CITATION: Abrahamson, D. (in press). Rethinking probability education: perceptual judgment as epistemic resource. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: presenting plural perspectives (PT: PPP).*

Rethinking Probability Education: Perceptual Judgment as Epistemic Resource

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ABSTRACT: The mathematics subject matter of probability is notoriously challenging, and in particular the content of random compound events. When students analyze experiments, they often omit to discern variations as distinct outcomes, e.g., HT and TH in the case of flipping a pair of coins, and thus infer erroneous predictions. Educators have addressed this conceptual difficulty by engaging students in actual experiments whose outcomes contradict the erroneous predictions. Yet whereas empirical activities per se are crucial for any probability design, because they introduce the pivotal contents of randomness, variance, sample size, and relations among them, empirical activities may not be the unique or best means for students to accept the logic of combinatorial analysis. Instead, learners may avail of their own pre-analytic perceptual judgments of the random generator itself so as to arrive at predictions that agree rather than conflict with mathematical analysis. I support this view first by detailing its philosophical, theoretical, and pedagogical foundations and then presenting empirical findings from a design-based research project. Twenty- eight students aged 9–11 participated in tutorial, task-based clinical interviews that utilized an innovative random generator. Their predictions were mathematically correct even though initially they did not discern variations. Students were then led to recognize the formal event space as a semiotic means of objectifying these presymbolic notions. I elaborate on the thesis via micro-ethnographic analysis of key episodes from a paradigmatic case study. Along the way, I explain the design-based research methodology, highlighting how it enables researchers to spin thwarted predictions into new theory of learning.

A few years ago, I was kicked out of a party. I had told the hostess that I study people's intuition for probability, and she wanted to hear more. So I did what I usually do in such situations: I asked her the two-kids riddle. Here's a rough transcription of our dialogue.

Dor: I have two kids. One of them is a girl. What's the sex of my other kid?

Hostess: I don't get it. It's just, like, 50-50—it's the same chance of getting a boy or a girl. What does your other kid have to do with it? They're totally independent events!

Dor: Indeed they *are* independent events, but together they make a compound event. There's double the chance that my other kid is a boy than a girl.

Hostess: Well excuse me, but that just doesn't make any sense at all. How could this possibly be true? *And* I resent your patronizing tone.

Dor: You see, two-kid families are either Girl-Girl, Girl-Boy, Boy-Girl, or Boy-Boy. That's all the possibilities. Now, we know that one of my kids is a girl, so that rules out Boy-Boy. So we're left with Girl-Girl, Girl-Boy, Boy-Girl. This means there's only one option with the other kid being a girl, but there're two options with the other kid being a boy. So it's 2:1. Boy wins.

Hostess: No you don't. That's just a mathematical model. In reality, it's still 50-50.

She became very upset and things got out of hand. She said that mathematicians should not pry their logical tentacles into that which is most sacred—the life of an unborn child. This was Berkeley, so I took it all with a grain of jalapeño. I went home and wrote a paper—by all accounts a favorable outcome for an untenured professor.

Still, probability has a bad reputation as being notoriously unintuitive—its assumptions opaque, its solution procedures arbitrary (see in Prediger, 2008; Rubel, 2009; Vos Savant, 1996). So much so, that Kahneman, Slovic, and Tversky (1982) suggest we should refrain from applying gut feelings in making decisions with respect to complex randomness situations, and Fischbein (1987) advises substituting error-prone "primary intuition" with mathematically informed "secondary intuition."

Is this to be our fate? Is randomness to be indefinitely outside the province of informal reasoning? Are stochastic phenomena beyond the reach of our limited hominid brains? Perhaps textbook chapters on probability should begin with the banner, "Abandon intuition, all ye who enter here."

Perhaps, rather, the issue is not so much about probability phenomena or content per se but with how we teach this mathematical topic. Perhaps we choose to abandon intuition, because prevalent pedagogical praxis structurates probability content as patently alien to our gut feelings. Perhaps we all have remarkable informal intuitions for probability, only that traditional curriculum does not enable us to make use of this intuition beyond the very obvious cases of a single coin, die, or spinner. Perhaps there are better ways of introducing learners to the more advanced ideas of probability. Perhaps there are alternative materials and activities that empower students to engage their informal reasoning so as to build trust with the basic notions of the formal analytic frameworks for probability. Perhaps, specifically, informal intuition could be leveraged in the more complex cases of *two* coins, dice, spinners. At the same time, perhaps we would need to design a new compound-event random generator that is mathematically analogous to a pair of coins but is better tailored to accommodate relevant perceptual capacity.

Granted, informal intuition stemming from tacit perceptual capacity is a curious epistemic resource. It may not be as accurate as stipulated by the disciplinary context, in that it yields qualitative heuristic estimates where precise quantitative indices are required. Moreover, intuition is susceptible to context, in that its successful application may require very particular forms of sensory presentation and, as such, does not transfer to semiotic systems typical of professional practice, such as alphanumerical symbols. Finally, intuitive reasoning may not lend itself to reflecting on the reasoning process and documenting it in forms that enable scrutiny of peers.

That said, intuitive reasoning can ground formal knowledge by evoking meanings, schemas, familiarity, and coherence that can be brought to bear as we approach unfamiliar mathematical analyses, tools, procedures, and displays. Furthermore, intuitive reasoning enables instructors and students to share referents—they can talk *about* properties of phenomena, such as chance events, even before these phenomena have been formalized. As such, informal reasoning can be a useful epistemic resource for learning formal models in discursive contexts. Yet for this to work, informal inference needs to align with mathematical analysis. Enter designers.

