violate something very visceral that now we're taking it again and putting it here, because we've already used it up. So maybe just put half here and half there, but it's violating something dormant or implicit or… just that we can't quite access, but it seems to impact the way that we're looking at this image. Did that sound vaguely right to you, or does it sound too fanciful to you?

Emily
[164] Oh, while you were saying that I was thinking, oh yeah, **what happens if you add all the ones on the bottom**? And then I got, like on mine, 86, and then I was like, that's not the same as the sum of <anyone> and that's not 280, and <u>yeah, so</u>…

Dor
[165] So, I think that what in our mind, and I'm not sure quite where this'll take me, but...[*begins sketching diagram on board*] I think we have a different kind of structure in our mind, as if it's… Well, it's not gonna be pretty… but as if it's something like that. You know? This plus whatever… uh, 3 and 4, 7, and 5 and 6, 11, so this is 18. [*enters values as he utters them*] I think something of that is coming in here as we're looking at the bigger structure. Because here there's no **double-using.**

Nora
[175] So this kinda showed me that there might be some kinda solution. But, so, if we had 1, and then 1, 1… [*enters first 5 rows of Pascal's Triangle*] So, if we put something here [*indicates {5,4}*]… like… maybe, so, **we want 72 additional up here** [*indicates {1}*], right? So **if we did it in this square** [*indicates {5,4}*], we would end up, maybe we would only have to do, like, **a fourth of 72**, or something. Because then, **by the time it gets up to the top, then you'll have the full 72 that you want**. But then I don't know if that takes into consideration… I don't know exactly how this ** but I think <it would fit it> somehow if you did **some fraction of 72**, then, you know, **it's gonna be multiplied here, there's gonna be… two of them up here, and then that'll count as one, and that'll count as two, so that's three.** [*indicates boxes above {5,4}*] **If we had some number here, then we'd have that number, then twice that number, then three times that number**, and then… <that same>… I don't know, something like that. ** <keeping anyone that could be more useful?>

Zoran
[177] I think you certainly have to distribute the 72, because we're not really, I mean, **a part of that 72 has to get**, cause right now the biggest problem is that we have those two 13s. I mean, that's what we first noticed, right, those two 13s that add up to 75? So the 72 right now isn't contributing to that 75 at all, but it has to be, just, I mean, it just has to be, because that 31… That 31, cause, see we have, two numbers on the, ** I guess I should get up <instead of just pointing from here>. [*moves to blackboard; Nora moves back to seat*] But we have, uh… We have these. I think <with this> we have this set here [*indicates {5,1}, {5,2}*] and this set here [*indicates {5,4}, {5,5}*]. And so, this set [indicates *left set*] is only going to go up to 71, I mean, this set is only going to go up to 31. And we said we already have a 72 here [indicates *right set*], right, and zeros [*indicates bottom row*]. So what you have to do is, um, **what, I think, we have to do is distribute 72 across, somehow**. To put it here [*indicates {5,4}*], because we have to make this 75. We're not going to be able to, well, we can put a part of this here [*indicates moving from {5,1} to {5,2}*], and then it'll contribute to the 75. **Cause this 31 is not contributing to this 75, right? Because it's only affecting this one, this one, this one, and this one**. [*indicates, with ascending motion, boxes above {5,1}*]. But if you put a part of it here, it'll, like, cascade this way. [*indicates upward effect emerging from {5,2}*] If you put part of the 72 here, it'll affect the 75. [*indicates upward effect emerging from {5,4}*] **So we have to use some parts of this to affect that 75**. But, uh… I think, I think this here, I think it's too complex. [*indicates Dor's diagram*] If it were maybe like had three layers, we might be able to do it. But I think that keeping everything in mind is, is just too complex without writing algebraically the restraints that we're dealing with. At least it's too complex for me to think about, oh, I can do this but I have to – **this number can only be so big, but it can't be**… **Like, this number, um, has to be 44 or more, but there's also a restriction on how high it CAN be**. So **there's a minimum bound, and there's also a maximum bound**. And keeping track

of all of that, I couldn't, I mean, I can't do it. I'm sure there's some people who can. It just seems, it just seems like it's too much work, to do… to do it, you know, purely arithmetically.

Nora
[185] Well, I don't know exactly how it works, but if we had… [*draws bottom row*] Oops. If we had **some number here** [*enters "a" into {5,2}*], then… [*draws next row*] OK, well, **kind of ignoring the numbers that are already in there**, once you go up here, so, well, it's like what you were saying, we feel like OK, **if the *a* is here then it's already taken care of, we don't need to worry about it again**, **but actually it's gonna – you have to count it again in this box and in this box.** [*enters a into {4,1}, {4,2}*] And then, when you come to the next level, I mean, this isn't all that will be in here, there'll be other things, but **you have to remember to count it here and here.** [*indicates places where a is already entered, completes next row*] And then, once you get up to this level, then **this *a* [*indicates {4,1}*] will be counted here [*enters a into {3,1}*], but then this *a* [*indicates {4,2}*] will also be counted here [*adds a into {3,1}*] and here *[enters a into {3,2}*]. And the, by the time you get up… [*draws next row*] Then you have, well you have this *a* + *a* [*enters a + a into {2,1}*], but then you have this *a* here [*indicates a in {3,2}, adds it to {2,1}*], and this a also here [*indicates a in {3,2}, enters a into {2,2}*]. And then, in the top one, you'll just have… [*enters a + a + a + a into {1}*] You'll have these three a's and this a, they're all being fed into there.** [*gestures upward towards top box*] Does that make sense?

[193] But that's what I was trying to say originally. **Well, let's say you had something here. [*enters b into {5,5}*] It would be counted here, here, here, and then up here.** [*enters b into all boxes above {5,5}*] Actually, that's why, that was Justin's reasoning of why he put it on the right-hand side, so that you didn't get all this double counting.

