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Adding Up to Multiplicative Concepts: The Role of Embodied Reasoning

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Citation:

Abrahamson, D., Negrete, A. G., & Gutiérrez, J. F. (2012, April). *Adding up to multiplicative concepts: the role of embodied reasoning.* Paper presented at the annual meeting of the American Educational Research Association (SIG Research in Mathematics Education), Vancouver, April 13-17.

Note: Rosa Lee was not a co-author on the original submission to AERA but contributed to the analyses and writing.

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Key phrases: additive reasoning, embodied cognition, embodied interaction, mathematics education, multiplicative reasoning, pedagogical design.

Abstract

22 Grade 4-6 students participated in a design study investigating the conjecture that the concept of proportionality can be grounded in perceptuomotor schemas. Students engaged in a non-numerical embodied-interaction problem-solving task, in which they bi-manually manipulated two virtual generic objects in parallel along a vertical axis on a computer screen, in an attempt to elicit a designated feedback. The solution involved moving the objects at different rates. Once they discovered simple solution strategies, the tutor introduced onto the screen a set of symbolic artifacts—a Cartesian grid, and then numerals. Students appropriated these frames of reference, yet in so doing they implicitly reconfigured their strategies so as to incorporate explicit attention to quantitative properties, resulting in strategies that we identified as either additive or multiplicative. We then asked the students to reflect on relations among these various strategies. The paper offers micro-ethnographies of three selected episodes, in which some students expressed logico-mathematical inferences in support of their claim that their additive and multiplicative strategies could be derived one from the other. Students' naïve deductions hinged on heuristic schematic links that enabled them to ground multiplicative phenomena in additive primitives. Formal coordination of additive and multiplicative conceptualizations of proportion, via elaborating the distributive property of multiplication over addition, may come only through algebraic and number-theory reasoning. We speculate on perceptuomotor intuitions and socio-epistemic norms underlying patterns in students' informal inferences.

Adding Up to Multiplicative Concepts: The Role of Embodied Reasoning

But there was yet another big problem at hand,
The distributive law they knew not!
So the use of the brackets could not understand,
Which to progress was clearly a blot.
Zoltan Diénès, *Why Johnny Can't Add*[1]

1. Introduction

In this paper we present and discuss findings from an analysis of data gathered in a design-based research study that investigated the microgenesis of mathematical concepts (n=22, Grades 4-6). Students' videotaped behaviors in this study are interesting for mathematics-education researchers and practitioners, because they create opportunities to revisit afresh unresolved theoretical and pedagogical issues respecting the cognitive constitution of multiplicative concepts and in particular relations between additive and multiplicative forms of reasoning as these pertain to important notions such as ratio, proportion, and rate.

An argument emerging from our findings and analyses is that students are able to ground multiplicative conceptualizations of proportional relations in simple additive operations, yet this grounding depends on a nuanced heuristic inference they make; while lending students a sense of understanding, this heuristic inference circumnavigates deductive steps that would be necessary for a formal mathematical inference. As theoreticians, we are impressed by the intellective creativity revealed in these children's quantitative reasoning. As designers, we are interested in articulating the pedagogical affordances of learning environments that support such creative reasoning. As policy shapers, we are concerned that such informal quantitative reasoning should be recognized and encouraged as pedagogically desirable.

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[1] See in http://www.zoltandienes.com/?page_id=130

1.1 Theoretical and Pedagogical Orientations

The theory of learning underlying our analysis of conceptual microgenesis in mathematics is motivated by Piagetian and affiliated views that sense making is grounded in goal-oriented embodied enactment (Bruner, Oliver, & Greenfield, 1966; Hwang & Roth, 2011; Kalchman, Moss, & Case, 2000; Maturana & Varela, 1987; Núñez & Freeman, 1999; Skemp, 1983). The activities we create and investigate are accordingly inspired by the reform-oriented pedagogical commitment that students should make sense of the formalisms they are expected to master (Dewey, 1916/1944; Freudenthal, 1986; Kamii & DeClark, 1985). As design-based researchers, we thus strive to build and implement learning environments that foster student grounding of mathematical concepts, and these learning environments, in turn, serve as empirical contexts for our inquiry into teaching, learning, and design practice (Sandoval & Bell, 2004).

1.2 Focal Mathematical Content and Background Theoretical Debate

The target mathematical content of the activity was proportional reasoning. A longstanding and unresolved debate among mathematics-education researchers and practitioners pertains to the cognition and instruction of multiplicative concepts and, in particular, to whether and how students should ground multiplicative notions in additive operations (e.g., Cobb & Steffe, 1998; Confrey & Scarano, 1995; Norton, 2008; Sherin & Fuson, 2005).

Whereas mathematicians will tell you quite flatly that multiplication is defined as iterated adding, for example 3*4 *is* 4+4+4, learning scientists challenge this axiomatic equating of disciplinary formalisms with mental schemes and wonder, instead, what cognitive and experiential resources students might bring to bear as they build and master what Vergnaud (1983) calls the Multiplicative Conceptual Field. This debate is fraught with underlying epistemological disagreement over the cognitive embodiment of mathematical concepts. For example, does it even make sense to speak of presymbolic, proto-quantitative notions of multiplicative concepts that are devoid of iterated-unit underpinnings (Confrey, 1995; Resnick, 1992; Thompson, 1993)?