In this chapter I propose a rethinking of how students should be introduced to the fundamental principle of classicist probability theory, the rule of ratio, as it applies beyond the simple cases of single events. I propose that students' perceptual judgment of the stochastic propensities inherent to random generators should constitute an epistemic resource for making sense of the classicist approach to probability, particularly in the case of compound events. I argue that *under auspicious conditions, students' perceptual judgment of the stochastic propensities of a random generator can play a similar epistemic role as do actual experiments with the device in terms of evoking sensations of relative likelihood that, in turn, can be linked to the distribution of possible outcomes in the event space*. In order to create these auspicious pedagogical conditions, I submit, educational designers should create materials and activities geared to accommodate humans' evolved perceptual inclinations, such as sensitivity to proportional relations in the visual field. The chapter attempts to promote this rethinking by furnishing intuitive, empirical, and theoretical evidence as support for its validity.

The conjecture that probability theory can be grounded in perceptual judgment drove a multi-year design-based research project. I will describe how the design was rationalized, engineered, and ultimately implemented over a set of iterated studies. In particular, the conjecture became refined through this designer's reflection on his own intuitive rationales for the particular materials, activities, and facilitation developed over the project (Schön, 1983; Vagle, 2010). These reflections emanated from analyses of empirical data from implementing the design, specifically video footage from clinical–tutorial interactions, in which twenty-eight 9-to-11 years old children and twenty-four undergraduate/graduate statistics students engaged the activities individually.

The more general form of my conjecture, namely that conceptual understanding should be grounded in perceptual reasoning, is hardly new to the learning sciences literature. I will draw on seminal theoretical frameworks to suggest that *unmediated, pre-analytic perceptual sensation from a phenomenon under scrutiny may be an equally or even more powerful epistemic grounding for a mathematical model of the phenomenon than mediated, encoded information about the phenomenon's behaviors*. For example, visually analyzing a random compound-event generator may better ground its event space than visually analyzing its actual experimental outcome distribution. This suggestion invites a potentially interesting dialogue with those who claim that students' inferences should be encoded directly in formal symbolic systems (Cheng, 2011). The suggestion also implies a central role for designers to customize phenomena so as to render them cognitively congenial (Kirsh, 1996) or cognitively ergonomic (Artigue, 2002), and specifically more conducive to the application of tacit perceptual capacity relevant to the targeted mathematical concepts (Abrahamson, 2006a; Abrahamson & Wilensky, 2007).

I begin, below, by focusing on students' difficulty with the logic of permutations, for example listing only [Heads–Tails] as the mixed result of a two-coin experiment rather than [Heads–Tails *and* Tails–Heads]. The conjecture evaluated in this paper was originally motivated as a response to students' difficulty with this particular situation.

1. Compound Event Spaces: Rethinking an Enduring Pedagogical Challenge

> Adaptors [of mathematics to the school level] do not trust their eyes, they cannot believe it is so simple, or if they can, they do not trust other people to be able to believe it. (Freudenthal, 1974, p. 277)

In 1814, Pierre-Simon Laplace put forth a theorem, by which the likelihood of an event occurring randomly is the ratio of the total number of outcomes *favorable* to the occurrence of that event and the total number of *possible* outcomes, where both uniqueness and equiprobability of outcomes are assumed. Implicit to a successful application of this algorithm is the construction of an event space that includes all possible, unique outcomes. Here lies the educational rub.

1.1 The Problem of Differentiating Outcomes in a Compound Event Space

Consider a fair coin tossed onto a desk. Students learning probability by-and-large readily accept that the chance of receiving a Heads is one-in-two. Formally, “one” is the total number of favorable outcomes (we discern only one way of getting Heads), and “two” is the total number of all possible outcomes (we discern exactly two possible outcomes: Heads, Tails). By and large, students familiar with the notion of proportions rarely encounter substantial difficulty in constructing a simple event space and applying the Laplace principle.

Now consider a pair of coins. What is the chance of receiving Heads-and-Tails? The event “Heads-and-Tails” is compound, meaning that its occurrence depends on the intersecting occurrences of (two) simple events, here a Heads *and* a Tails. A pair of coins has a total of four possible outcomes. This number can be calculated by multiplying the total number of possible outcomes in each coin, so 2 x 2 = 4. These four possible outcomes are [HH, HT, TH, TT], and two of them, HT and TH, are favorable to the occurrence of the compound event in question Heads-and-Tails, whose chance is thus 2/4, that is, 1/2.

An enduring educational challenge lies in having students list a complete event space for compound-event experiments built of *identical* random generators, such as two pennies. Students often do not distinguish between variations, that is, possible outcomes that differ only in the order of constituent singleton events, such as HT and TH. Of course, students can *see* that “HT” and “TH”—the actual Roman characters written on paper—are different symbol strings, and they can learn to *enact* the combinatorial analysis algorithm by which such variations are generated. The issue is not visual or procedural per se but logical, conceptual. That is, in the context of analyzing a random generator, students typically claim that variations are redundant because the variations do not indicate different worldly eventualities; they believe that only one of these variations should be listed in the event space. For example, students build for the two-coin experiment an event space of only three possible outcomes—[HH, HT, TT]—and consequently determine via the rule of ratio a 1/3 chance of receiving “Heads-and-Tails” (Abrahamson & Wilensky, 2005b). This form of reasoning, which does not agree with probability theory or empirical results, is considered an impediment to the teaching and learning of probability. The ubiquity and perseverance of this reasoning throughout the school years and into college makes it a major pedagogical challenge (Batanero, Navarro-Pelayo, & Godino, 1997; Fischbein & Schnarch, 1997).

