Toward a Taxonomy of Design Genres: Fostering Mathematical Insight via Perception-Based and Action-Based Experiences

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ABSTRACT

In a retrospective analysis of my own pedagogical design projects over the past twenty years, I articulate and compare what I discern therein as two distinct activity genres for grounding mathematical concepts. One genre, "perception-based design," builds on learners' early mental capacity to draw logical inferences from perceptual judgment of intensive quantities in source phenomena, such as displays of color densities. The other genre, "action-based design," builds on learners' perceptuomotor capacity to develop new kinesthetic routines for strategic embodied interaction, such as moving the hands at different speeds to keep a screen green. Both capacities are effective evolutionary means of engaging the world, and both bear pedagogical potential as epistemic resources by which to build meaning for mathematical models of, and solution processes for situated problems. Empirical studies that investigated designs built in these genres suggest a two-step activity format by which instructors can guide learners to reinvent conceptual cores. In a primary problem, learners apply or develop non-symbolic perceptuomotor schemas to engage the task effectively. In a secondary problem, learners devise means of appropriating newly interpolated mathematical forms as enactive, semiotic, or epistemic means of enhancing, explaining, and evaluating their primary response. Whereas my analysis distills activities into two separate genres for rhetorical clarity, ultimately embodied interaction may interleave and synthesize the genres' elements.

Categories and Subject Descriptors

H.5.2. [**Information Interfaces**]: User Interfaces—input devices and strategies; user-centered design.

General Terms

Design, Human Factors, Theory.

Keywords

Design-based research, design framework, educational technology, embodied cognition, mathematics education.

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1. OBJECTIVE: SYSTEMATIZING PEDAGOGICAL DESIGN

This essay shares the results of one design-based researcher reflecting on his cumulative practice. My hope is to develop useful constructs and perhaps some humble theory that may promote productive dialogue with fellow scholars interested in deepening our collective understanding of educational design—its art, craft, and theory [34, 39, 87].

Design-based researchers, members of a community at the intersection of learning theory and practice, generally find it useful to articulate, disseminate, and debate among themselves philosophical, theoretical, and practical aspects of their métier [17, 42, 52, 81]. One particular aspect of this dialogue that tends to draw the attention of industry, and not only academe, is the building and refinement of empirically evaluated heuristic design frameworks for creating effective learning materials [14, 36, 50, 54, 73, 82]. Specifically, the following essay is on principled frameworks for designing mathematics learning activities geared to foster student re-invention of conceptual cores that the designer identifies for the targeted content domain. Essentially theoretical, this retrospective essay will draw on a body of empirical work to support and exemplify two proposed design frameworks as well as demonstrate their commonalities and hone their distinctions.

The motivation for sharing the current reflection is that I have noticed structural consistency as well as variation across a set of intuitively conceptualized pedagogical designs I have been investigating over the past two decades. In hindsight, I am striving to make sense of this similarity and contradistinction vis-à-vis educational-research literature. In particular, I am spurred by a tension between, on the one hand, what appears to be quite cohesive an approach to mathematics pedagogy and, on the other hand, apparently different ways of implementing this approach. I am thus looking to develop a useful taxonomy of what I propose to call *design genres*, such as perception-based design and action-based design. This taxonomy, which would avail of critique, elaboration, and expansion, is couched in learning-sciences nomenclature in an attempt to build a coherent account of relations between mathematics-education theory and practice in a way that may inform the work of other researchers. As such, though this budding taxonomy cannot be exhaustive, it may indicate routes toward charting some design waters in the ocean of reform-oriented mathematics education. To the extent that this effort bears appeal to fellow designers and design-based researchers, we may thus all be better equipped to help mathematics students navigate conceptual transitions along meaningful continuums [61].

2. MODUS OPERANDI: THE DESIGNER AS A REFLECTIVE PRACTITIONER

Winograd and Flores [101] view scholarly discourse on design as part of a larger, interdisciplinary intellectual pursuit that goes beyond how to build this gadget or another to encompass an inquiry into the human potential to navigate transition:

> In ontological designing, we are doing more than asking what can be built. We are engaging in a philosophical discourse about the self—about what we can do and what we can be. Tools are fundamental to action, and through our actions we generate the world. The transformation we are concerned with is not a technical one, but a continuing evolution of how we understand our surroundings and ourselves—of how we continue becoming the beings that we are. (p. 179)

Design-based researchers embrace the above urge to perceive the practice of design not only as a compliant operationalization of extant theoretical models of human learning but also as a proactive, critical agent of change that can inform and transform these models. Technology plays a particularly vital role in stimulating reflection on what it means to know, because its architectures, encodings, and encasings often dictate the decoupling of naturalistic form and content, sensation and cognition, semiotic systems and meaning—technology tends to mirror and unpack for us implicit aspects of our reasoning and lay them bare for scrutiny and improvement [37, 68, 100]. As Marshal McLuhan [59] wrote:

> The hybrid or the meeting of two media is a moment of truth and revelation from which new form is born….The moment of the meeting of media is a moment of freedom and release from the ordinary trance and numbness imposed….on our senses. (p. 63)

In like spirit, I am inspired by the prospects of reconceptualizing mathematics education via identifying within our community's inventions and empirical data such mechanisms and processes that may challenge our field's implicit assumptions about how students can and should learn as well as how, accordingly, instructors can and should teach.

