DISCOVERY RECONCEIVED: PRODUCT BEFORE PROCESS

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This article is motivated by a commitment to the ideas underlying discovery learning, namely the epistemological notion of grounded, meaningful, generative knowledge. It is also motivated by concern that these ideas have been implicitly misinterpreted in curriculum and instruction, ultimately to the detriment of students. Accordingly, I discuss an alternative, empirically based, theoretical articulation of discovery pedagogy that addresses the criticisms it has faced. The research question framing this alternative approach is, "What exactly about a mathematical concept should students discover via discovery learning?"

I will pursue this question by reflecting on two case studies of children who participated in activities of my own design. Empirical data from these and other studies have served me over the past decade as contexts for inquiry into the cognition and instruction of mathematical concepts, an inquiry that, in turn, keeps feeding back into further design and articulation of design principles. In this essay, I will use these data to offer an empirically grounded "centrist" answer to the question of what students should discover, at least with respect to a particular class of mathematical concepts (intensive quantities) as embodied in a particular type of design (perception-based learning). First, though, some further clarification of terms is due.

The rationale of my proposal hinges on a common distinction between process and product in mathematical learning activities. By *process*, I am referring to a general problem-solving sequence: (a) construing, parsing, and modeling a realistic situation along dimensions relevant to goal information; (b) determining targeted values within these dimensions (by enumeration and/or measurement); (c) manipulating these extracted values algorithmically with the aid of further mathematical instruments, tools, forms, and media, such as inscribing and developing an algebraic formula; and (d) reinterpreting obtained values or inferences in light of the source situation (*e.g.,* Verschaffel, Greer & De Corte, 2000). By *product*, I refer primarily to any of the milestone mathematical displays created through engaging in the process, including inscriptions, such as diagrams, and multimodal utterance, such as speech or gesture. An example of a product would be the event space of a probability experiment, which is created through combinatorial analysis of a random generator—a novice can be guided to build this product, yet only an expert can infer from it an anticipated outcome distribution.

I do not cast my articulations of the terms "process" and "product" as offering in-and-of-themselves any unique insight into the nature of mathematical cognition. I am not referring to epistemological debates about arithmetical operations as processes or objects (*e.g.,* Confrey & Costa, 1996; Sfard, 1991). Similarly, by process, I am not referring to schematic mental activity with operational closure (Varela, Thompson & Rosch, 1991, p. 139) but to the cultural rote algorithmic practice that can be enacted piecemeal and imitatively with little to no grounding or clear goal-based orientation. By product, I am not referring to an interim state of self-adapted schemes resulting from situated enactment, but to external or externalized information, such as inscriptional artifacts or vocalized or gestured expressions that come to exist in the world, in forms that Norman (1991) might call "cognitive artifacts" or what Hutchins (1995) might view as means of instrumenting and distributing collective cognitive activity.

Finally, my objective is not so much to survey or evaluate the variety of pedagogical philosophies in dialogue with constructivism (see Cobb, 2005; Freudenthal, 1991; Norton, 2009; Radford, 2005). Nor will I comment on whether mathematical knowledge is formed as a cognitive schema, social practice, semiotic system, *etc.* Rather, I am exploring whether the exacting tenets of the more radical forms of constructivist mathematical pedagogy may be upheld, yet adapted, so as to include instructional situations in which the locus of reinvention is the solution product rather than process.

From process to product: relocating the locus of meaning making

What is common to the instructional designs discussed in this paper is that all of them begin with some carefully engineered situation—a collection of stimuli in the perceptual field—that the instructor invites the learner to examine with the objective of determining some quantitative or logical property or anticipated outcome. What is further common to these designs is that the learners are perfectly capable of responding to the instructor's prompt with an answer. They do so without mathematical analysis, using only perceptual judgment, heuristic inference, and common sense. They might thus offer qualitative, rather than quantitative, responses, and yet these responses are all interpretable by experts as agreeing with a mathematical view on the same situation. Once the learners have offered their inference, the instructor walks them through a formal analytic process appropriate to the situated task objective. The compliant learners do not yet take this new, secondary task to be vitally pertinent to the initial, primary task, essentially because the new process attends to and parses the situation in unfamiliar ways that do not appear to align with the child's intuitive inclination (see Bamberger & diSessa, 2003). Moreover, the new process introduces structure and computation where the child may not have used any, at least not explicitly. It is only once the formal analysis process is complete and the child has created a product that the instructor considers to be a suitable model of the situation relative to the task objective, that the instructor can help the child to adopt a way of seeing this product as aligned with their naked-eye inference from the source situation.