Educators have sought to help students make sense of the Laplace principle as it applies in the case of compound events. In particular, designers and teachers have attempted to support students in understanding *that* and, ideally, *why* listing all variations on an event, such as both HT and TH, is critical to building the event space of a compound-event random generator. However, success has been mixed (Jones, Langrall, & Mooney, 2007; Shaughnessy, 1977).

A prevalent design rationale for introducing the analysis of compound events is to have students witness its predictive utility. That is, it is hoped that students accept the importance of including permutations in the event space of a compound-event random generator via engaging in activities wherein event spaces that include variations turn out better to predict its actual experimental outcome distributions than do event spaces that do not include variations. Below, we first elaborate on this design rationale and then critique it and offer an alternative design rationale and solution.

1.2 Empirical Experimentation as Epistemic Grounding for Classicist Analysis

Many education researchers believe that students best learn classicist theory when combinatorial analysis activities are combined with empirical activities (Steinbring, 1991). In practice, students operate a random generator and reflect on the results vis-à-vis their expectations (Amit & Jan, 2007). Some researchers use computer-based media to build microworlds wherein learners can author, run, analyze, and modify stochastic experiments. These chance simulators typically include animations of actual or hypothetical random generators as well as a variety of standard forms, such as graphs, tables, and monitors, that display cumulative results in real time (e.g., Abrahamson, 2006b; Iversen & Nilsson, 2007; Konold, 2004; Pratt, 2000; Stohl & Tarr, 2002; Wilensky, 1995).[1]

The underlying assumption of this empiricist approach is that learners are fundamentally rational beings. Learner are expected to accept experimental results as a valid epistemic resource bearing on the problem; accordingly, in the face of results that patently contradict their expectations, they will acknowledge the inadequacy of their reasoning and seek an alternative form of reasoning that better accounts for these results; in particular, they will recognize that compound event spaces *with* variations are better fit to the world and thus should be preferred over spaces without permutations (cf. Karmiloff-Smith, 1988; Koschmann, Kuuti, & Hickman, 1998; von Glasersfeld, 1987). For example, students whose erroneous analysis led them to expect that only a 1/3 of the two-coins experimental outcomes will be of the type Heads-and-Tails may take pause when simulations of this experiment repeatedly converge toward 1/2 (Wilensky, 1995).

[1] Some of this design work bears the concomitant far-reaching conjecture that cyber media can and should *transform* mathematics—not only serve mathematics as we know it—by offering alternative cognitive access to phenomena in question (Wilensky & Papert, 2010). For example, as human practice becomes increasingly "wired," brute-force frequentist simulation could render classicist analysis a redundant anachronism shelved away with other precyber artifacts and procedures, such as logarithmic tables, slide rules, and long division.

In sum, students are expected to endorse experimental outcomes as veridical and accordingly seek to adjust their reasoning that has proven erroneous.

Being proven wrong is a formative learning experience, because it stimulates reflection, which may result in conceptual change. As such, experimental empiricism appears to provide an epistemic resource that promotes student reasoning in ways that agree with the overall pedagogical objective.

Yet being proven wrong might bear also implicit effects that transcend the content itself and result in pedagogically undesirable habits of mind (see Bateson, 1972, on deutero learning). That is, being proven wrong may cause students to distrust their intuitive judgment as an epistemic resource and develop epistemological anxiety with respect to the formal procedure (Wilensky, 1997); students may learn to rely solely on empiricism, which lends a sense of certainty but not a deeper sense of causality (Harel, 2013); students will consequently forget a procedure they studied that is "not tied to lived reality with strong bonds" (Freudenthal, 1971, p. 420). Given these potentially deleterious effects of being proven wrong, are alternative pedagogical approaches feasible?

Perhaps not. Some researchers believe that humans' primary intuitions for compound-event probabilistic situations are inherently fallible (Kahneman et al., 1982). It follows that humans must perforce resign themselves to developing secondary intuitions better fitted to rational models of stochastics (Fischbein, 1987).

However, it could be that humans' primary probabilistic intuition is in fact well aligned with mathematical theory; studies that concluded otherwise may have been inauspicious to primary intuition by using materials and tasks that did not accommodate perceptual reasoning (Gigerenzer, 1998; Xu & Garcia, 2008; Zhu & Gigerenzer, 2006). More emphatically, I maintain that naïve reasoning, no matter how incompatible it might seem with respect to formal knowledge and practice, should be conceptualized as a resource—not an impediment—to designing, teaching, and learning (Borovcnik & Bentz, 1991; Bruner, 1960; Gigerenzer & Brighton, 2009; Smith, diSessa, & Roschelle, 1993; Wilensky, 1997). More broadly, if somewhat lyrically, mathematics is a human invention—it was created in the image of (wo)man—and so other humans are biologically and cognitively equipped to share these images (Núñez, Edwards, & Matos, 1999).

I have thus furnished theoretical, pedagogical, and philosophical critiques of the prevalent approach to introducing the mathematical content of compound event spaces. These critiques suggest that empirical experimental evidence might not be the *sine qua non* epistemic resource for students to ground the logic of compound event spaces. I wish to propose perceptual judgment as an epistemic resource alternative, or at least complementary, to experimental empiricism. Given appropriate design, I maintain, perceptual judgment may enable learners to ground compound event spaces in correct rather than incorrect naïve judgment and thus relate to this content, beginning with its introduction and even on beyond.