Emily

[220] … **Say you just have your numbers down here: *a, b, c, d, e.*** [*enters letters as she utters them into bottom row, in that order*] OK, **there's one of each of those**, I'm just saying they're… distinct, or they could be <one>, I don't know, um, **they're variables**. So, to get here, you do a+b, and this is b+c… Can Zoran see back…? OK, **this is c+d**, and I'm sure everyone already thought about this in their head, and now I'm up here drawing it. [*enters appropriate sums into second row*] But, whatever, you know, it's all…

[227] So, yeah, **this is sort of becoming the, uh, Pascal's Triangle in reverse**. And **this one would be b+2c+d.** [*enters value into {3,2}*] **This would be c+2d+e.** [*enters value into {3,3}*] And up here, I need to start making the bricks bigger. And now, **we have a+3b+3c+d**, I think. [*enters sum into {2,1}*] Someone correct me if I'm wrong. **And then b+3c+3d+e**. [*enters sum into {2,2}*] And for the last one… Make it a little bigger…! [*expands first box borders*] OK. **a+4b+6c+4d+e.** [*enters sum into {1}*] And then, <see> **you have 1, 4, 6, 4, 1**, just like this guy. [*indicates Pascal's Triangle fragment on the board*] **And this is just like coefficients.**

Dor
[231] So, there's a… a Canadian researcher called Luis Radford, and if he were here now, if I sent him this little film here, he would be ecstatic, because what we've <shown> here, as we, uh, we're

not yet done here, I just want to point something out. We saw here, sort of a birth of… algebra, and I'll tell you what I mean. There was this critical moment.

[232] **Emily**, **Nora:** [laughter]

[233] **Dor**: Sorry?

[234] **Nora**: I told her she just birthed algebra.

[235] **Dor**: She birthed algebra. Yeah. Are you in post-partum depression now? [laughter] There was-

[236] **Emily**: It was an immaculate conception.

[237] **Dor**: …there was a moment, the reason that I called Nora up again was because initially you were, I'm not sure if you were looking here or here [*indicates Justin's pyramid and Pascal's Triangle fragment*], but you said something like, well, if there was a number here, it would be here and here… What's happening at that moment is… you were speaking about some number, OK, but it's not necessarily saying it's a variable and has different values. We don't need those terms, because it's very natural to say some number. <We know> what it is. It's, just, unknown, but it's a number, let's call it a number. What we do know that the same number will be again here. So that's, I think young kids can be pretty comfortable with that. There's some number, not telling you what it is, but here it is again, and it's there… It's kind of a certain level of comfort that is kind of between just saying what a number is, it's five and it's five, on the one hand, and all the way on the other hand, maybe say, you know, *a* and *a*. It's somewhere in between.
There's some number, I do know that it's here and here. And then there are, kind of related to what Zoran was also saying, that he wants algebra, it's just getting too much, other people maybe can do it, but it's just too much stuff to hold together, it's like, you know, Rubik's cube while juggling, or something… [laughter] I can't do either, so… So, you want some kind of aid, you want some kind of help. And, after some point, whether it's because you don't remember what you did and you want to just remember, wait, what did I say guys, and, I think it was… let me just keep, make some kind of notation, let me just make some record of what I've done, all right? So you might say a dot, or an *x*, or… whatever, you called it *a*, fine, call it *a*. You have some kind of record. But then you can refer to it, and you put the same thing somewhere else. And by the token of its iconic identity, the same kind of sign you make, it's taken to stand for the same number. We don't know what it is, but <there is> some number. So it's really, Radford calls it sprouting. It sprouts out of, just naturally sprouts out of speech. There's some number here, <there's> some number there. Let me just make some sign. Everyone's just looking at you, so you make some sign, so you don't even have to negotiate, it's just conventionality, it's everyone knows it's taken to be the sign. But, you're actually speaking, you're using your fingers this way, as a gesture. I'm not sure <the frame or> if it was so much number <or> value, I'm not sure. Number here, and there, then it's there, and then goes there. [*gestures on Justin's Pyramid*] And then when I called you up, I thought you'd come here.[*indicates Justin's Pyramid*] But actually what you did, you went to the next phase and said, all right, let's, there's the number [*indicates a in bottom row on Nora's pyramid*], and here it is [*indicates a in fourth row in Nora's pyramid*], and you already actually used a particular sign, actually a symbol, which is algebraic, kind of mathematical convention to use lower case English characters as to stand in for some number. It's like, so a guy goes into a bar. Wait, what is his name? That doesn't matter! [laughter] It's like, what? A guy. And then the guy says… Wait, what's his name? It's the same guy, it's all you need to know! Call him Bob, I don't care, you know.So, it's here, and then it's there, and it's there. [*indicates a in Nora's pyramid*] And suddenly, by doing that, by sort of offloading that idea of the guy goes, yeah, some number, then suddenly this thing becomes part of our thinking. And the term that cognitive science has been