One way of contributing to this debate is "from the lips of children," that is, through investigating learners' authentic attempts to ground multiplicative notions in additive operations. Thus, coming into this analysis, we expected to learn from the students by listening closely to their voices (Davis, 1994). In so doing, we hoped indirectly to incorporate student voice as co-participants in our future design (Engeström, 2008). The research question guiding our work is: *How do students coordinate additive and multiplicative conceptualizations of proportion?*

1.3 Overview of Pedagogical Design

Students were presented with a problem that could be solved only hands-on—they were to determine interaction patterns governing a mystery artifact, where the only form of interaction was bimanual and dynamical. The design intentionally primed additive rather than multiplicative schemas as initial solution hypotheses, because our intervention was designed to simulate and treat within a presymbolic microworld student forms of reasoning that anticipate their famously inappropriate additive strategies for symbolic expressions of multiplicative problems (Post, Cramer, Behr, Lesh, & Harel, 1993; Van Dooren, De Bock, & Verschaffel, 2010). We thus intentionally fostered cognitive conflict between students' legitimate additive solution hypotheses and the automated proportion-based feedback they would receive from the interactive device, and we wished to observe how students might resolve this conflict. For example, we wondered whether and how they might assimilate multiplicative solution procedures as situated complexifications of additive reasoning or, rather, view these approaches as fundamentally incompatible.

1.4 Study Focus: How Do Students Coordinate Complementary Operatory Schemas?

To hone our investigation, we compared across selected samples of students who struggled with varying degrees of success to explain how two different solution strategies they had generalized nevertheless result in the same effect: (a) an iterated-unit recursive strategy (begin from 1:2 and 2:4 and then, noting the respective differences between corresponding elements, keep adding +1 on the left-hand side and +2 on the right-hand side to produce, 3:6, 4:8, 5:10, etc.); and (b) a closed or explicit multiplicative strategy (begin from 1:2 and 2:4 and then, noting that the right-hand side is always double as much as the left-hand side, determine more pairs, such as 4:8, 10:20, etc.). How might students coordinate these would-be disparate strategies? As we now explain, our methodological rationale is based on a theory of learning.

1.5 Coordinating Meanings as a Stimulus and Index of Mathematical Learning

Godino et al. (2011) offer a pedagogically oriented view of mathematical objects that emphasizes the requisite challenges of building flexible conceptual structures. One of their examples for what a flexible conceptual understanding should encompass is the individual's capacity to "consider two propositions that correspond to different configurations to be equivalent" (p. 257). From this perspective, we can appreciate the difficulty students experience as they attempt to understand and express why both the recursive-additive and explicit-multiplicative solution strategies obtain to the embodied-interaction problem. Moreover, we can appreciate that coordinating these different configurations bears important pedagogical benefits.

Following a Methods section, we will present findings from this explorative study.

2. Methods

2.1 Micro-Ethnographical Studies

Our chief methodological orientation toward empirical data is collaborative micro-ethnographic analyses of multimodal behavior (Nemirovsky, 2011; Schoenfeld, Smith, & Arcavi, 1991; Siegler & Crowley, 1991). Optimally, these analyses give rise to empirically grounded ontological innovations in dialogue with literature germane to our emerging research questions (diSessa & Cobb, 2004; Strauss & Corbin, 1990). As such, we select and study very closely brief excerpts from videotaped interactions between students and instructors. We are particularly interested in monitoring the referents of verbal/gestural utterances produced in these interactions as well as the interlocutors' apparent subjective sense for those referents, given their distinct and asymmetrical frames of reference, and how these loose discursive agreements are challenged, repaired, and leveraged (cf. Mariotti, 2009; Newman, Griffin, & Cole, 1989). For this particular study, we transcribed and analyzed a set of episodes in which students had just discovered multiplicative solution strategies for a situated problem and were then asked to make sense of these strategies in light of additive solution strategies for the same problem, which they had explored earlier.

2.2 Data Sources: a Design-Based Research of Embodied Mathematics

Our data corpus consists of videotapes and field notes gathered during 20 clinical interviews with 4^{th} – 6^{th} grade students who participated voluntarily in the implementation of an experimental design for proportion (Abrahamson & Howison, 2010; Reinholz, Trninic, Howison, & Abrahamson, 2010).

The research project was conducted in accord with the design-based research approach (DBR), in which learning theory and instructional materials are co-developed simultaneously, interdependently, reciprocally, and iteratively. Typical of DBR studies, our experimental design was driven by a conjecture (Confrey, 2005). That is, we hypothesized a certain cognitive mechanism that is usually dormant, at least within most prevalent instructional contexts designed to support the development of our targeted concept, yet whose activity could potentially support the desired learning. Drawing inspiration from the embodied/enactive approach (Goldin, 1987; Pirie & Kieren, 1994), our conjecture was that some mathematical concepts are difficult to learn because our everyday experiences do not occasion opportunities to embody and rehearse the body-based dynamic schemes underlying those specific concepts. Specifically, we conjectured that students' canonically incorrect solutions for rational-number problems—"additive" solutions (Behr, Harel, Post, & Lesh, 1993)—indicate students' lack of multimodal kinesthetic–visual action images to ground proportion-related concepts.

The pedagogical program of augmenting student experience via engaging them in activities with carefully designed objects bearing mathematical semiotic potential harks back at least to the dawn of kindergarten (Froebel, 2005). Whereas this study employed electronic technology, our objectives should not be viewed as any different from those of Maria Montessori, Zoltan Diénès, Caleb Gattegno, and other mathematics-education design luminaries.

2.3 Theoretical Rationale and Pedagogical Design of The Mathematical Imagery Trainer

At the center of our instructional design is the Mathematical Imagery Trainer for Proportion (MIT–P; see Figure 1, below). MITs are computational devices supporting embodied-interaction inquiry activity by which students are to discover, rehearse, and thus embody presymbolic dynamics pertaining to a mathematical concept (Abrahamson & Trninic, 2011). The goal is that the manual actions students perform in the air inscribe the very image schemas that we wish for them to develop as the epistemic substrate for constructing the meaning of the targeted mathematical concepts (see Lakoff & Johnson, 1980; Lakoff & Núñez, 2000). This particular MIT device, the MIT–P, was engineered for students to discover, rehearse, and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation.