1.3 Perceptual Judgment of Likelihood as Epistemic Grounding for Classicist Analysis

Even before one analyzes a random generator so as to build its event space, one is sometimes able to offer estimates for its stochastic propensities. In order for these naïve judgments to ground the event space, though, they must align with probability theory. Lo, have we not been arguing that students' naïve probabilistic reasoning about compound-event random generators is the very source of their difficulty to accept the combinatorial analysis procedure?

Let us interrogate the context inherent to that argument. In particular, I will attend to the particular types of *materials* and *tasks* related to that argument, in an attempt to understand how these dimensions affect the compatibility of informal and formal probabilistic reasoning. I begin with the task.

When earlier we discussed an apparent incompatibility between students' naive reasoning and formal probability theory, the task that students were erring on was combinatorial *analysis* of a compound-event experiment. Still, students' *pre-analytic* reasoning, and in particular their visual judgments about the random generator's stochastic propensities, may be compatible with formal probability theory. If so, then perceptual judgment might constitute an epistemic resource for grounding formal treatment of these experiments. The formal event space thus would not conflict with perceptual judgment. Rather, the event space would triangulate, explain, or elaborate perceptual judgment.

The materials of the learning task—that is, the particular random generator—are also of relevance. In our earlier expository discussion, the compound-event random generator was a pair of coins. Perhaps students' erroneous analyses of this particular device are due to it being non-conducive to pre-analytic perceptual reasoning aligned with mathematical theory. Perhaps there are alternative devices that are conducive to humans' primitive statistical inclinations. If so, perceptual judgment of these devices' propensities might complement or even improve on frequentist experimentation as epistemic resources for grounding compound event spaces.

The next section of this chapter describes a design-based research project that explored perceptual judgment as an epistemic resource for grounding combinatorial analysis of compound event spaces. Coming into the study cycles, I asked: *What structural forms might a random generator take that would elicit from students perceptual judgments in agreement with mathematical theory of probability?* Later in the cycles, when I had a working answer for this design question, a new question emerged: *Once we elicit from students perceptual judgments that agree with mathematical theory, how might the students come to express these judgments in mathematical form?*

These questions are important to mathematics education theory and practice. Findings from a research program pursuing these questions may imply that dice, coins, spinners, and slot machines—ecologically authentic random generators that played formative historical roles in the development of probability theory—do not make for optimal *introductory* materials for this mathematical subject matter, because they do not afford pre-analytic epistemic resources aligned with classicist theory as it obtains in the case of compound events.

Up to now, I have discussed an *intuitive* rationale for the proposed "perceptual approach" to grounding probability. Subsection 1.4, below, bridges toward the *empirical* part of the chapter by explaining how theory evolves in design-based research projects. Section 2 then describes a design-based research project that investigated probabilistic reasoning. Interleaved into the design narrative is a Piagetian–Vygotskian framework that evolved through the course of analyzing empirical data gathered over that project. The framework underscores the role of pre-analytic perceptual judgment in learning mathematical content. Section 3 draws on *theoretical* resources from the learning sciences to further support the framework. Section 4 integrates these intuitive, empirical, and theoretical supports for the chapter's central conjecture and offers conclusions.

1.4 The Evolution of Theory Through Cycles of Design-Based Research Studies

The conjecture pursued in this chapter is that students may bear pre-analytic cognitive resources that are suitable for grounding fundamental notions of probability theory, only that curricular materials and tasks commonly employed in introductory probability units do not enable students to draw on these resources; by understanding the nature and function of these subjective resources, we may be able to engineer tasks and materials that better accommodate and leverage these resources. This chapter describes a research project that put the conjecture to the empirical test (see e.g., Confrey, 2005, on the design-based research approach).

The rethinking of probability pedagogy proposed in this chapter evolved through design-based research in the domain of probability, and the assertions offered here were originally couched in terms of particular materials and activities developed in that project. More specifically, the rethinking is the researcher's articulated reconceptualization of his own design in light of multi-year analyses of empirical data gathered in its iterated implementations.

I will support the rethinking via tracing its evolution along project milestones (Collins, 1992). Each milestone is an instance where, having wrestled with an enduring challenge of analyzing empirical data gathered in our studies, I realized the purchase of a hitherto unconsidered theoretical model and methodology, typically from the learning sciences literature. Yet even as the research team embraced these theoretical models as tools applied to the empirical data, the models' evident analytical utility impelled us to reconsider our theoretical assumptions. Namely, when multiple models are all taken to bear on the same data, one can promote theory via attempting to resolve tension among the models' epistemological underpinnings (Artigue, Cerulli, Haspekian, & Maracci, 2009; Cobb & Bauersfeld, 1995; Sfard & McClain, 2002).

Indeed, I will offer a view on probability education that integrates cognitivist, sociocultural, and semiotics frameworks. I will propose a theorization of mathematical learning as guided, heuristic–semiotic coordination of sensations from two different types of epistemic resources: (a) tacit gestalts rooted in perceptual primitives; and (b) formal analysis mediated by artifacts.

From a constructivist perspective, this view of mathematics learning implies a pedagogical dilemma that revisits Plato's *Meno*: (1) on the one hand, students require gestalt perceptions of the phenomena they are studying—they need to know what they are learning *about*—even before they engage in analyzing it (Bereiter, 1985; Pascual-Leone, 1996); on the other hand, (2) these gestalt meanings ironically cause students to resist formal analysis of these phenomena, because the analysis carves the phenomena along dimensions that appear to the child irrelevant to the intuitive gestalt perception (Bamberger & diSessa, 2003). In particular, how might students ground "order-based" compound-event spaces (e.g., [HH HT TH TT]) in "order-less" gestalt sensation of event representativeness or anticipated plurality?