using to call that in the past few decades is, it's like your thinking is distributed, it's called distributed cognition. It's like this is, it's actually part of your thinking now. [*indicates Nora's pyramid*] Without this, it would collapse. You actually need this to think. So it's, some call it a cognitive artifact. It's sort of, it's extending our thinking. We need this in order to think. We can't think without it. It is not, doesn't have a mind of its own. Nobody says that it is working, it is not collaborating with us, it's… Our thinking is distributed so that we look here, now we come back here, this is still there, so when we come back, it'll still be there. [*gestures on Nora's pyramid*] So we don't need to kind of hold it up in our… Have you heard of working memory? They use different words for it. So, <this is a> person called Allan Baddeley <has called it> working memory. It's like this, how much, um, bits of information, if you will, can you keep just in mind. <As> people say, ** oh wait I need to write this down. People say <that> because they have this feeling that you just can't hold that much. So you're using other media in order to keep everything in mind. In order to keep the working going. In order to offload from your working memory onto the paper so that you can do other things with your mind at that time. But the other things incorporate that. It's already been encoded on paper. It's already <been> put there. Especially using the conventional signs so that you'll remember what it is. You won't have to go, wait, *a*, *a* for apple or, why did I put an *a*? Oh, it's an *a*. Other people will also know what that is. Now, the moment you're here [indicates Nora's pyramid] you certainly have, what, uh, yet other researchers, they call this the, um… when you're creating some kind of diagram. So let me just put some of the name I've mentioned here, so… I've mentioned, uh, Allan Baddeley, that's working memory, and Edward Hutchins speaks about this distributed cognition in other people. [*writes names on blackboard as he utters them*] And…so, the, they're kind of, he's 70s, 80s, 90s person, he's more kind of 80s, 90s, 2000s person. But, going back more than a century to American to the, uh, American… logician, pragmatist, mathematician Peirce [*writes name on blackboard*] * heard of Peirce? So he spoke about how we work with these kinds of things, like diagrams or what have you, and he said that we first create them, and just by looking at them we notice new patterns. And his fancy word for that was, uh, hypostatic abstraction. [*Writes term on blackboard*] Hypostatic abstraction. It's when you glean patterns that were not available for your attention before, because they were encoded separately, and yet by virtue of putting them on paper you suddenly realize connections between objects. But those connections emerge to you just by doing that. And so then you see these, and you can write that up [*gestures on Nora's pyramid*]. And so, my point is that, this is a very… The way we move from saying if I had arrows before and kind of tracing Emily's work <it would be like> OK, we start at this guy, then it's this, but because of that then that is that, but because of that, that is that [*gesturing, emphasizing on, marking Emily's first pyramid*]. <It would> just kind of be something like that, right? Then that, then that, then that goes down. If we had like little arrows tracing the work, then bingo, right? Then we might kind of say this thing <affects> that, and what happens if we change this one, how would it change that? Then we can speak further, do we, can we come up with a rule for how, for what happens in this black box when you decide to put in one number, what it means, what the other number will be? And is Emily's method, is it feasible to go through that and come up with a rule? Maybe. Maybe we can…. what happens here, then how does that… pile over <pour over> what happens to the next one and so on? And does it matter what order <we go>? But, so these are the kind of questions we're pursuing, and then we have these kinds of tools that are related and they have certain benefits. After working with it a number of times we can start noticing benefits and saying, well, there's this kind of thing we do here where we can, where we no longer have to hold all these things in our mind. And so, but what really got me was how we speak about numbers as a group. We, as Betina calls it, we're thinking aloud together. We're thinking about numbers, and at some point there was a need to speak about this particular number. And it started in speech and in gesture, and then somehow this thing was born. This algebraic thing was born. And once it was there, it kind of led up to, I think maybe * Emily, you kind of maybe continued from there, or, I'm not sure how <it came in, did that come before?>

Zoran

[250] And now I'll make these bigger, OK. So I did pretty much the same thing that's here, except I already used the number. So I just had 13, this is 31… and 280. [*enters givens into the pyramid*]. I think this is all the numbers. So, **at this point I just called this one *a, b, c*, and *d*.** [*enters a into {5,1}, b into {5,2}, c into {5,4}, and d into {5,5}*] **And then I just, because I, all I have to—I mean, why didn't I pick like this one [*indicates {2,1}*] and call it *a* or something? Because I could build all the rest of them using these numbers**. So, you know, on the side, I was sort of doing this * I said, you know, **31 = a + b so now I can express either one of these in terms of the other.** And then I just build up all the way through. **So this is *b*+13, for example. This is *b*+44. This one is *b*+90, or +119.** So **I did the same thing on the other side, except, you know, it's a little more complex because here is *c* + *d*, and building through.** So, on the side I was doing a tally. **Whenever I have a real number, like, for example, this one [*indicates 75*], not a real number, an actual integer that I know what it is. So this is 13+*c*, so, for example, I can now express, oh 75 = *b*+13, that's this one, plus 13+*c*. So now I can express *b* in terms of c and c in terms of *b*. And I can already express b in terms of a, so I knew that, at this point I knew that if I could just express, uh, the last one, *d*, in terms of either *b* or *c*, then I can express each in terms of the other so I can just reduce it to one unknown, essentially, and express them all with that unknown. And if you work all this way through, you get that *d* in there, so can express 280 in terms of, um… I don't know where I wrote it down, but 280 can in the end be expressed in terms of *d* as well. So now if I can express 280 in terms of *d* and *c*, *b*, and *a*, then I can just go back and, you know, pick my favorite one of these and express every term as *a* or *b* or *c* or *d*. Does that make sense, that approach? So in the end I picked *a*, because if I have a restriction on *a* I know that *a* has to add up to, with *b*, to 31, so *a* can be any number between 0 and 31. So, once I did that, I looked back and I said, oh, like, I really kinda wasted my time in a sense, because all I really did, to myself, is prove that *a* can be any number between 0 and 31. So there are 32 solutions in that case. So all the algebra was just a way of showing, oh, it is, if I pick, whatever I pick for *a* will immediately determine the rest of the solutions.** So that was, uh, was it, Emily, did [] you went first?

Nora

[279] So **I picked this box right here, just cause it was touching so many other things I figured I could get a lot of information out of it.** [*places x in {4, 2}*] So **this one is 31+x, and then this is 106+x, and then this is 75-x, and this is, let's see, x-13**, and then… etc, etc. [*enters values as she utters them*] Do you want me to do the whole thing?

[404] **Emily**: Are there other pyramids? Can you just make any old pyramid you want and make sure you leave the three center ones and one on the outside? I don't know.

[405] Dor: **Interesting. So now we're stepping back and speaking about the design of the space of problems and what they give, what they afford, and what they don't. So, I mean, certainly if you… well, you can just check it. If you take away the 31, for**

instance, do you still have sufficient constraints? Is it redundant? Do you have- It's like, as you, as Nora said about the multiplication table, do we have more solutions or not? All right? You can start working down. What do you think?

[406] Zoran: We'd have two unknowns if we didn't have a 31 there.

[407] Emily: Yeah.

[408] Dor: **Why?**

[409] Zoran: Because you wouldn't be able to express the bottom, um, left one. <Do we have> 44-*x*, we wouldn't be able to have that, we wouldn't be able to express that number in terms of *x* if we didn't have a 31 there.