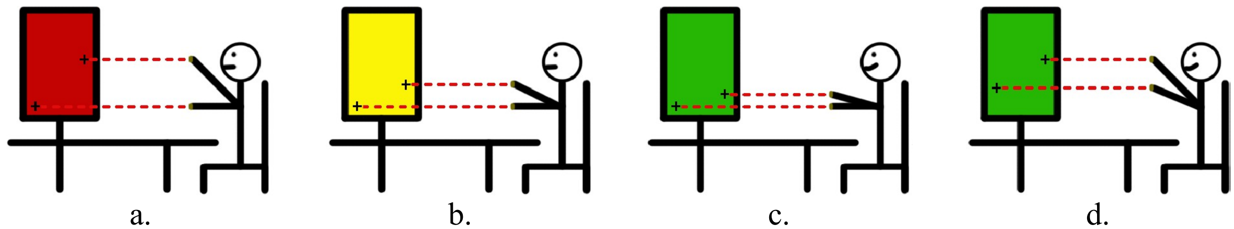


Figure 1. The Mathematical Imagery Trainer – Proportion (MIT-P) set at a 1:2 ratio, so that the right hand needs to be twice as high along the monitor as the left hand. A schematic interaction sequence: (a) incorrect performance (red feedback); (b) almost correct performance (yellow); (c) correct performance (green); and (d) another instance of correct performance (green).

Using remote-action sensor technology (see Figure 2a, below), the MIT–P sensors measure the heights of the users' hands above the desk. When these heights (e.g., 10'' and 20'') match the unknown ratio set on the interviewer's console (e.g., 1:2), the screen is green. If the user then raises her hands in front of the display at an appropriate rate, the screen will remain green; otherwise, such as if she maintains a fixed distance between her hands while raising them from a "green spot," the screen will turn yellow then red. Participants were tasked first to make the screen green and, once they had done so, to maintain a green screen even as they move their hands.

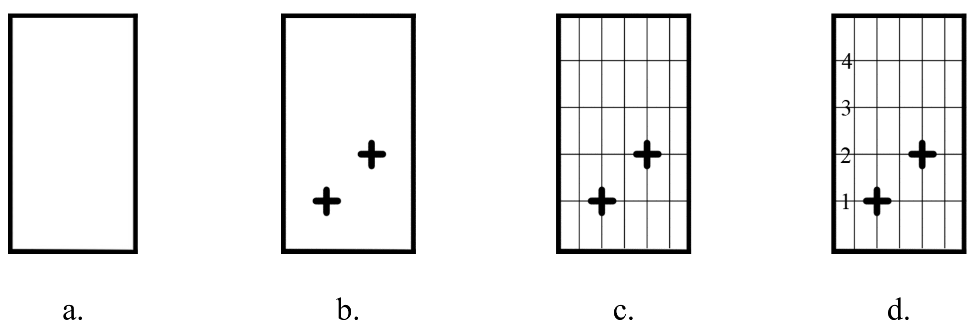
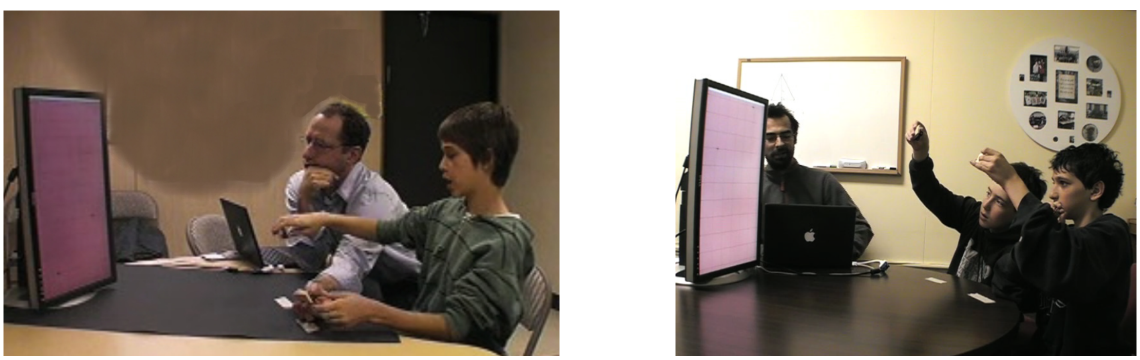


Figure 2. MIT-P display configuration schematics, beginning with (a) a blank screen, and then featuring a set of symbolical objects incrementally overlain onto the display: (b) crosshairs; (c) a grid; and (d) numerals along the *y*-axis of the grid.

At first, the condition for green was set as a 1:2 ratio, and no feedback other than the background color was given (see Figure 2a; this challenging condition was used only in the last six interviews). Then, crosshairs were introduced that “mirrored” the location of participants’ hands (see Figure 2b). Next, a grid was overlain on the display monitor to help students plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions (see Figure 2c). In time, the numerical labels “1, 2, 3,…” were overlain on the grid’s vertical axis on the left of the screen to help students construct further meanings by more readily recruiting arithmetic knowledge and skills and more efficiently distributing the problem-solving task (see Figure 2d).[2] In addition to the 1:2 ratio, we worked with students on 1:3 and 2:3 ratios.

2.4 Participants

Participants included 22 students from a private K – 8 suburban school in the East San Francisco Bay Area (33% on financial aid; 10% minority students).