From sociocultural perspectives, this incompatibility does not present a dilemma, because learning in the disciplines necessarily involves conceptual reorganization

(Newman, Griffin, & Cole, 1989). Notwithstanding, I maintain, mathematical models of situated phenomena require pre-analytic gestalts as their *aboutness*, that is, as their core epistemic resource.

The following section elaborates on this conceptualization of learning in the context of the design-based research project wherein the conceptualization evolved.

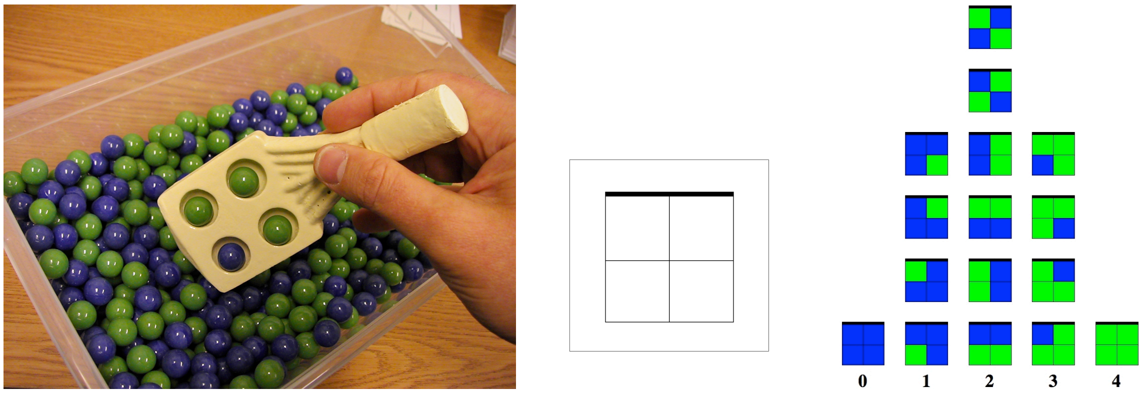
2. Pursuing a Design-Based Research Conjecture to Build Learning Science Theory

This chapter argues for the epistemic role of perceptual judgment in grounding probability notions. The thesis evolved through a set of empirical studies carried out over a multi-year educational research project that investigated issues of teaching, learning, and design pertaining to probabilistic reasoning. The project, initially named *ProbLab*, began during my Postdoctoral Fellowship at the Center for Connected Learning and Computer-Based Modeling at Northwestern University (Wilensky, *Director*) and continued as *Seeing Chance*, a NAE/Spencer Postdoctoral Fellowship at the Embodied Design Research Laboratory at the University of California, Berkeley, which I direct.

The design-based research project drew inspiration from the *connected probability* work (Wilensky, 1993, 1995, 1997). The project grew into a three-pronged effort to develop products, theory, and frameworks (see Abrahamson & Wilensky, 2007, p. 25). Embarking from prior findings (Abrahamson & Wilensky, 2002, 2004, 2005a, 2005b), I: (a) built mixed-media materials and activities for learning fundamental probability notions (Abrahamson, 2006b) and evaluated their pedagogical affordances (Abrahamson, 2007; Abrahamson & Cendak, 2006); (b) developed explanatory models for the roles of perceptual reasoning in conceptual learning (Abrahamson, 2009b, 2010, 2011, 2012b; Abrahamson, Gutiérrez, & Baddorf, 2012); and (c) created a grounded mathematics design framework (Abrahamson, 2009a) and contributed to reflective discourse on design-based research practice (Abrahamson, 2009c, 2012a).

The project was formative in shaping a theoretically balanced perspective on mathematics learning. The perspective emerged gradually through cycles of analyzing video footage gathered during the *Seeing Chance* tutorial interactions (n = 28, Grades 4 – 6; n = 24, college seniors and graduate students). The collaborative analyses focused on episodes within these data that culminated in students expressing meaningful links among resources in the learning environment in ways that we evaluated as pedagogically desirable, in that they promoted the didactical objective of the interaction. Our theory development was motivated by a sense that the analytic tools we were using were not affording productive interpretations of these data episodes. As such, the theory development process was contingent on the researchers acknowledging the limitations of their analytic tools and, more deeply, problematizing the theoretical assumptions tacitly informing the selection of those tools.

At it broadest, I wanted to understand how people articulate tacit judgment in mathematical form. I therefore showed students perceptual displays bearing quantitative relations, asked them a framing question related to those properties, and then introduced semiotic resources for them to express their inferences. In creating the perceptual displays, I sought to enable students to infer qualitative judgments that agree with mathematical theory.



a. *b.* *c.*

Figure 1. Materials used in a design-based research project investigating relations between informal intuitions for likelihood and formal principles of an event space: (a) a "marbles scooper," a utensil for drawing out ordered samples from a box full of marbles of two colors; (b) a card for constructing the sample space of the marbles-scooping experiment (a stack of such cards is provided, as well as a green crayon and a blue crayon, and students color in all possible outcomes); and (c) a "combinations tower," a distributed event space of the marbles-scooping experiment, structured so as to anticipate the conventional histogram representation of actual outcome distributions.