[410] Dor: **Mhmm.**

[411] Zoran: You could express all the other ones, though, but not that one.

[412] Emily: Well, by nature of not *******

[413] Nora: Wait, which one? I don't think you can have, I don't think you can have <just one box>.

[414] Zoran: [*moves to blackboard*] This one. This one. I mean, we wouldn't be able to get these ones as well, but if I didn't have 31 here- [*indicates {4,1} on Nora's second pyramid*]

[415] Nora: OK, now I agree with what you're saying, but you said only one box and I didn't agree with that.

[416] Zoran: Oh, yeah, yeah, you would *… [*indicates left side on Nora's second pyramid*]

[417] Nora: But the whole side, I know.

[418] Emily: Then, all the way around, yeah.

[419] Zoran: But, you'd just put a y here, and then you'd have everything in terms of *y*. [*indicates {5,2}*]

[420] Dor: **I think a point always to bear in mind when you're, you know, thinking about these kinds of problems, doing these kinds of problems, in a classroom is that… there are certain nuances, I think important nuances, in comparing a set of problems of this kind and kind of classic set of content or lessons. And I think the difference is that… when we build, uh, when we build, when we plan lessons or units, then we make sure that all the bits of content <that we need> are already there. But with these kinds of problems, and especially when you shift from thinking about mathematics as content to think about it as an activity. OK, when you shift from thinking about math as a noun to mathematizing as a verb. So, when you shift from thinking about kids learning stuff to kids learning to DO stuff, then also your thinking shifts towards creating opportunities for them to practice the doing, the problem solving. And with that comes all kinds of, um, unknowns, like algebra, ah, where you don't quite know where the lesson will go to, you don't quite know what the kids will guess, you don't quite know- I mean, unless you know the kids pretty well, it could be that the first person comes up, "Uh, here's my algebraic solution," you know. And then, like, OK, everybody else feels really stupid, "Uh, I've got this number, um, you know, I only… my solution doesn't have *x* and *y*, do you still want me to come up?" Right? So much stuff can happen, right? And we're not really talking here, I mean, at the end of the day some traditional, maybe, curriculum person <will say>, "Well, what did they learn?" "Show me the, do they now know how to write an algebraic equation?" "Do they know how to solve an algebraic equation?" "Do they know…" I don't know, if this were, if we were kind of in the classroom and this is, we went through this now, * probably, you know, we <wouldn't> have more sophisticated things there, but, what**

would you say that kids learn? What is it that, for you as a teacher, if you were thinking "My learning objective is…" and you need not even write it, but just… what is the, what gives you the, the kind of, <what is your>, the spine to come into classroom and say and feel great about, "Wow, I'm going, <I'm kind of gonna do> Brick Pyramid with my kids today, I'm really excited." Because I know that teachers who have done it are really psyched out about it. And I just wonder what, coming in, what do you sense is… and I guess in a way by asking that I am also like undermining some of the classical way of thinking about lesson plans. "Today the vocabulary is… the concepts are… by the end of this lesson kids will be able to…" Which is very useful in some way of controlling and, you know, is very accountable for parents and all that. But I am just curious, what would say, how, what would you feel coming into a classroom with a problem like that? And, in a sense, I guess I'm speaking to the possible gap between the visceral sense of "Oh my God, we were doing real math today. This is, like, good stuff!" and then, kind of other persona, kind of, "Yeah, but, um, can we now…?" Do you know? Do you see what kind of space I'm in here?

[421] Zoran: I would be excited about exposing students to algebra and having them work with it. I mean, that would be my take on it.

[422] Dor: **Well, exposing to, just open your, you know, open your notebooks, *a* plus *a* equals?<u>****</u>**

[423] Zoran: <u><But, right, but,></u> but working on a problem. Which… I'm trying to use a word you've used before… necessities.

[424] Dor: **Mm.**

[425] Zoran: Which… sort of naturally progresses to the use of algebra. And I wouldn't have, I think that a class is too small of a unit for me to say they are going to be able to do this and this at the end of this class. I can say, throughout this, maybe throughout these two weeks the students will have gained familiarity with using algebra, or something like that. I don't think I can just say that about one class. Because I didn't learn algebra in a day. I learned it in… <u>years.</u>

[426] Dor: **<u>I mean, Rome</u> wasn't built in a day, you didn't *…**

[427] Zoran: Right, exactly. [*laughter*] I mean, it took me years and years to make sense of algebra. So, I… I would say that you're just building on putting down these bricks. Maybe that's a bad analogy, but you're building up the students' understanding and familiarity and willingness to work with the content, or- Even just that, even just not having them, "Oh no, this is algebra, I can't do this" or "This is too difficult, this doesn't make sense." Even just breaking that down would make ME happy as a teacher.

[428] Dor: **Hm. More thoughts?**

[429] Justin: I like how this… creates the need for algebra by extending arithmetic. And so, I mean… yeah, showing a kid *a* plus *a* equals 2*a* is just kind of cramming it down their throats, whereas, like, you could get to that later, but first-, I mean, maybe not exactly presented that way, but, um, having and creating a need for their motivation for having symbols is really important, I think. And I think this does a good job of it.

Appendix B – Online Discussion Following the Collaborative Solving of the Brick Problem

Following the Brick Pyramid lesson, the instructor posted the following questions for the forum:

1. I would like for each one of you to reflect on your personal experience in the Brick Pyramid lesson we just shared. Yet I'd like you to do so through the lenses of the three readings (as well as any of the previous readings, if you so wish). For example, which, if any, aspects of symbol sense (Arcavi) did we witness? what forms of pre-symbolic algebraic reasoning (Reeuwijk & Wijers)?

2. What, if any, were evident relations between your own thought processes and those of the group? Actually, is there such as thing as "the group" that is larger than the sum of its human parts? *We* apparently progressed from one pyramid to the next, but who is this 'we'? Also, on a number of occasions, we've discussed the potential benefits of entertaining a variety of solutions – your own, others'. How, if at all, might last Thursday's experience contribute to this discussion?