(a) (b)

Figure 3. (a) Individual semi-structured, clinical–tutorial interview. Child and tutor co-remote-control two cursors on a computer monitor. The child is attempting to accomplish the designated task objective of making the screen green. The tutor is attempting to steer the child toward discovering effective interaction strategies, articulating them in terms of quantitative principles, and coordinating functionally commensurate strategies into a coherent conceptual field. (b) A paired version of the same interview, in which children problem-solve collaboratively.

[2] Not treated in this paper yet key to the designed learning trajectory is yet another structure layered onto the screen, namely an interactive ratio table for effecting green numerically rather than gesturally.

Students participated either individually (17 of the 20 interviews; see Figure 3, above) or paired (the last 3 interviews) in a semi-structured interview (duration: mean 70 min.; SD 20 min.). For this study, we are drawing on the last 15 interviews, wherein our protocol had stabilized.

3. Findings: Selected Vignettes of Students Coordinating Additive and Multiplicative Views

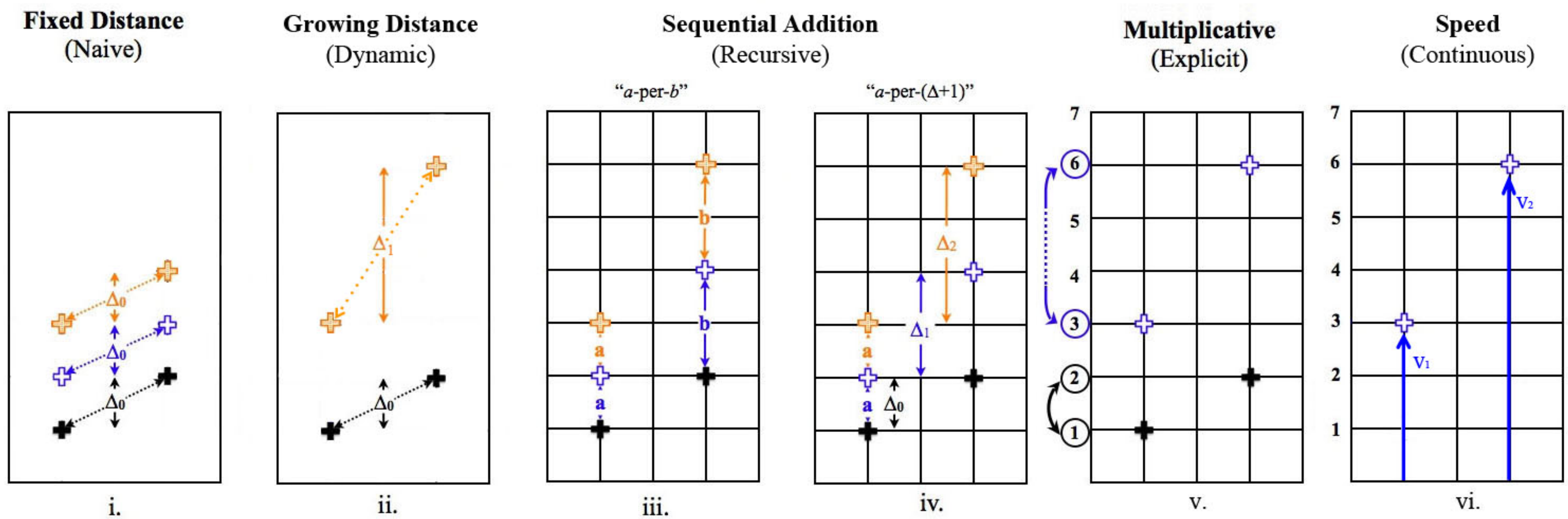
Students discovered a variety of strategies for "performing green." Whereas these strategies are schematically different, they are mathematically commensurate. That is, the subjective experience varies across the strategies, and yet technically speaking each strategy could be derived from the other. Most important for our study, the iterated-adding strategy ("beginning from 1:2, go up 1 on the left and 2 on the right, then over again") and the multiplicative strategy ("wherever the left hand is, the right hand is double as high") are compatible ways of generating a sequence proportionally equivalent ratios (1:2, 2;4, 3:6, etc.).

We begin this section by describing students' main solution strategies (Section 3.1). Next we present three episodes illustrating how students transitioned between and coordinated among the various strategies and, in particular, how they connected embodied additive strategies and arithmetic multiplicative operations (Section 3.2).

3.1. Student Generated Schema

Figure 4 (see below) offers schematic representations of the main solution strategies we observed across all the participants for the MIT–P problem "make the screen green" (for a more exhaustive list of strategies, see Howison, Trninic, Reinholz, & Abrahamson, 2011; Reinholz et al., 2010). For illustrative clarity, all the diagrams depict solutions for the same ratio, 1:2, that is, the right cursor needs to be twice as high along the monitor as compared to the left cursor.

All participants articulated a strategy relating the crosshairs' elevation (how high along the screen the pair was) and the vertical spatial interval between the left and right hand crosshairs (the distance between the hands, henceforth "Δ"). Initially hypothesizing that Δ should remain constant ("Fixed-Distance"; Figure 4i), ultimately they inferred that Δ should vary with elevation (Figure 4ii-4vi).