Specifically for the subject of probability, I created a set of recourses (see Figure 1): (a) a concrete random generator; (b) media for building an event space via combinatorial analysis of the random generator; and (c) an innovative structural form for organizing the event space so as to make it more conducive to heuristic perceptual inference. In addition, I designed and built a suite of computer-based simulations of the experiment featuring schematic models of the random generator. In this chapter I do not feature these computer-based simulations, because my thesis here pertains primarily to students' perceptual judgments of the random generator itself and, in particular, how students coordinated these judgments with their guided perceptions of the event space. But I will briefly mention the simulations in the general discussion so as to compare and contrast perceptual judgments and empirical experimentation with respect to their epistemic contributions to content learning.[2]

I first had students briefly examine the random generator's experimental mechanism (see Figure 1a). Immediately after, I asked them to offer their estimations for its expected outcome distribution. Importantly, *the students did not conduct any actual experiment at all*, so that their responses were based only on sensory perception and reasoning.

The particular random generator designed for this project was such that students tended to offer likelihood judgments that agreed with mathematical theory. In particular, they predicted a plurality of two-green-and-two-blue samples (hence, "2g2b"), a rarity of both 4b and 4g, and, in between, the events 1g3b and 3g1b. When asked to support their responses, students referred to the equal number of green and blue marbles in the bin.

Next I guided students to build the experiment's event space. Instead of having them represent the possible outcomes as a list or tree of symbols on a single sheet of

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[2] Strictly speaking, the marbles-scooping experiment is hypergeometric, not binomial, because as each marble is captured by a concavity in the scooper, there is one less of that color in the bin. However, the fairly minute ratio of the sample (4) to the total number of marbles in the bin (hundreds) enables us to think of this experiment as quasi-binomial and, for all practical effects, as actually binomial.

paper, I provided a stack of cards (see Figure 1b) as well as two crayons, and students used these media to create iconic representations of the possible outcomes. Typically, students organized the construction space on the desk by clustering the cards into five emergent groups, with 1, 4, 6, 4, and 1 items, respectively (Abrahamson, 2008). I then guided the students to assemble the sixteen cards according to these five event classes in a spatial configuration that highlighted the different number of outcomes per each event (see Figure 1c).

In line with my theoretical stance coming into the study, I had conceptualized the event space as a formal concretization of intuitive reasoning. I therefore expected that the event space would stimulate students to abstract and encapsulate the schemas they had tacitly employed in judging the likelihoods of the device's possible outcomes (Piaget, 1968). Specifically, I expected the students to recognize the variations—such as [gggb, ggbg, gbgg, bggg]—as articulating their pre-analytic judgments. In so doing, I envisioned and hoped for a smooth continuity from naïve to informed views of the experiment; I did not envision tension between these views.

My expectation was largely informed by the great care I had taken in the design to build a sampling device (the scooper) that configured each sample in a particular order. That is, I had designed "order" as an inherent structural property of the sampling device, because I had thought that this feature would impress upon students the uniqueness of the variations. I therefore implicitly took it for granted that the students were attending to the dimension of order. It never occurred to me that they might see the scoops different from how I saw the scoops; that they could construe gggb and ggbg as "the same thing."

However, throughout the construction of the event space, students tended to resist the variations as redundant objects. They thought that the event space should consist of five cards only, with one card per each of the five events 4b, 1g3b, 2g2b, 3g1b, 4g. They accepted the complete event space only once the combinations tower had been assembled. Specifically, *students endorsed the variations only once they were able to perceive the event space's five vertical projections as respectively signifying their sensation of the five events' differential likelihoods per their earlier perceptual judgments of the marbles box*. Only after having thus made sense of the event space in its totality as mapping onto their informal inference did the students retroactively accept the analytic procedure by which the event space had been constructed.

As such, my implicit assumption that tacit judgment can be directly articulated in disciplinary form quickly became problematized. In particular, my research team came to acknowledge our implicit assumption and recognize that it apparently built on a lopsided constructivist conceptualization of learning. We thus gradually began to realize the overwhelming constitutive role of artifacts in mediating cultural forms of reasoning (Wertsch, 1985) as well as instructors' formative role in framing and guiding this mediation process (Newman et al., 1989).

Prior to this sociocultural calibration of our design rationale, we had focused exclusively on the student as the locus of education. We had regarded the phenomenon of learning essentially as the child's solipsistic developmental process, which the researcher merely catalyzed for the purposes of the study. As such, we had construed the experimenter's actions as necessary methodological means of eliciting from the study participant a targeted response. Yet the experimenter's procedural action, we came to realize, was in fact simulating a culturally authentic interaction between a tutor and student. In particular, we now surmised, *learners do not articulate tacit judgment directly in mathematical form; rather, they objectify these presymbolic notions in semiotic means made available to them in the learning environment* (Radford, 2003). Moreover, student expression cannot always be direct articulation of the tacit judgment, because the available cultural media often require a parsing of the source phenomenon in ways that are very different from naïve perception (Bamberger & diSessa, 2003; Bamberger & Schön, 1983).

Still, if our study participants were struggling to objectify presymbolic notions in cultural forms, what was the source of these presymbolic notions? How did the students perform perceptual judgments in the first place, during the initial phase of the experiment, when they essentially gazed at the experimental utensils? Research on infants' "statistical" reasoning, in empirical settings strikingly analogous to ours (Xu & Garcia, 2008), suggests that our study participants drew on early, unmediated perceptual capacity to judge the relative chances of random events. It could be that the order of singleton outcomes is ignored or downplayed in these judgments and only the color ratios are perceived.

Students thus appear to be working with two resources, tacit reasoning and cultural artifacts. *When tacit inference is aligned with mathematical theory, instructors can guide students to appropriate the cultural resource as a means of supporting and empowering their tacit inference*. This insight into design bears concomitant insight into theory of learning, as follows.