3. Any other thoughts about the pyramids that you wish to share?...

EMILY (Sep 30, 2008 10:11 PM)

1. The Brick Pyramid lesson was more easily solved and grasped than some of the other problems, I felt, and I enjoyed how much we could elaborate and explore beyond the main notion of "the answer." We continued well after I provided a numerical solution, generalizing and noticing patterns as well as "birthing algebra." At first, with my simple solution of "If I just pick a number, say, 17, and put it in this box, then I just keep filling out the whole pyramid until it is all done, and I never need to choose another number because they have all been dictated by my choice of 17," we could have said this was arithmetic reasoning, which "worked" just fine, and left it alone. But we continued beyond the plug-and-chug. Justin gave his solution of "The value in this brick must be less than the value of the brick above it, and therefore the brick adjacent must also be less, accordingly, and all the way around the entire pyramid with conditions on the values in each brick." This was a step toward using unknowns; this was pre-symbolic (unless you count the less-than or greater-than signs, which were not in the place of number values, so I don't count those) algebraic reasoning. There were unknown values, but not variables. Directly after Justin's solution, Nora came to the board and began to generalize the case where I had used 17, and instead, I introduced a symbol: 'x'. The birth of algebra! As Arcavi states, "Having symbolic sense includes the relevant invocation of algebra; or in other words, having symbols readily available as possible sense-making tools" (25). Clearly Nora has symbol sense. She put the variable 'x' in for a certain value, and proceeded to complete the chart in these general terms, in terms of the symbol interacting with the defined values. All of us classmates oohed and ahhed, showing that we too had a bit of symbol sense ("… we claim that symbol sense should include… the appreciation of the elegance, the conciseness, the communicability and the power of symbols to display and prove relationships in a way that arithmetic cannot" (Arcavi, 26)).

I did want to bring up a point that I noted from Reeuwijk, that said: “Realistic situations are very critical to start the development of mathematical concepts with” (10). I do not know how “realistic” the brick pyramid problem is. I could not correlate this activity with a real-life example, and thus I wonder at the validity of the statement. We showed in our classroom setting that we can take a mathematical object, “Brick Pyramids”, and birth algebra, a complete abstraction and mathematical concept, though it did not have a specific tie to real life. Or perhaps the tie to the pyramid problem ‘is’ the realistic situation, from which the algebra is springing.

2. Most of the class jumped to the algebraic symbolism in their solutions, seemingly immediately after receiving the problem. I, on the other hand, spent a short moment thinking “I should solve this for a general rule…” then just went ahead with a specific numerical case instead. As we discussed together, and other classmates demonstrated their more abstract thoughts and generalities, I joined the bandwagon and wrote out a case using five variables, one in each brick of the base, which I then shared with the class to show the appearance of Pascal’s Triangle in the value of coefficients. The group cannot exist without the human parts, so in some ways it is not greater than the sum: <u>we each came up with a solution and presented each to the rest of the group. But as we began sharing our ideas, new ideas arose that had not been present in the separate parts of the group –ideas that arose from the interaction with other members of the group- and in that way the sum was greater than the parts. Once we came together we got farther, faster than we did alone.</u>
   There are many benefits to viewing and exploring and explaining various solutions. As we saw with the Pyramid problem, sometimes a solution works when we are flying solo, then with the rest of the group we can see it does not do as we had planned, but it took more eyes and minds to reveal the issues. Also, as with the case of myself, I only solved a specific numerical solution, and did not explore the variable symbolic ideas until other solutions were presented. Thus, it is highly beneficial to share different solutions with the entire group, that some may see “better” ways of solving, or can correct their mistakes, or that the group can grow and develop as a whole.

3. This is not of much importance, but I tried explaining the Brick Pyramid problem to my dad verbally (no visual aids or gestures or anything) last weekend, and it was thoroughly confusing to convey! <u>It is not actually pyramidal, because it is only two-dimensional, so the title and terminology were throwing wrenches into my explanation because he was visualizing three-dimensional bricks and pyramids.</u> Eventually I think we came to an understanding, and that I would show him my worksheet if I see him in October or at Thanksgiving.

Dor ABRAHAMSON (Oct 2, 2008 12:54 PM)

1. Emily, in one of the earlier readings, an author bemoaned the ills of translation. ‘Realistic’ does not mean, in the Dutch source, "*from* reality,’ but rather -- ‘that which can be *realised*", that is... imaged/imagined; if you will, the numbers should come from or come to life as a thing, and then... relations between the things. That said, I would agree that there is some confusion in the math-ed research community w/r/t what a ‘situation’ might be. If you’re

interested, I could refer you to a brief document from a symposium I chaired on specifically this!

Oh. And another point about the birthing of algebra: do you remember the moment when Nora used only gesture so as to suggest relations among the cells? But then that got just too much to remember, so one reason for algebra was the sheer load of "stuff to remember." As Zoran put it, maybe someone else can hold all those relations in her/his head AND operate upon them, but he can't...

Finally, it would be interesting to think further about Justin's remark that -- unless we're including negatives -- numbers get smaller as we go down. I'd say that that too is a form of algebraic reasoning, b/c it defines the relations among unknown numbers. In fact, it necessitates a reference to unknowns, such as using pointing.

2. Yes, and I think that often our new ideas come from... *mis*understanding each other, not only from understanding each other. Do you (mis)understand this point?...

   Oh, and I wanted to add that your way of approaching the problem -- beginning with one actual example -- is in fact a powerful way of going about things -- surveying the terrain and scanning for emerging relations. In fact, trying an example if one of the problem-solving heuristics advocated by Polya in "How to Solve it."

3. Interesting. So the co-occurrence of the name "Pyramid" and the graphics does seem to work. But just the term "pyramids" on its own is unconstrained. There are psycho-linguists who study these issues, but my quick guess is that <u>the property of 'pyramid' that actually does contribute to understanding the setup of the problem is the sense of vertical accumulation (that it's not, e.g. a triangular form lying on the ground). This "stacking" conceptual metaphor lends the notion that each brick lays *upon* the other two bricks, and the support they lend feeds into the notion of addends and sum.</u> I find these issues fascinating, especially when thinking through them informs my design work (both in creating and then, later, in figuring out the data).