Note: RH = right hand crosshair; LH = left hand crosshair; Δ = spatial interval between the LH and RH crosshairs (construed as either vertical or diagonal)

Figure 4. Student generated schema for performing green (the case of a 1:2 ratio, hands rising either simultaneously or sequentially): (i) Fixed Distance—maintaining a constant Δ (recursive, simultaneous); (ii) Growing Distance—increasing Δ in relation to the height of the hands/crosshairs above the baseline; (iii) Sequential Addition—LH and RH sequentially rise at respective constant values *a* and *b* ("a-*per*-b"); or (iv) LH rises by *a* (usually 1), RH by 1 box more than the previous Δ; (v) Multiplicative—directly expressing green hand-pair locations as a function of the height of only one of the crosshairs; for example, determining LH *y*-axis value, then multiplying (doubling or self-adding) to find RH, or determining RH value and dividing (halving); and (vi) Speed—LH and RH rise simultaneously yet with different velocities, v_1 and v_2 respectively, so as to maintain green, where velocity may or may not be expressed quantitatively, either as a comparison between velocities or as two independent rates.

We shall now provide a brief overview of each of these strategies.

3.1.1. Fixed distance. While exploring the MIT–P in the "continuous space" or "crosshairs" conditions (see Figure 2a-2b, above), students articulated a relation between the following two spatial dimensions as associated with the feedback: (a) the hand-pair elevation (i.e., height from the baseline); and (b) the distance between the left and right trackers (whether the actual diagonal distance or the vertical projection of that distance, what we might call the *y*-axis interval). In particular, once students had found an initial "green spot" and were prompted to find another, all participants expressed a situated theorem for each of these dimensions. Initially, students found that the right crosshair should be higher than the left yet believed that this distance should remain constant as the crosshairs move. A typical statement a student made was, "I think it's just, like, you stay the same distance apart"—a *fixed*-distance theorem.

However, upon further exploration all students eventually discovered and articulated a theorem regarding the *covariance* of height and distance, that is, a *changing*-distance strategy. As we elaborate below, the changing-distance strategies (Figure 4ii-4vi) can be sub-divided into five main types: growing distance (dynamic); sequential addition (recursive), of which there are two types: *a*-per-*b* & *a*-per-(Δ+1); multiplicative (explicit); and speed (continuous).

3.1.2. Growing distance. Students inferred from the red feedback that Δ, the distance between the two hands or between the two crosshairs, co-varies with respect to height above the baseline. They stated, for example, "The higher you go, the bigger the distance." This growing-distance strategy is characterized as dynamic or continuous in nature and is typically articulated in qualitative terms.

3.1.3. a-per-b. Students often resorted to two types of "recursive" strategies, which indirectly express the height of the hands/crosshairs as a function of their previous locations.

First, "*a*-per-*b*" emerges as a proto-ratio, addition-based strategy. This strategy involves sequential hand motions, in which each hand separately moves up or down according to its respective quota. Students state, for example, "For every one I go up on the left, I go up two on the right."

3.1.4. a-per-(Δ+1). The second recursive-addition strategy we observed is "*a*-per-(Δ+1)." Again, each next green spot is found by moving the cursors sequentially, yet whereas the left hand moves by a constant value, the right hand then moves by one more unit as compared to the previous interval. For example, when asked if they had noticed anything about the distance between the crosshairs, students typically responded by stating that "the distance grows by one every time."

3.1.5 Multiplicative. We have observed students employing a strategy that can be characterized as multiplicative, whereby green hand-pair locations are determined directly as a function of the height of only one of the crosshairs. This strategy need not attend to previous pair locations and offers the "green" right-hand location for any given left-hand location and vice versa. Students determine a green hand-pair location by multiplying or dividing their *y*-axis values, either by multiplying directly (e.g., a student will state, "[you should] double the number that the left one is on, and you put the right one on that number") or by self-adding units (e.g., a student will state, "one plus one is two, two plus two is four" and so forth).

3.1.6. Velocities. Finally, we have observed students resorting to notions of "speed" as a means of expressing a strategy for enacting green, whereby the left and right crosshairs move simultaneously, up or down, each with a different respective constant velocity. Some students analogized their hands to moving vehicles, whereby each hand moves at a constant speed, v_1 and

v_2 respectively, yet stating that the "right hand moves faster than the left," or that "Like this [LH] one's going 20, and this one's [RH] going 50. And they have to keep on going…"

3.2 Case Studies—Transitions and Coordinations: From Additive to Multiplicative Strategies

In Abrahamson, Trninic, Gutiérrez, Huth, & Lee (2011) we analyzed for mechanisms underlying students' guided discovery of effective additive and multiplicative schemes. We are now attempting to determine the nature and quality of any links students may have constructed *between* these complementary operatory schemes, and in particular between *a*-per-*b* (see Figure 4ii) and the multiplicative rule (see Figure 4v). We will thus focus on paradigmatic data episodes exemplifying students' success or failure to coordinate these additive and multiplicative strategies. Specifically, in this section our analyses focus on transitions from iterated-adding embodied schemes (Figure 4iii-iv, above) to multiplicative strategies (Figure 4iv, above).

Per our interview protocol, once students had formulated different solutions to a single problem—the problem of keeping the screen green while moving the hands—we guided them to reflect on relations among these different solutions. In particular, we invited the students to evaluate the mathematical commensurability of different strategies for generating a sequence of number pairs that we think of as proportional (e.g., [1,2], [2,4], [3,6], [4,8], etc.). As mathematics educators, we see value in these activities, and so we are interested in how students attempt to accomplish the coordination of these different strategies.

Each of the subsections below presents a vignette from our videographed interviews so as to demonstrate and explain students' accomplishments in coordinating additive and multiplicative strategies. Emerging from our analyses of these vignettes is a common form of reasoning across the vignettes: students determined the commensurability of the strategies via discovering a heuristic link between them. As we shall see, the link may be a schematic element common to the two strategies, such as the notion of "half," or it might be imported from other knowledge domains, such as coordinating distance-based and velocity-based strategies by importing "time" and the knowledge that time is inversely related to velocity. Thus what is common to the three following vignettes is that they all portray learners heuristically leaping from additive to multiplicative strategies. The vignettes differ in the leaps' contextual catalyst.