The sociocultural perspective, which underscores the mediating role of cultural artifacts in conceptual development, appears to require a constructivist complement so as together to afford a comprehensive explanation of what students are doing when they link tacit and cultural resources. We were thus heartened when diSessa (2008) called explicitly for "dialectical" theoretical work that seeks to combine and possibly synergize cognitivist and sociocultural models of learning. The call resonated with other urges to view the work of Piaget and Vygotsky as compatible (Cole & Wertsch, 1996; Fuson, 2009; Stetsenko, 2002).

Once we had aligned our theorizing with the sociocultural perspective, we could infuse into the swell of our data analysis seminal neo-Vygotskian contributions to the modeling of mathematics teaching and learning (Bartolini Bussi & Mariotti, 2008; Saxe, 2004; Sfard, 2002, 2007). Consequently, *we came to view mathematics learning as the elicitation and acculturation of subjective meanings via guided, goal-oriented, and artifact-mediated activity.* As a result of this dialectical theorizing, we could offer a sociocultural interpretation of abductive inferential reasoning (Abrahamson, 2012b) as well as a reconceiving of discovery-based learning (Abrahamson, 2012a).

This section answered our chapter's research questions regarding the prospects, materials, tasks, and process of grounding a compound event space in perceptual judgments of random generators. Having outlined the evolution of our perspective on mathematics learning, what remains is to reflect on the epistemic nature of perceptual judgment and then compare perceptual judgment to experimental empiricism as complementary epistemic resources for grounding compound event spaces.

3. Discussion: Learning Sciences Views on Perceptual Judgment as Epistemic Resource

What is the role of perceptual judgment in mathematical learning? Is it desirable for students to reason about properties of concrete objects, given that mathematical texts are symbolic? What are the epistemological and cognitive qualifications of a proposal that perceptual judgment of random generators can serve as an epistemic resource for understanding compound event spaces?

Drawing on a broad reading of the learning sciences literature, I will now build the argument that perceptual judgments, and more generally multimodal dynamical images, are essential for conceptual development—they are what mathematical texts are *about*.

The human species developed via natural selection the capacity to quantify aspects of phenomena relevant to their survival (Gigerenzer, 1998). Some of these enabling constraints on perception (Gelman, 1998) pertain to quantities whose mathematical modeling features in school curriculum, such as the intensive quantities of slope, velocity, chance, and aspect ratio (Suzuki & Cavanagh, 1998; Xu & Garcia, 2008). Do we, therefore, like the slave in *Meno,* know the concepts before studying them?

Not quite. Perceptual capacity cannot be directly translated into mathematical knowledge. First, perceptual judgments are holistic and pre-articulated, such as when we perceive the gradient of a sloped line, whereas mathematical modeling is analytic and symbolic, such as when we measure and calculate "rise over run." Second, perceptual judgments are tacit—the neural mechanisms of these cerebral faculties are cognitively impenetrable (Pylyshyn, 1973), and so we are conscious not of our perceptual process itself but only of our operatory reactions and contextual inferences that result from these tacit processes. Referring to the relation between object constancy and proportional reasoning, Piaget and Inhelder (1969) wrote, "However elementary they may be in the child, these concepts cannot be elaborated without a logico–mathematical structuration that....goes beyond perception" (p. 49).

From a neo-Vygotskian perspective, this logico–mathematical structuration of perceptual judgment is achieved via social mediation. In particular, individuals appropriate cultural forms as means of realizing personal goals for solving collective problems (Saxe, 2004). In interactive contexts, such as tutorial or classroom activities, these personal goals may be discursive. In particular, the semiotic–cultural perspective (Radford, 2003) highlights the role of students' presymbolic notions in educational interaction: students develop new mathematical signs by objectifying presymbolic notions using available semiotic means.

Educators can thus play vital roles in students' conceptual development by strategically placing pedagogically desirable cultural forms in the learning environment and steering students to re-articulate their naïve views by these particular semiotic means (Abrahamson, 2009a; Mariotti, 2009; Sfard, 2002, 2007). Students may appropriate cultural forms also as means of accomplishing enactive goals, and not just discursive goals, and in so doing they may bootstrap new operatory schemas by reconfiguring their naïve strategies (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011).

I thus discern across a range of constructivist and sociocultural theorists a loose consensus, by which meaningful learning occurs when individuals instrumentalize cultural forms to accomplish tasks (Vérillon & Rabardel, 1995). When the tasks are embedded in pedagogical activities and the forms are mathematical symbolic artifacts, students become acculturated into mathematical discourse and praxis.

Implicit to this dialectical process by which personal sense meets cultural form, is the question of the cognitive form of personal sense, prior to its acculturation. Many scholars believe that personal sense is embodied as multimodal images. For example, scholars in mathematics (Davis, 1993), semiotics (Hoffmann, 2003; Peirce, 1867), developmental psychology (Arnheim, 1969), creativity (Getzels & Csikszentmihalyi, 1976; Hadamard, 1945), and philosophy (Barwise & Etchemendy, 1991) believe that perceptual reasoning, such as visual or imagistic pattern recognition, is the *sine qua non* of conceptual development. Moreover, according to theories of cognition (Barsalou, 1999; Glenberg & Robertson, 1999) and cognitive linguistics (Goldin, 1987; Johnson, 1999; Lakoff & Johnson, 1980; Lakoff & Núñez, 2000), all reasoning is perforce imagistic by virtue of the fundamental cerebral architecture and mechanisms of reasoning and perception. It is therefore hardly provocative to explore mathematical pedagogy that fosters perceptual grounding for symbolic text (Kamii & DeClark, 1985).