JUSTIN (Sep 30, 2008 10:41 PM)

1. *Reflect on the brick pyramid through the lenses of the readings.*

   Our representation of the pyramid took a surprisingly steady and natural progression from the arithmetical to the symbolic. Because the progression was so fluid, I am going to speak of the transitions between representations as a construction, although the development is undoubtedly more complex. Emily began by exploring the pattern, using the pyramid as a concrete model of its specific configuration. Then my "conceptual mathematization" of the situation brought me to the pre-symbolic representation of 'constraints' which proved employ symbolism that was too difficult to use for the situation (much like the stacked tiles problem from Reeuwijk &

Wijers). Nora then began the Arcavian friend making with symbols, representing the inverse Pascal-like additive properties of the pyramid, that made my representation problematic. After introducing this new choice of symbols, Emily then used conventional mathematical symbols to generalize the pyramid model, which allowed for flexible representation, allowing the 'emergence' of the binomial coefficients, bringing further insight into the up-side-down-Pascal-ness of the situation. Zoran then used Emily's generalization as a model for the original configuration, this time taking the symbols outside of the pyramid, and therefore mathematizing previous thoughts, like a pre-image of the original situation. After several choices of variables (from no variables, to one, two, five, four) it was interesting to see Nora's symbolic representation using only one variable. This representation I think clarified what I had initially intended, and also revealed concrete information about other squares in the pyramid (specifically the 24 on the far right). While I had no trouble extending my thinking to the other representations, the self-validation gave me a sense of ownership of the pyramid problem. And (paraphrasing Zoran) mathematics can not be learned without that sense of ownership.

2. *Who is we?*

At this point, I wish that I could point to ideas proposed by the great thinkers in the discipline of collective intelligence, but I don't know any of their names. So, from a personal perspective, I think that the progression from arithmetic to symbols was too smooth to be coincidental. During our dynamic interactions, ideas went in and out of alignment and reflection and we contributed to group dynamic while simultaneously building our own understanding. Our internalization of the group's ideas as well as the incorporation of our own created a sense of ownership in the variety of solutions that would not have existed otherwise.

3. *Further commentary*

I wish I had handwriting like the rest of you.

Dor ABRAHAMSON (Oct 2, 2008 1:17 PM

1. *Evolution*: So it seems like algebra was introduced as a possibly useful tool for mathematizing the situation, and then it took some exploration, constraining, pruning, and calibrating until the tool actually served as a model for reifying the pre-symbolic notions. Some of the clunckiness (?) of the process was due to the collaboration, and yet the collaboration is what made this possible in the first place. Betina is currently completing a fascinating paper in which she claims -- and here I'll totally misrepresent her... -- that the "togetherness" of thinking should be regarded as an enabling constraint, just as media and representations are. That is, you need to parcel your emerging notions into the parameters of discourse, just as you need to parcel algebraic reasoning into algebraic notation. These 'parceling' processes are not easy, b/c we are bound by conventions forms, such as the English language, and yet once parceled -- in diagrams, in language -- these "texts" we have created are accessible to others, who appropriate, process, and re-parcel, and so on.

2. *We is who*: So, yes, ownership is important, and -- given that the group was a key mobilizer -- participation becomes key to ultimate ownership. It is then that individuals' self-perception comes into play, and lacking the nerve to speak out only exacerbates any negative self-perceptions, in a vicious cycle that is damaging to learning. I think that for successful TAT-ing (Thinking Aloud Together), the teacher needs to cultivate a culture in which errors are not only pardoned but are encouraged. (Because when we think privately, we are continuously rejecting false starts; When we present only completed work, it is error-free, hopefully; So TAT-ing requires a certain loosening of the ego... For this to happen, the objective must shift from individual solving to group solving. Yet, many students might resent this practice, and they should be respected. Hence, the issue becomes -- as in comedy -- that of timing, That is, when to initiate whole-classroom discussions.

3. *Calligraphy*. My hand-writing is typing.

ZORAN (Oct 1, 2008 10:02 PM)

Let's start from the middle:

2. I find these questions fascinating. Not from a socio-cultural perspective - blah - but from some type of agent-based-cognitive-hivemind perspective. I first noticed the phenomenon of thinking as a group when I started working on mathematics homework with other students in graduate school. When stuck, one of us would eventually have a breakthrough - breakthrough was shared - another stumbling block - another breakthrough - and so on... Sure, some of us contributed more than others, but all of us ultimately felt *smarter* together (and I hate to use such cheap words, but that's the best way to describe it).

   Later, when I attempted to analyze such episodes, what stuck with me was the efficiency with which the information is disseminated. Once one of the group members made a move forward, he would share it with the rest of the group not by discussing how he (just guys in our group, unfortunately) arrived at this knowledge, but by sharing the relevant piece of knowledge and how it fits in with the problem. Now, occasionally we would share the motivations behind these insights, but those were exceptions. In this sense, individuality is lost and only the mathematics is shared for sake of efficiency.

   Finally, on the topic of entertaining others' solutions, I found the sharing of solutions on Thursday very helpful. While solving the problem wasn't difficult per se, the structure of the problem was challenging, and seeing different approaches helped me better understand the problem itself. For example, seeing how putting a in bottom row contributes to the value in top row opened the problem beyond simply "having the solution." Why might this be helpful? Well, you asked one question about a problem that could entertain many questions. If I encounter anything similar in the future, I'll already have experience dealing with a problem of that type.

1. Speaking of Arcavi, those "squares" didn't deserve to be called magic squares. That said, if having symbol sense is having "symbols readily available as sense-making tools" - so I might have too much symbol sense. In a sense (pardon the pun), I brought too much symbolism to my solution - so much that I trampled over certain nuances to the problem.