This new seeing is a heuristic semiotic achievement by which learners discover a means of perceiving the product as signifying their own intuitive inference from the situation it is said to model. Through the completion of this heuristic semiotic coordination, the child retroactively comes to accept the process (Abrahamson, 2009). In a sense, the child has been guided to reinvent the mathematical concept. However, unlike cognitive-developmental perspectives by which all conceptual learning is subjective invention or construction (Piaget, 1968) and unlike pedagogical philosophies by which students are guided to reinvent solution processes (Freudenthal, 1983), here, it is not the process that the child reinvents, but a way of accepting its product as meaningful and valid, even though the product may initially have made no sense at all or even *because* it appeared to imply a conflicting inference.

Two vignettes

In the remainder of this article, I will present a set of instructional materials and problem-solving activities that I have used in my research on discovery-based mathematics instruction, as well as brief vignettes of children who engaged in these activities guided by an experienced investigator–tutor. This design involves intensive-quantity concepts, such as geometrical similitude, density, chance, and slope. My focus on proportionality appears not, in retrospect, to be coincidental. Rather, it may point to an emerging conjecture that the human mind is pre-equipped to perceive certain intensive quantities as phenomenological gestalts (Gelman, 1998; Suzuki & Cavanagh, 1998; Xu & Vashti, 2008). This conjecture may challenge conventional mathematics pedagogy by suggesting we reverse the learning trajectory so as to begin with gestalt embodiments of intensive quantities rather than their elements. At the same time, Piaget and Inhelder (1969, pp. 46-49) contend that early perceptual capacity does not engender mathematical knowledge directly, but only through action and reflection on this action. As such, the constructivist pedagogical task becomes to find ways of supporting the learner in coordinating this naked-eye gestalt seeing with its counterpart mathematical analytic seeing (Abrahamson, 2007).

The following two vignettes present proof of concept for the plausibility of a product-before-process approach to guided reinvention. The particular designs presented in the vignettes resulted from study cycles that began from close listening to children's reasoning (Confrey, 1991; Davis, 1994; Ginsburg, 1997). My approach is to nurture children's intuitive reasoning as their epistemic foundation of mathematical knowledge. I do not wish children to capitulate their intuitive reasoning in favor of mathematical analysis but to sustain their reasoning and coordinate it with mathematical analysis.

The objective of the vignettes is not so much to sketch a comprehensive view of instructional interaction. As such, while I acknowledge the tutor's vital, constitutive role in the process of knowledge mediation, I have chosen strategically to "dampen" this voice and focus on the child's perspective.

Finally, I will not (and cannot) make any categorical claims about these students' ultimate learning. What I am looking for in these data are fragile moments, in which students abduct mathematical rules or notions that I view as kernels of understanding.

Vignette 1: Seeing chance in a sample space

Li, 11 years old, was one of 28 Grade 4–6 students, who each took part in a one-hour semi-structured one-to-one clinical interview designed to accomplish two objectives. I wished both to investigate young students' cognitive resources for making sense of compound random events and to evaluate the prospects of a set of instructional materials to support content learning in probability (Abrahamson, 2009). The students had not studied probability formally.

Participants were presented with an urn-like random generator (see Figure 1a)—a box full of hundreds of marbles of two colors, green and blue, with half of the marbles being of each color. A "marbles scooper" was used to draw from the box a random sample of exactly four marbles that each fell into a concavity, so that samples appeared in fixed spatial configurations. Participants were first asked to guess the results of experimenting with the marbles and scooper. I guided the study participants to perform only as many scoops as were necessary to understand the nature of this mechanism. By comparison, designs for probability usually have students conduct extensive empirical experimentation with random generators. *I skipped this "empirical" phase*

Figure 1a. A marbles scooper.