3.2.1. The heuristic role of "half." In this first section we present and analyze video data from a paired-student interview, in which the dyad collectively discover, articulate, and verify *a*-per-*b* and multiplicative strategies for enacting green and, moreover, coordinate between these strategies, thus explaining how these two strategies are not only related but in fact equivalent.

Eden and Uri, two Grade 6 male participants, were selected for a paired interview on the basis of compatible mathematical achievement (both had been identified by their teachers as "high achievers"). Their interview was conducted by an apprentice researcher (DT), with the lead researcher (DA) occasionally intervening. The students sat side by side in front of the remote-action sensor system and computer display, and each operated one of the two tracker devices (right-tracker device [RT] and left-tracker device [LT]). The students were presented with the task of making the screen green under an unknown 1:2 ratio setting, and then under a 1:3 ratio setting.

Working under the 1:2 ratio setting and in the grid condition (see Figure 2c, above), Uri and Eden's collaborative hands-on problem solving enabled them to notice and explicitly articulate a co-variant relation between the crosshairs' height and distance. Specifically, by using the grid as a means to measure the spatial interval between the left and right crosshairs at various green locations, they concluded that "the higher you go, the more boxes it is apart." Furthermore, the dyad's exploration with the grid shifted them from the "higher–bigger" strategy toward a proto-ratio *a*-per-*b* strategy.

We posit that instrumenting themselves with the grid's quantification affordances enabled Eden and Uri to reconceptualize space, giving rise to the *a*-per-*b* strategy. In particular, they co-discovered that in order to maintain green, they should progress at coordinated vertical intervals of ½ (Eden) and 1 (Uri), either both going up the screen or both going down. The dyad concludes that, "For every box he [Uri] goes up—you [Eden] have to go up half." Similarly, once the *y*-axis numerals were overlain on the display (see Figure 2d, above), Uri and Eden reformulated their recursive *a*-per-*b* strategy as a closed/explicit formula, that of "halving" (i.e., the left crosshair is half the height of the right crosshair), thus they collectively transitioned from an embodied additive scheme to a multiplicative one.

The transcription that follows documents the dialogue immediately after DA asked Eden and Uri whether their additive and multiplicative strategies are related.

Eden: Uh, well, I think they basically mean the same thing, because if I go up one half [unit on the left] and he goes up one [unit on the right], it's the same thing as he being up twice as much as me.
Uri: Yeah.
Eden: So if he's up at ten and I'm up at five, I still move up half as much as him.
DA: Ah, I see. So there're two ways of speaking about this idea of "half."
Eden: So you can say I move up this much or he moves up that much. It's like I move up half or he moves up twice as much as me. Maybe he moves up two times faster.
DA: Do you see what's happening with this word "half?" It's very interesting. Sometimes I think you use the word "half" to mean *half a unit*,… [gestures deictically toward the computer display and indicates iconically half of a unit]
Eden: Yeah.
DA: …and then almost in the same sentence you say "half," when you mean *half as much as him* [referring to Uri].
Eden: Well…
DA: Do you see what I mean? What I am doing with this word "half?"….
Eden: So it's like, well, with the units part, it's like he moves up one unit [RH] and I move up half a unit [LH]. Where the other way to say it is he moves up two times faster than the normal speed, I move the normal speed.
DA: Uh-huh. Wow, that's even another thing about this—

Uri: Or *I* go up the normal speed, *he* goes up half the speed.
DA: Well one of you is normal… [laughter erupts]

The transcription suggests that the boys first "leapt" from the *a*-per-*b* strategy to the multiplicative relation ("if I go up one half and he goes up one, it's the same thing as he *being up twice as much* as me") and only later connected these conceptualizations via reinterpreting *a*-per-*b* as two different velocities ("he *moves up two times faster*").[3]

Both insights are remarkable and represent the types of connections that we believe underlie fluency in multiplicative concepts. Looking in particular at the first insight, we note the function that "half" played in moving from the iterated-adding strategy to the multiplicative strategy. "Half" is a polysemous construct that encompasses, among other meanings, unit measures (e.g., half an inch) as well as fraction-as-operator (e.g., half as much of/as something else). When Eden transitioned from half-a-unit to half-as-much, his achievement was in loosening the grid's visual grip so as to see a half not of a single unit but of a discretized number line. Reciprocally, if you will, he was able to see the discretized number line as a large unit.

Looking at the second insight, we note—perhaps over-fastidiously—that the notion of rate-based velocity only "feels" as though it logically connects *a*-per-*b* and $2x = y$. This type of loose generalization, which may be a necessary stepping stone en route to a formal generalization, is what Radford (2003) might call "contextual" rather than "symbolical." Perhaps only an algebraic formulation would adequately derive x and y values from an *a*-per-*b* rate. That is, the formal proof would demonstrate algebraically that equal sets of a and of b, respectively, sum up to the number-pair *ka*-and-*kb* that is proportionate to the pair *a*-and-*b*. Less fastidiously, though, we believe that these students' insight is the conceptual dawning of proportion.

Several minutes later, yet still working in the grid-and-numerals condition, Uri and Eden

[3] For a discussion of cognitive–developmental relations between rate and speed, see Thompson (1994).

were presented with the new task of making the screen green under a different, unknown 1:3 ratio setting. The dyad moved adroitly between a 1/3 *of a unit* and 1/3 *as much as*, and they were able to explain coherently how their solution was analogous to the 1:2 case.