   Let me explain. After examining the problem, I realized that if it has a solution, that solution can be achieved by labeling each unknown in bottom row and solving equations that emerge. <u>In one sense, my problem solving was an absolute success - I solved the problem in the most brutal way possible. But using that level of algebra obliterated any chance of opening up the problem for closer inspection - and it wasn't until I saw other students using different approaches that the problem opened up (though I already knew everything about the solution!).</u> Even Nora's algebraic solution, weaker (not a bad thing) and more elegant than mine, revealed more than my brute-force method.

   Now, I think that all 4 (five?) of us have a fairly high level of symbol sense. But what I found most interesting was the degree to which some students held back on algebra, while others (me especially, Nora somewhat) just went all out with algebra. And as I've mentioned earlier, it isn't clear to me that - if my purpose was to understand the problem - my approach was praiseworthy. To be more general, if the purpose of algebra education is to endow students with ability to use algebra as a means of sense-making, than teaching *prudence* in algebra use might not be such a bad idea.

Dor ABRAHAMSON (Oct 2, 2008 1:39 PM)

2. *Hive*: I find very interesting your observation that the collaborating problem-solvers did not (necessarily) share *how* they arrived at a bit of information, only the information per se. What might be an individual's tacit criterion for sharing or not sharing their approach? I'd expect this has something to do with how many steps they took. One could say of an ambiguous figure, which all see as a Duck, "Look what a cute beak." But, if you suddenly see it as a Rabbit, you probably would not say, "Look what cute ears." You more probably first say, "Well, if we look at it as though it were a Rabbit --- see, everyone?.. this is how I see it... -- then we see it has cute ears." Again, the criterion is whether or not we sense that everyone is "on the same page." I think that the human capacity to asses the group's same-page-ness is fascinating and worthy of research.

1. *Interesting*: So you took pause to step back and examine the problem more closely. Quite a mix of metaphors here. Arcavi, of course, speaks of that too. Then so does Schoenfeld, openly citing Arcavi. Your notion of the problem "opening up" is very interesting, too. Along which dimensions does it open up?

NORA (Nov 30, 2008 8:34 PM)

1. When I solved the problem, I went almost directly to picking a brick whose value I could represent as x, then writing expressions to show how every other brick was related to that one. By doing that, I figured I'd be able to see if all the bricks were related to each other or if the values of some bricks were independent of other bricks. It seems obvious in retrospect that all the boxes should be "related" and we should be able to express each in terms of any other brick (since any given brick is dependent on the bricks around it, which are touching other bricks, which means they should all be dependent). So apart from symbols, I was reasoning about how the bricks are related in some way and about which brick I would have the most information about. I chose the second brick from the left in the second row from the bottom to be x. I picked this because it was touching the greatest number of bricks with given values. I figured this would give me the most information about x as possible. It should also be noted that I didn't know at first whether there would be more than one solution, and if there were only one solution, then finding the brick with the most information right off the bat would be the most useful. As I began filling in other bricks with their value in terms of x and I started filling in bricks closer to the top, I wasn't sure if the values near the right side of the pyramid could also be expressed in terms of x, so I chose another brick (right most brick on third row from bottom) to be y and used that to fill in the bricks above it. However, I soon realized those bricks could also be expressed in terms of x. <u>In the beginning, since I thought there might only be one solution to the puzzle, I thought of x as a fixed value rather than a symbol to represent a set of values which could satisfy those conditions. After I filled in all the bricks, I saw that really there isn't only one way to fill in the bricks, and that the pyramid would work out no matter what I filled in for x (and filled in the rest of the bricks accordingly).</u>

2. One compare/contrast situation I saw during our discussion was the discussion about Pascal's Triangle. I was trying to express that (jpg 3) if you put a number in the second from left, bottom square, by the time you've added it up all the way to the top you'll end up having counted it four times (twice in the second row in left, and second to left on second row; then three times in the next row, then four on the next two rows). So I looked at it as Pascal's triangle, where each number represents how many times each box is counted in the top brick (or by time it gets to the top brick). However, others in the group ended up bringing up the inverted Pascal's triangle, as Emily did in jpg 4. This ended up being our group's model to use and discuss.

   While I think the group did progress in our discussion, I think our discussion's progression from non-symbolic to symbolic was not completely unplanned, as I suspect Dor was asking particular people to present at particular times (i.e. students whose solutions didn't use symbols come first, then students with slightly more, etc.). Within that structure, though, our conversation did grow and change. I think that what was considered to be the "we" was formed by each person's contributions of what they individually thought, combined with others' descriptions of their understanding of what the presenter said as well as their own related thoughts. I think what was said and done collectively was definitely more than just the sum of the parts, however, in some ways I think the greatness of what was done in the collective is usually too great for any one individual to recognize/absorb in the moment. What I mean is that in those group interactions, there is probably enough great points/ideas that could be combined together to make something great, but we won't necessarily be able to understand how valuable/important every part is just within ourselves. As we're reaching a conclusion and putting things together, it's not so much that each individual is creating a coherent image of the solution in his own head, but each person is able to make particular key connections, and the sum of those connections gives an awesome outcome. I also saw this in the plank problem we worked on. It's not that one of us visualized the entire solution ourselves, but we each attended to particular parts that we found important or were particularly puzzled, then convinced each other that it was worth their attention (and then tried to explain exactly what the problem was). For example, certain people's fixation with the balancing of weights. One person picked that thinking up, then dropped it, then another picked it up. Anyway, the main idea is that no one person put the whole picture together themselves. Of course this isn't always the case (if the five of us in PDMPS were asked to find the solution to 2+2, we would each be able to individually see the entire solution), but for particular difficult problems, the collective we is much greater than the sum of the parts.

3. I really liked Justin's reasoning of 0, 13+, etc., which described conditions on each square. Although we stopped, I think we might have been able to continue on with that and narrowed conditions sufficiently to get a solution, and if not an exact solution, at least a good idea of one. There was just a lot to keep track of with the conditions.