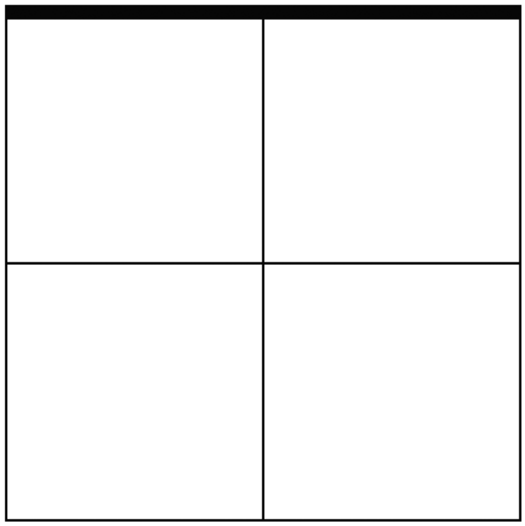

Figure 1b. A card for constructing the sample space of the marbles-scooping experiment (a stack of such cards is provided, as well as crayons, and students color in all possible events).

entirely. Next, a stack of cards, each bearing a 2-by-2 matrix (see Figure 1b), as well as a blue and a green crayon, were provided for conducting a combinatorial analysis in accord with the classicist procedure. Participants were guided to create all sixteen (2^4) possible outcomes of the experiment and then sort the sixteen events into five stacks by the number of green and blue cells in each card. I analyzed for any connections participants discerned between their initial guesses and features of the event space and how they reasoned about these connections. [1]

I elicited predictions from the study participants for the experiment's long-term outcome distribution. By and large, they stated that a 2-green-and-2-blue event is the most likely draw from the box, a 4-green scoop would be rare, *etc*. These qualitative predictions agree with mathematical analysis. They were not, however, based on empirical experimentation but on perceptual analysis alone (see Xu & Vashti, 2008, for babies' analogous behavior on an age-appropriate version of this task). It thus appears that the participants parsed the situation as five aggregate events: 4-blue, 1-green-and-3-blue, 2-green-and-2-blue, 3-green-and-1-blue, and 4-green (hence, 4b, 1g3b, 2g2b, *etc*.). Indeed, they tended to build sample spaces comprising only five cards as the case of Li, below, illustrates.

The children did not initially consider this construction activity as a means of modeling the likelihoods they had just predicted. More specifically, they did not explore order-based variation on the five combinations as potentially signifying their stochastic properties. Rather, they insisted that five cards exhaustively showed all possible outcomes. Nevertheless, once the combinations tower was completed, all but one of the children eventually realized that the tower could be used to support their earlier anticipation of outcome distribution. For example, they asserted that samples with 2g2b marbles would be drawn most often *because* there were more elemental events in the 2g2b column than in any other column. They therefore expanded their five-object parsing of the experiment into a five-*set* parsing, so that all sixteen elements in the space—not only five of them—took on the signification of their naïve sense of outcome distribution.

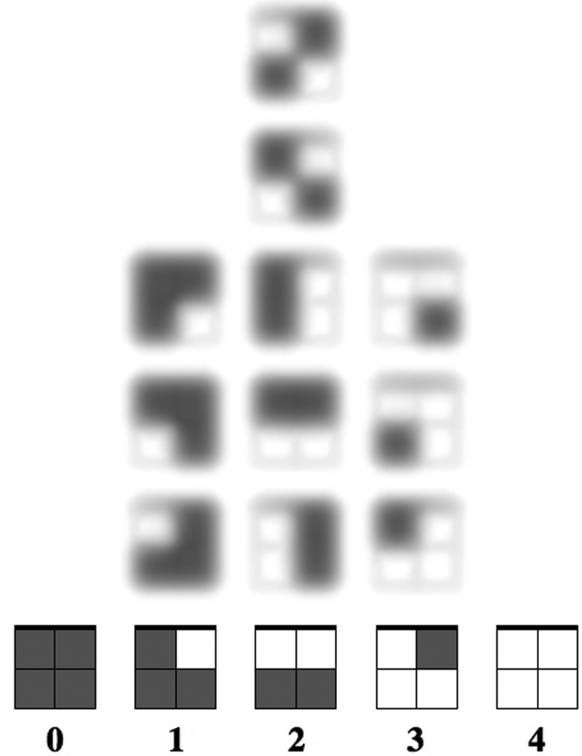

Figure 2a. Li's mental construction of the sample space as a model of the marbles-box experiment: five equally likely events (bottom row) and eleven redundant duplicates (above the bottom row).