3.2.2. Working with two perceptual frames. Next we present the case of a student, who used the linking phrase "half is bigger" to successfully coordinate between a dynamic changing-distance scheme and a halving multiplicative scheme.

Liat, a Grade 6 female participant (identified as medium-achieving by her teacher) was seated between lead researcher (DA) and apprentice researcher (DT) in front of the remote-action sensor system and computer display. For most of the interview, Liat operated both the right-tracker device (RT) and left-tracker device (LT), with the researchers DT or DA occasionally taking control of one of the trackers in order to prompt Liat's emerging strategies.

In the initial conditions of the 1:2 ratio, with only the crosshairs visible, Liat commented that RT should be higher than LT. Only once the grid was overlaid on the screen did Liat articulate a growing-distance scheme: "If it's farther up, then it has to be... They [the trackers] have to be more apart." Working accordingly, Liat then manipulated LT and RT up and down the screen, creating a continuous green feedback. When the numerals were overlaid, a researcher probed Liat's reasoning by holding the RT fixed at a certain value and asking Liat to predict the corresponding location of the LT that would produce the green feedback. For instance, when the RT was held at 10, Liat predicted that the LT should be "a little bit higher than 6." After some further structured exploration, Liat initiated the multiplicative strategy "half of the number."

Subsequently (see transcription, below) Liat discovered a logical relation between the changing-distance ("farther up....more apart") and multiplicative strategies ("half of the number"). As we shall see, this coordination hinged on construing the vertical distance between the crosshairs—the “apart” interval that we have earlier referred to by Δ—as “half the number.”

DA: So when this one [LT] is at one, where will that one [RT] be?
Liat: This [LT] is at one [lifts LT, moves it to horizontal grid Line 1], and this [RT] would be at two, I think. [lifts RT, moves it to Line 2; screen turns green]
DA: Ok. Now, if you wanted to go up—
Liat: Oh, then it would—and then it gets—then it gets farther apart, because the number gets bigger.
DA: Which number gets bigger?
Liat: The number—well. Hmmm. ‘Cause it’s farther apart, every time it goes...the...the...the right nu...The number that the right is on, it gets higher, so they have to be farther apart, because half of it is bigger than the number before it.
DA: Half of it is bigger than the number before it?
Liat: Er...If this one was at...
DA: Could you, could you kindly use the word “left” and “right” when you say “this one”? I’m not sure which one you mean.
Liat: Because if this one [RT] is at four, then this one [LT] has to be at two because it’s half, and then if I were to put the right one on five, then, and I would have to, I’d have to put it [RT] on three and a half... or *two* and a half, and then, so it gets farther apart because there is a bigger difference between the numbers.
DA: Ah, so the one on the right went up, like, went up one?
Liat: Yeah, so there is a bigger space between the left and the right.
DA: Huh, is that something that keeps on happening? Or not? Oh. Uh huh.
Liat: So I go to six [LT], and then to three [RT]. So there is a bigger space between the right and the left one.

Liat was successful in coordinating between the growing-distance and halving strategies, because she blended their elements in building a logico–mathematical inference. In particular, she attended to a spatial element from the growing-distance strategy—the distance itself—as being in flux *because* it is always half as much as the greater number. We find this analysis intriguing, because it supports the notion that some mathematical learning relies on seeing one and the same distal perceptual element in two different ways, each of which is mathematically sound (Godino et al., 2011) yet both of which give rise to new understanding (Abrahamson, 2006; Abrahamson & Wilensky, 2007; Abrahamson, Gutiérrez, & Baddorf, 2012). In this case, the understanding resulted not from a need for reconciliation of explicit conflict as much as from an intellectual need (Harel, in press), partly mediated by the tutor, to condense competing problem-solving strategies into succinct coherent form.

3.2.3. The heuristic role of ratio. Shani, a female Grade 5 student, who had been identified by her teachers as mathematically low-achieving, was seated between researchers DA and DT in front of the remote-action sensor system and computer display. During most of the interview, only Shani operated the trackers, though DT and DA occasionally controlled one of the trackers to prompt and structure her reasoning.

During the pre-grid interview phase, Shani located three "green" hand positions. A researcher then structured Shani's inquiry by holding RT at specific locations on the screen and, for each successive location, asking Shani to move LT such that the screen displayed green. During this interaction, Shani first commented that performing green relies on controlling the distance between the trackers (and in particular, the diagonal distance; see Figure 2ii).

The researchers then laid the grid over the screen and kept structuring Shani's inquiry, this time by controlling LT. Recall that per each unit that LT rises under the 1:2 ratio setting, Δ grows by *one* unit whereas RT rises by *two* units (compare Figures 2iv and 2iii). As she controls RT in an attempt to make the screen green, Shani's attention vacillated between Δ and RT itself, and her multimodal speech–gesture utterances accordingly demonstrated a strategy shifting from *a*-per-(Δ+1) to *a*-per-*b*.

The researchers next drew Shani's attention to the numerical values of the "green" locations by asking her to recount which pairs had generated a green screen. In response, Shani operated the *a*-per-*b* strategy, reading out each successive number pair she thus generated. Yet in so doing, she became aware of the within-pair multiplicative relation and consequently shifted from an additive to a multiplicative conceptualization of proportional progression, as follows.

DA: So, what else can you say about those numbers? One and two...

Shani: One and two, then two and four, three and six. Hey wait. Um, oh, it's... [fidgets body, becomes animated] It's all doubles. The bottom number, like time... times two is the top number. [points at monitor] We had, like, one and two, then three and six, then, um, then four and eight, then five and ten.
DT: Mmhm.
DA: Interesting. Huh. So you've said...
Shani: Because they're each going...'cause this [RT]... this one's always going up by two, and this [LT] one's going up by one, which would mean that...
DA: Which would mean that what?
Shani: That, uhm, this one [RT] is always double this [LT].