Appendix C – A (Graduate) Course Participant Reports on his Tryout of the Brick Pyramid

When the bell rang, in frantic haste I (regretfully) made the split second decision to give them revealing hints and reveal an algebraic solution. Earlier, we had placed the number 20 in the bottom left, and calculated the remaining bricks from that single choice, so I asked them if it were possible to generalize the "bottom up" strategy using algebra, if we could find a solution for *any* number in place of the 20. We then placed an x in the bottom left brick, and the remaining two minutes were spent calculating bricks in terms of x. I made this move to bring closure to the lesson, to allow the students to leave the room feeling like they had learned some "math" as opposed to simply learning to solve a puzzle, but now I realize that this decision may have been borne from my traditional quick-fix mathematics background. The students reacted in a way that said "oh, he's giving us the answer…" and their thinking stopped. A better move would have been to give them a couple of minutes to reflect, as I had originally planned. In the end, students left feeling weary, but I think that they began to understand the dynamics of the pyramid. They worked through the problem with arithmetic, set the conditions to the problem, and began to compare basic strategies. With the short amount of time that we had, I set the conditions for the reinvention of algebra, but did not have the time to let the embers ignite.

Thoughts about what happened

My lesson may have been brought to a premature end, but that does not change the fact that students were sincerely engaged during those thirty minutes. The problem naturally surfaced an interesting conversation about fundamental mathematical principles in a collaborative problem-solving environment. If I had more time I might have been able to allow their ideas to surface, and guide discourse towards their reinvention of algebra, but in the ten minutes that we shared as a group, their thinking was not entirely trivial. I do not want to claim that the lesson was complete success, but I do want to emphasize how it kindled embers of the reinvention process.

Students began this problem entirely in "arithmetic space" by plugging two numbers into the top two bricks and subtracting to fill in the bricks below. They were changing numbers to see "what would happen" and not "how the numbers change". By comparing "top down" to "bottom up" strategies students began to objectify their previous solutions, by the end of the lesson Oscar was able to describe the difference between the top down solution and the bottom up solution, and which one he preferred to use and why. By discussing strategy Oscar started to take a more generalized approach to the Brick Pyramid, making strides towards algebra, which fed back into the collective thinking.

While collaboratively working through her solution, Valerie hesitated, and received help from Oscar to change her strategy. Their interaction, while only five seconds in duration, represents the spontaneous and collective shift in thought that occurs in a an environment of discourse.

Also, foundational conversations about mathematics surfaced naturally from the pyramid problem. While questioning the use of negative numbers, Oscar asked, "How could you make a pyramid with a *negative* block?" Implying that in some sense, the idea of a "negative brick" is absurd. How can the substance of brick represent the absence of number? The implicit existential character of a brick channels student intuition to make basic assumptions about the problem. Pointing this out led to an interesting discussion about setting the conditions to a problem, which is in my experience, is a foundational mathematical topic.

Refining Teaching Practice

If I were to teach this lesson again, I would certainly devote more time to the problem. During our ten-minute lesson the students only began to develop strategies, let alone support their arguments for why they worked. A thorough exploration of the problem requires at least twenty or thirty minutes of group discussion. I think an hour would have given ample time for students to work individually, work as a class, and reflect. I also think that the problem would be more successful with a larger group of students. Variety in student thinking is an important feature of an interactive classroom dynamic and having more than four students would have increased diversity in thinking.

I also see the potential to extend this problem into a larger "problem of the week" style project, starting by solving the problem as a puzzle, then "figuring out" the puzzle with algebra, then changing the fixed numbers, and then finish with by algebraizing a blank pyramid. I think that this would be a great introduction to binomial coefficients while remaining grounded in a single mathematical artifact.

However, if I had only an hour with eleventh grade students, I think that I would pose the Brick Pyramid in its original form. Adapting the problem into my "puzzle-like" version did little more than force students to initially try numerical solutions. The scaffolding may have been useful if I had more time to let their thinking develop, but I could have effectively channeled student knowledge about puzzles during the initial launch of the problem. I could have given students the original problem, and simply said "here is the pyramid puzzle, the goal of today's activity is to figure it out. Meaning, find all of the ways to fill it in." I think that this would have worked better than to explicitly integrate the "puzzle-ness" into the structure of the problem.

More than the refinement of this particular lesson, conducting this teaching experiment helped me to develop sensitivity to complexity in the classroom. While transcribing video, I became aware of the subtlety of student interactions while listening closely to students' practically inaudible conversations. At one point during the lesson, the students began to quietly talk with each other while I prepared a poster and this is what they said during this thirty-second transition:

Valerie: "Did someone else get another way?"
Oscar: "What? [other than] 150, 130? Nah… I got 150, 130 but at first I did 140, 140 and ended up with… I was only off by ten, so I just scrapped that"
V: I was only off by four. See? (Points to paper)
O: Yeah I know… If it was only ten more I would'a had it. So I scratched that, … then I tried switchin' it, and it didn't work.
A: This one, wait… this one would work if you put a negative number in there…
O: Yeah, if you want to use negatives…
Teacher: So, does someone want to step up and show us what they were thinking?

In this thirty-second conversation they discussed many of the main points that were covered during our class discussion and it all went unnoticed. Reviewing this as well as other subtle interactions, made me realize that I was simply blind to certain aspects of classroom interactions. I never imagined that conversations comprised of whispers can actually have substance. This knowledge among other classroom focuses will affect my awareness in future lessons. By taking a close look at what students are saying and how students are thinking my "teacher feelers" have become more sensitive and the next time that I teach this lesson, those feelers might just pick up something new.

Some may argue that teaching a discourse centered classroom requires less work than teaching a traditional classroom because there is less direct instruction required. However, teachers who lead discourse must be sensitive to the thoughts of their students at every given moment, be attuned to where those thoughts might lead, decide whether those routes are productive and intervene without extinguishing the embers of reinvention. A RME instructor must be sensitive to classroom dynamics and student thinking during instruction, and must truly live in the moment.

Although the lesson was cut off prematurely by my pre-conditioned desire for closure, I feel like the lesson was successful for the amount of time that was available. Interesting conversations about the nature of mathematics surfaced, students began to engage in discourse, and I became more aware of the type of problems that surface during RME. This was a good start, and if I had more time it would have had a good finish.