The Li data are particularly helpful in supporting the notion of product-before-process guided reinvention, because his trajectory through the design was by-and-large typical of all participants, yet he was exceptionally articulate in his deliberations around the product. [2]

Like all participants, Li initially parsed the experiment as five possible events (with 2g2b being the most likely) and, therefore, created a sample space of only five cards (one of each of the five combinations or aggregate events). Gazing at these five cards, however, Li changed his original prediction, stating instead that these five events all have the same chance (*i.e.,* a uniform distribution, in which the five combinations or aggregate events are equally likely). Consequently, Li engaged reluctantly in the process of combinatorial analysis—he certainly did not initiate this process and he did not see how the new products it would generate (the variations) could serve the task objective as he perceived it. Once he had created the remaining eleven cards, Li said emphatically that they were not relevant to modeling the expected outcome distribution and were thus redundant (see Figure 2a). But by now the entire event space had been created, and so the ensuing conversation was not about the process of combinatorial analysis but about its product.

In the course of the interview, Li was subsequently guided

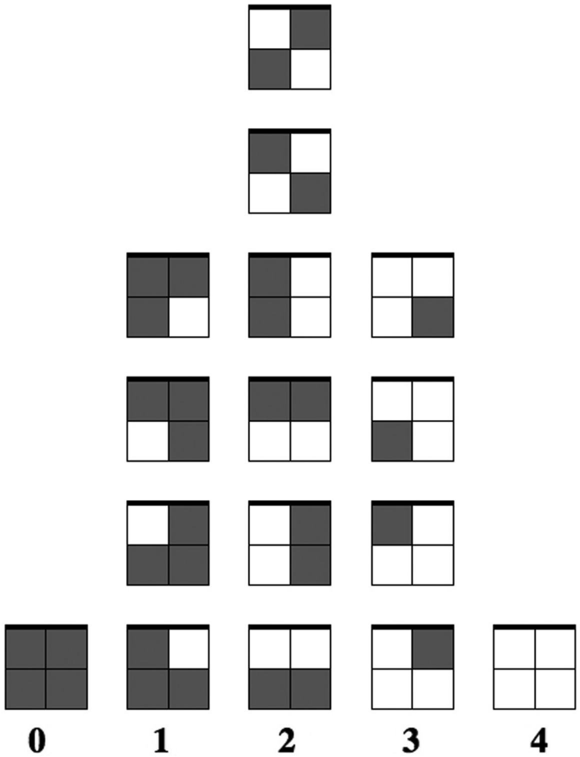

Figure 2b. Li's mental construction of the sample space as a model of the marbles-box experiment: sixteen equally likely events sorted into five sets.

to revisit the marbles box, whereupon he re-affirmed his initial prediction of five heteroprobable events. Li consequently changed his view of the combinations tower: he now looked anew at the sample space in its entirety with the active objective of seeing it as signifying a heteroprobable, rather than an equiprobable distribution. The 1-4-6-4-1 structure now appeared to Li as affording an expression of his naïve sense of likelihoods. The eleven cards thus shifted in their semiotic status from being irrelevant objects to bearing information critical to the signification of the re-evoked tacit inference (see Figure 2b). Specifically, the vertical projections of the five columns came to signify the events' expected relative frequencies.

Li was thus guided to invent a way of seeing the product as expressing his qualitative notion of chance. Li invented a way of seeing chance in a sample space. Importantly for my thesis, Li did not invent the process of combinatorial analysis. Rather, only after engaging in this process and generating its product in the form of new material elements in the learning environment did Li experience a guided opportunity to reinvent the sample space.

In Piagetian terms, one might say that Li accommodated his naïve, "combinations only" schema so as to assimilate the mathematical "variations also" analysis. Yet the catalyst of this equilibration activity was not cognitive conflict in the classical sense of experiencing evidence that challenges an expectation. First, unlike in teaching situations that often give rise to cognitive conflict, Li's initial inference was evaluated by the researchers as mathematically correct rather than incorrect. Second, Li never regarded those eleven supplementary cards in the event space as evidence challenging his *a priori* holistic sense of the outcome distribution, because their epistemic quality was not empirical but arbitrary. Third, Li ultimately did not forsake his naïve view—his struggle was not to abandon it but rather to reconcile it with mathematical analysis. As such, the empirical data gathered in the context of implementing this particular product-before-process design do not lend themselves too well to modeling the situation as a simple case of cognitive conflict. Rather, the Li vignette is a case of socially constructed, pedagogically oriented conflict, by which asymmetrical interlocutors negotiate meanings for symbolic artifacts in their shared discursive space (Sfard, 2007).