4. Conclusion: Perceptual Judgment Grounds Classicist Analysis

Perceptual judgment of random generators enables students to meaningfully ground the products of formal classicist analysis procedures. As such, perceptual reasoning constitutes a viable pedagogical entry into fundamental probability content, and particularly into compound event spaces.

I began this chapter by illustrating the need for effective probability curriculum. I then underscored the importance of designing materials and tasks appropriate to leveraging the epistemic resource of perceptual reasoning. Next, I demonstrated the plausibility of my thesis via describing milestones in a decade-long empirical design-based investigation of probability learning. Finally I supported the conjecture with seminal theory from the learning sciences literature.

The conjecture that this chapter has sought to promote should not by any means discourage educators from employing frequentist approaches in the instruction of probability. *A fortiori*, the empirical data discussed in this chapter represented only two of my three design phases, where the third phase consisted of running computer-based simulations of the stochastic experiment. In those activities, I guided students to draw on their intuitive sensations both from the static random generator and its event space so as to make sense of actual, “imperfect” outcome distributions that resulted from the experimental runs (Abrahamson, 2007, 2010). Linking intuitive, analytic, and empirical probabilistic activities appears to support a coherent and connected perspective on probability (Wilensky, 1993).

Perceptual judgment of random generators and empirical experimentation with random generators play different *curricular* roles in terms of the conceptual content they explore. For example, perceptual examination of a random generator is a condition for its combinatorial analysis, whereas conducting experiments with the random generator creates opportunities to encounter randomness and sample size as they relate to variance. However, perceptual judgment and empirical experimentation play similar *epistemic* roles in understanding event spaces (see Figure 2): both activities evoke sensations that resonate with the distribution of possible outcomes across events; both activities may result in implicating the event space as explaining the random generator’s propensities that we sense or witness; in both cases, adopting the event space is mediated by tacit or direct sensation of relative magnitudes.

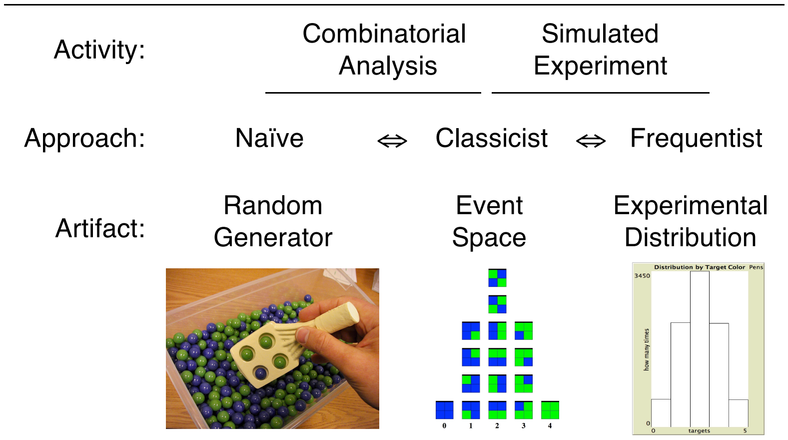

Figure 2. Probability design trialogue.[3] Conceptually critical coordination via two activities across three artifacts in a design for the binomial. Activities bridge complementary conceptualizations of the stochastic phenomenon. Double arrows indicate that learners need to interpret a new artifact they encounter as signifying meanings they had established for a previous artifact. The original device suggests its own stochastic propensity, the event space models the propensity, and the experimental distribution exemplifies the propensity. Both naïve and frequentist conceptualizations ground the classicist artifact. Accepting the event space retroactively grounds the combinatorial analysis procedure by which the space was built.

Notwithstanding, naïve perceptual judgment of random generators is different from experimental activity, in that perceptual judgment directly evokes presymbolic notions of the property in question, whereas experimental outcome distributions indirectly evoke these notions. This difference between immediate and mediated notions may confer upon perceptual judgment a unique advantage over experiments in relation to selecting introductory grounding activities for probability designs. Elsewhere, we have demonstrated the extensibility of these introductory activities toward incorporating symbolic displays as well as cases of heteroprobable outcomes (Abrahamson, 2009a).

Students possess natural capacity to perform powerful perceptual reasoning pertaining to the study of probability. Designers, teachers, and researchers may greatly avail themselves by leveraging this power so as to support the learning and continued investigation of this chronically challenging subject matter.

Author's Note

[3] The term *trialogue* is borrowed from Wilensky (1996) who, shifting attention to artifacts rather than activities, writes, "By engaging in computational modeling—this trialogue between the symbolism, the program output and the real world—and, then, reflecting on the feedback obtained, learners can make meaningful connections" (p. 128). Wilensky's assertion was expressed in the context of constructionist activities, wherein learners themselves create the computer-based models, and so the "symbolism" element of the trialogue refers to alphanumerical expressions in the modeling language (the "code," such as "forward 10"). Nevertheless, Wilensky's notion of a trialogue among inquiry artifacts obtains in the case of ready-made models, too, such as in the case of *Seeing Chance* design discussed in this chapter. Therein the "symbolism" element refers to inscriptions generated via analyzing the real world, such as diagrams, icons, and concrete displays (i.e. the "combinations tower" event space).

I wish to dedicate this chapter to my dear friend Ólafur Elíasson, whose noble recording of the Adagio from J. S. Bach's piano concerto in F minor inspired me as I wrote. Thanks to Maria Droujkova for excellent formative comments on an earlier draft. I really do have two kids: a girl and a boy (in that order).

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