In analyzing Shani's insight, above, one is liable to gloss over the tentativeness of her inference. Why, precisely, do successive *a*-per-*b* actions "mean" a constant *b/a* relation? Just because the iterated left and right quotas relate by *b/a*, does that "mean" that their respective running totals necessarily relate similarly? The reader is invited to sketch a convincing argument from Shani's premise to her inference. Can one do so without resorting to the distributive property of multiplication over addition? Is not that property a critical inferential link that is absent in Shani's reasoning or, at least, in her verbalization?

Granted, as constructivist designers we are the first to laud Shani's achievement. At the same time, as educational researchers we are intrigued by the nature of this inferential reasoning. Whereas we do not have evidence to support the claim that Shani's reasoning was pedagogically valuable, we believe that her efforts reflect precisely the type of quantitative reasoning called for by progressive mathematics-education researchers (e.g., Arcavi, 2003; Thompson, 1993). That is, Shani's sense that particular repeated-adding actions implicate particular multiplicative relations could serve as an epistemic substrate grounding the formal proof she is yet to learn. Still, we wonder how much of Shani's capacity to reason thus is a cognitive primitive—say a search for cause–effect patterns—and how much of it is a cultural imposition of "how we think."

The interview continued with a researcher physically guiding Shani's hands in performing a fluid 1:2 enactment of green. As her hands moved up, Shani noticed that RT moved faster than LT. As we see in the transcription below, Shani successfully linked this scheme to the recursive-additive scheme.

> DA: ...I want to see if we can—if I can track your hands all the way up in green. I'm going to find a green place... [guides Shani's hands such that she performs green while moving up the screen.]
> Shani: So this one should be... so my right should be moving faster.
> DA: Oh, I see.
> Shani: So that it can make... be going up two spaces on the grid....while the other one is only going up one.
> DA: Oh, I see.

We wish to draw attention once again to the inferential structure inherent in Shani's reasoning. In particular, note how Shani conceptualizes the greater velocity of RT as compared to LT as *enabling* the enactment of the *a*-per-*b* strategy: "...my right should be moving faster....so that it can....be going up two paces on the grid....while the other one is only going up one." As in her earlier inferential reasoning, here too a critical element—*time*—is omitted. The notion of time, so intrinsic to the coordination of two hands moving simultaneously at different rates, still awaits liberation:

> Confrontations such as these can serve....to "liberate" previously unseen, unsaid, and unused aspects of the phenomena, while at the same time creating the powerful potential for students and teachers, children as well as adults, to choose selectively and effectively among them, depending on when, where, and what they want to use them for. (Bamberger & Ziporyn, 1991, p. 55).

Shani's positioning of $V_{LT} < V_{RT}$ action as enabling *a*-per-*b* enactment is further intriguing, given that earlier she positioned *a*-per-*b* as causing the constant *b/a* (doubling). We move at different rates...so as to complete different quotas...so as to maintain a constant multiplicative relation. Where does this causal hierarchy originate? We return to this issue in this next section.

4. Conclusion

Under auspicious conditions, students are able to leverage their rich naïve knowledge to connect mathematical notions in ways that go beyond their learning. These notions may be pre-symbolical and, from a strict mathematical perspective, ill articulated. Likewise, students' inferential reasoning can be informal, in the sense that it glosses over "invisible" situated prosperities, such as time, and it omits logical steps, such as the distributive property of multiplication over addition. These rhetorical lacunae can be implicated in the distributed nature of student reasoning, which heavily relies on tacit embodied skill and cues in material objects. In particular, students are able to build rich dynamical images anticpating proportional reasoning.

It thus appears that underlying student engagement in embodied-interaction mathematics learning activities is a host of unsaid cognitive primitives and socio-epistemic norms. If we are to use learning environments such as the Mathematical Imagery Trainer, in which mathematical ideas are plotted as dynamical transformation that students learn to enact and reflect upon, we should better understand the relation between perceptuomotor intuition and inferential reasoning. By developing theoretical models that are geared to explicate the perceptuomotor grounding of mathematical reasoning, we would be better geared to foster and evaluate student learning. In particular, we would better be able to conceptualize the roles of heuristical inferential leaps in students' quantitative reasoning (Abrahamson, 2009; Abrahamson, Gutiérrez, & Baddorf, 2012).

Given the situated nature of students' embodied enactments, several theories of learning come to mind as bearing analytic traction on our empirical data. One theory is "Knowledge in Pieces" (diSessa, 1988), because it offers a framework for charting the evolution of embodied knowledge toward formal reasoning as a graduate "tuning toward expertise" of phenomenological primitives. Another is the semiotic–cultural theory of objectification

(Radford, 2002), because it offers tools for interpreting students' multimodal utterance as the appropriation of available forms in the learning environment as semiotic means for expressing emerging pre-symbolic notions (see also Saxe, 2002). Furthermore, DNR (Harel, in press) critiques the intellectual needs of students engaged in mathematical problem solving, thus offering lenses for understanding students' motivation to construe cause and effect in mathematical relations. Finally, Sfard (2007) underscores the problem of learners needing to adopt the leading discourse that incorporates new ways of looking at situated phenomena.

In our own work, we are attempting to integrate these diverse views into an emerging framework for the construction and analysis of mathematical learning (Abrahamson, 2012, in press). This framework will inform our iterated development of the Mathematical Imagery Trainer, both for proportion and for other concepts; hopefully, it will guide our own heuristic leaps as we attempt to make sense of student sense-making.

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