Vignette 2: Seeing object constancy in proportional equivalence

Aliya, 8 years old, took part in a series of clinical interviews conducted in the context of a design-based research study that bore two objectives. I wished both to explore a particular perceptual capacity that could potentially serve as the grounding for proportional reasoning, and to evaluate the utility of a set of instructional materials to support the mathematization of this tacit knowledge. In her schooling, Aliya was just studying multiplication and could quickly recall several simple multiplications.

Aliya was presented with a set of picture cards of graded sizes that all featured two cartoon characters standing side by side (*e.g.,* Danny and Snowy, see Figure 3, overleaf). A subset of these cartoons were geometrically similar—they were in fact print-outs of the same image, so that their contents were identical, but they were of different physical sizes (see Figure 3). The other cards in the pool—the distracters (not shown in Figure 3)—featured the same two characters, but their heights were not related by the same ratio as in the target subset. For example, Danny was only half the height of Snowy. Aliya was asked to select from the pool of cards a subset whose common property would be that they were "the same only bigger/smaller".

To assist in determining this "sameness," Aliya was guided to perform the "eye trick" (Abrahamson, 2002). Shutting one eye so as to eliminate stereoscopic vision, you hold up in front of your open eye two cards whose similarity is in question, one card in each hand (see Figure 4). You then move the larger card away from your open eye and the smaller card nearer and adjust back and forth in an attempt to calibrate the retinal images. For example, once the Dannys in the smaller and larger cards appear to be of the same height, then, keeping your hands at those fixed distances, you check whether the Snowys, too, are of like height. If so, then the two cards are deemed to be "the same." During the enactment of the eye-trick, at the moment when the two objects project equally sized retinal images, the self-induced optical illusion is so powerful that even a mathematically informed person may momentarily construct the two objects as identical, not only similar.

Using the eye trick, Aliya tried matching different pairs of cards featuring Danny and Snowy, until she had selected out of the pool of cards a set that she judged as being "the same" as each other. Aliya was then asked to repeat the task

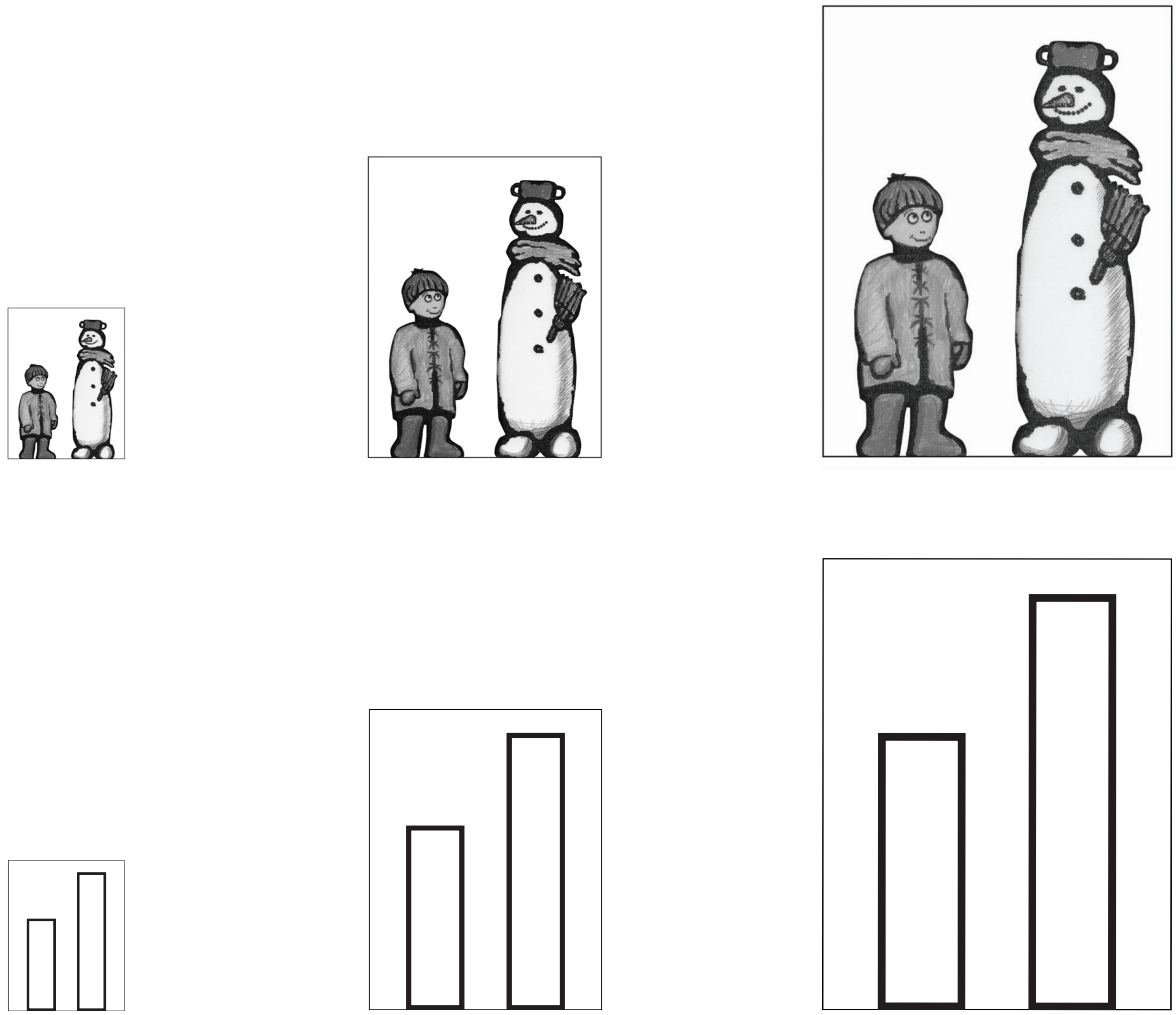

Figure 3. Materials used in Vignette 2. Two sets of cards depict pictures that are geometrically similar, and each set shows a pair of vertically oriented elements: Danny and Snowy (top), "buildings" (bottom). Within each pair, there is a 2:3 ratio between the heights of the two elements. In both sets, the measured heights of the elements in the three cards are [2, 3], [4, 6], and [6, 9] units, respectively.

with a pool of cards that featured "photographs" of a pair of "buildings" (see Figure 3, bottom row). Again, Aliya successfully used the eye trick so as to identify within the pool a subset of cards that were "the same." The initial inferences having been established, the interviewer then moved on to the mathematical analysis process. [3]

The interviewer produced a ruler and guided Aliya in measuring the physical sizes of the short and tall buildings in the smaller card. Aliya determined their respective heights as 2 and 3 units, and the interviewer drew her attention to the difference of 1 unit between these measurement values. The interviewer then asked Aliya to compare the smaller card to the middle-sized card, and Aliya said that they are of different sizes but depict the same picture. She then spontaneously measured the heights of the "buildings" in the middle-sized card and was very surprised to find that these heights were 4 and 6 units, for a difference of *2* units. In response, she stated that the pictures cannot be "the same."

Aliya had expected the difference between the measured values in the middle-sized card to be the same as in the small card. For Aliya, it appears, the experience of the cards' overall "sameness" implied that all their corresponding internal figural magnitudes, too, are necessarily "the same." That is, Aliya was not yet mathematically equipped to differentiate actual and proportional equivalences. Next, Aliya completed the process and reflected on the product, as follows.

The interviewer handed Aliya the next card in the series. Taking the card, Aliya said that if the corresponding difference in this third card turned out to be of 3 units, then the two first cards could still be "the same." She measured the difference and indeed found it to be of 3 units. Excited, Aliya proclaimed, "The bigger the picture gets, the more units apart it is!"

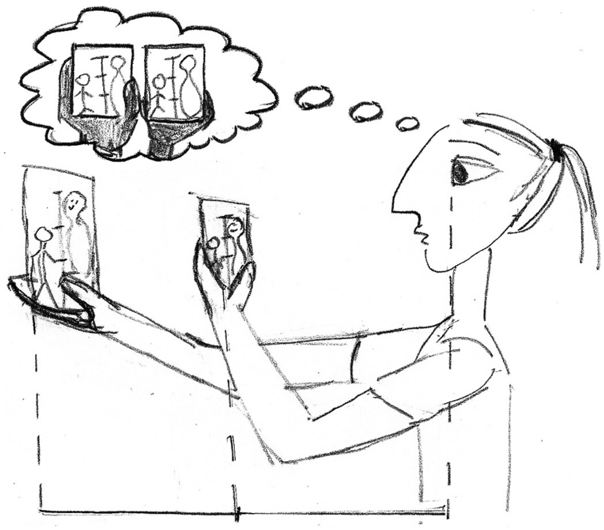

Figure 4. The "eye trick". This person's right eye (not shown) is shut. She will determine that the two cards are "the same." The small card is exactly half the size of the larger card and is held half as near to the eye as the larger card. Note the difference in hand sizes in the "see" bubble.

Aliya did not initiate the measuring procedure. Rather, this process was prompted by the interviewer's questions and literally modeled by the interviewer, who provided the ruler, demonstrated its function, and applied it specifically to determine the figures' respective heights. Thus, it is not the case that Aliya reinvented the process of investigating and representing the proportional equivalence of geometrically similar images. Nothing in the situation remotely suggested to Aliya a need to measure, because the eye trick had been framed as an appropriate and sufficient criterion of "sameness." Indeed, the analysis products (the measured values "1" and "2") surprised Aliya, because she had expected them to be equivalent but they were not. This perturbation drove Aliya to accommodate the core schema for identity (sameness) so as to assimilate a variant case, by which larger-yet-identical images bear greater dimensions. Namely, once she found a new numerical pattern (the consecutive counting numbers 1, 2, 3, …) in lieu of her anticipated constant (1, 1, 1, …), Aliya had a new grip on reality. Once again, we see an intervention, in which the child was steered by the instructor to appropriate a particular way of re-seeing a situation through the lens of cultural forms that sustain their initial informal inference.

Comparing the two vignettes

The cases of Li and Aliya have much in common. Both cases depict pedagogical interaction centered on a design for an intensive quantity. They are also alike in that each child's initial qualitative judgment as to quantitative, logical, or relational properties of a situation was in agreement with mathematical analysis. In both cases, the tutor guided the child through the enactment of an analysis process that the child had neither initiated nor appeared to view as relevant to investigating the properties in question. In both cases, the product of the analysis process initially caused the child to re-evaluate their original inferences, which they then reaffirmed only by referring back to the source situation and re-evoking their initial judgments and inferences. Finally, in both cases the child invented a way of seeing the product, which the tutor was offering as a model of the situation, as supporting their own naïve inference from that situation. In so doing, the children articulated conceptual notions that were the pedagogical objectives of the designs. Table 1 (overleaf) summarizes this comparison. [4]

The children's respective insights in the excerpts might appear to the reader to be somewhat haphazard or too "convenient." Namely, if the green and blue marbles in the box were not equal in number, then the combinations tower could not signify the naïve inference. Similarly in the case of Aliya, if the third card did not indeed bear the appropriate spatial interval between the two figures (*i.e.* 3 units), her anticipation of that particular numerical value would have been thwarted, and she may not have arrived at her insight. I can only concur and submit that students' perceptions, inferential reasoning, and insights are didactically contrived, in the sense that they are anticipated and designed for.

Finally, a vital question arises as to what these children have learned, in the sense of what they have developed as some new applicable skill. Is it enough to have re-invented a mathematical principle that reconciles a product as expressing one's intuitive inference for a situation? I can only hope that beyond the interactions I had with these children, I have stimulated their appreciation for the new process.

Conclusion

Charged with the formidable task of creating learning environments that optimize children's conceptual development, instructional designers informed by the learning sciences are receiving ambiguous messages. On the one hand, staunch radical constructivists warn of providing students with ready-made forms as supports for problem-solving tasks, because to do so, they believe, deprives learners of the most precious cognitive rewards of inventing strategy bottom-up. On the other hand, sociocultural theorists object that a blanket "don't show them" fiat is absurd, given that children are naturally immersed in a top-down world of artifacts, procedures, values, and norms that are implicit in any social practice in which they participate, first peripherally and then more centrally. Given these apparently opposing bottom-up and top-down views, what is a designer to do? Might best practice lie somewhere in between, or would a reconciliatory compromise vitiate the functional integrity of both views? [5]

The vignettes presented in this paper demonstrate the plausibility of synthesizing rather than compromising constructivist and sociocultural approaches. In the proposed design framework, the instructor: (a) elicits and validates students' intuitive perceptual judgments for the properties of a situation; (b) engages the students in analyzing the situation using a formal process that results in a product, whether a display or a multimodal utterance; and (c) supports the students in seeing the product as resonating with their own common-sense judgment of the situation.

Designs based on this framework should structure with care both the situation source and its modeling resources: (1) a situation should be selected or constructed that affords informal inference evaluated by mathematically knowledge-

Dimensions	Case Studies	
	Li	Aliya
Content	Binomial distribution	Proportional equivalence
Intensive Quantity	Likelihood	Geometric similitude
Source Situation	A box full of green and blue marbles of equal number; device for scooping out four marbles	A set of cards featuring the same image printed at different sizes
Tacit Perceptual Construction	Infers heteroprobable five-event distribution in the marbles box	Eye-trick comparison of the two cards instated them as "the same"
Process	Combinatorial analysis: built and organized sample-space display	Applied a measurement tool to determine values for the cards' respective internal dimensions
Product	A sample space consisting of 16 possible outcomes organized in five adjacent "stacks" (1-4-6-4-1)	A set of values of 1 and 2 units for the measured lengths of respective elements in the cards
Naïve Framing of Product	Saw only five objects in the sample space as relevant	Anticipated that the respective values would be equivalent
Conflict, Loss of Tacit Perceptual Construction	Rejected the expanded sample space, posited an equiprobable distribution instead	Suspected the different differences, wondered if cards are indeed "the same"
Re-Evoking Tacit Perceptual Construction	Re-attended to the marbles box, once again saw distribution as heteroprobable	Re-handled the two cards, once again saw them as "the same"
Reconciliation, Reinvention	Articulated the rule of ratio (chance is "favorable outcomes / possible outcomes")	Articulated a qualitative rule of proportional equivalence relating overall and internal magnitudes
New Perceptual Construction	Saw a single card as an elemental event, not an aggregate event; expanded his attention to additional objects—from five objects to five object sets	Learned to orient toward internal physical magnitudes of the cards; saw different differences as (proportionately) the same

Table 1. Comparison of Vignettes 1 and 2.

able persons as correct, even if naïve and/or qualitative; and (2) the analysis process and product should catalyze and hone juxtaposition and eventual alignment of naïve and scientific views. This conceptual change requires a tutor—whether actual, virtual, simulated, or embedded—a cultural voice that frames, challenges, furnishes, and guides the novice's heuristic–semiotic leaps to new, more complex stability.

The framework has not been evaluated beyond its application to intensive quantities, where it might be uniquely suited. As such, evaluations and elaborations of guided-reinvention should both continue to mark their contexts of research and keep exploring the expansion of these contexts.

Finally, the objective of this paper was not to suggest that all mathematical learning emulates or should follow product-before-process trajectories. I have drawn on a decade of design, in which my rationale was to ground students' mathematical meaning in *perception*-based tasks. Student experiences in the activities discussed in this paper are grounded in their intuitive perceptual judgments for properties of a situation under scrutiny; the students then re-articulate these judgments via mathematical tools (Abrahamson, 2007). More recently I have been exploring designs that ground meaning in *action*-based tasks. These designs engage students initially in non-mathematical problem-solving activities that involve remote-controlling virtual objects on a computer screen. Once the students devise effective interaction strategies, the tutor layers onto the screen symbolic artifacts, which the students recognize as conferring strategic and discursive advantages. By using these artifacts, the students come to re-articulate their qualitative solutions in mathematical form (Abrahamson, Trninic, Gutiérrez, Huth & Lee, 2011). Thus, both perception-based and action-based designs enable students to bootstrap mathematical ways of seeing, thinking, and speaking.

Notes

[1] Strictly speaking, the marbles-scooping random generator is a hypergeometric *approximation* of the binomial, because the four singleton events are not independent—each scoop is a without-replacement experiment,

because as a marble settles in place, there is one less of its color in the box. However, the very small *n*-to-*N* ratio makes this point practically negligible, so I shall henceforth treat the experiment as truly binomial.

[2] See Abrahamson (2009) for transcriptions. A 3.5 minutes video can be viewed at http://tinyurl.com/DorLi-FLM.

[3] A brief video can be viewed at http://tinyurl.com/DorAliya-FLM.

[4] I have witnessed similar interaction sequences and insights in empirical data from the implementation of designs for other intensive quantities. For example, Thacker (2010) investigated a design for fostering student grounding of the mathematics of slope in perceptions of steepness. In Abrahamson and Wilensky (2007) we describe students reinventing sampling as a means of reconciling a global estimate of color density with local enumerations of color elements (see Abrahamson, 2007, for further cross-design comparison).

[5] I speak somewhat loosely here, because I wish to hone actual dilemmas confronted by educators, both designers and instructors. In fact, pedagogical traditions emanating from the Piagetian and Vygotskian schools are certainly not as disparate as they are sometimes caricatured (Cole & Wertsch, 1996).

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