By reflecting specifically on the actual designs themselves that we build, we may also be able to face undertheorized aspects of our creative process and, in so doing, both acknowledge and demystify this process, which is difficult to describe let alone document empirically [15, 96]. That is, just because we do not always understand how we invent new instructional devices and lesson plans, we need not ignore, misrepresent, or romanticize this process [3, 83]. As Schön [87] cautions,

> mystification consists in making knowledge-in-practice appear to be more complex, private, ineffable, and above all more once-and-for-all, more closed to inquiry, than it needs to be.... [D]emystification is not a showing up of the falsity of the practitioner's claims to knowledge but a bid to undertake the often arduous task of opening it up to inquiry. (p. 289)

Finally, by exploring unknown aspects of how designers design, we may illuminate corresponding unknown aspects of how students learn. The rationale here is that design, as an educational enterprise, is enabled by designers and students sharing in biology and cognition [67]. As such, by reflecting on our own designs as projections of our mathematical knowledge—phenomenalizations of our tacit schemas [73]—we may better understand, share, and foster core yet covert aspects of this knowledge [53, 68].

The designs discussed in this essay were conceptualized intuitively. They resulted from my efforts to build materials and activities that concretize my core tacit images for the targeted mathematical notions [72]. My design process thus begins by introspecting, in an attempt first to elicit, capture, and articulate my own multimodal dynamical schema underlying the target notion, then to embody the schema in forms that learners can engage and utilize meaningfully in guided goal-oriented activities. In parallel, I perform cognitive domain analyses with the objective of retro-rationalizing my own intuitive design, and I iteratively evaluate and tweak these analyses vis-à-vis learning theory and consultation with peers as well as pre-pilot empirical results that I gather concurrently. The project then continues to ascend spirally through cycles of implementation, reflection, and modification [13].

The objective of this particular essay is to step back from the creative process so as to survey and sort the products of this process in terms of commonalities and differences in materials, tasks, and facilitation methodology. In an attempt to ground this taxonomy in the learning sciences, the reflection will draw from several theoretical resources, as follows.

I practice design-based research by integrating perspectives from constructivist, sociocultural, and semiotic–cultural approaches. This struggle to hold together under a single auspices perspectives from schools of thought that are often viewed as antinomous [33] has been described as the "dialectical approach" [38]. As such, in analyzing the multimodal behavior of children who participate in implementations of my designs, I attempt to articulate what primitive cognitive mechanisms children bring to bear [32, 48], how these mechanisms inform students' sense-making as they co-enact cultural practice with instructors [65], and how instructors steer students to objectify presymbolic notions in mathematical forms [57, 76]. The two design genres surveyed below share in a conceptualization of mathematical content learning as emerging through the students' efforts to enhance, communicate, or substantiate aspects of their implicit perceptuomotor schemas—a guided process that is mediated and formulated by the cultural tools students are encouraged to utilize as the means of accomplishing their ad hoc objectives.

3. A TALE OF (WHAT SEEM TO BE) TWO DESIGN GENRES

This section lays out what I am proposing to view as two related yet distinct design genres for creating mathematics learning activities, the perception-based and action-based design genres. Both genres can be viewed as sociocultural interpretations of radical constructivist pedagogical philosophy [5], in the sense that they abide with the more tempered accounts of what resources and guidance teachers should provide in fostering student reinvention of mathematical ideas [97]. Namely, per both of my proposed genres students begin from what they can see or do in coping with a problematic situation; yet then this naturalistic capacity enters in dialogue with analytical discourse on the same situation, as embodied in the lesson's media, symbolic artifacts, and teacher voice and positioning; via this guided dialogue, the students are encouraged to negotiate, coordinate, and reconcile the spontaneous and scientific perspectives [86, 89].

The objective of this section is not, and perhaps cannot be, to describe in detail a set of design studies. Rather, I wish to explain and exemplify design genres. Where the reader may wish to learn more about rationales and findings, I provide references to other publications. Finally, any taxonomy per force draws broad brushstrokes—it condenses complex activities into particular essential elements. Yet in practice, elements of these and other genres may often intermingle.

3.1 Perception-Based Design

In [2, 5], I surveyed a set of designs for students to ground mathematical concepts via coordinating tacit and analytical views on situated phenomena. These designs have all been evaluated empirically via semi-structured clinical interviews, and microgenetic analyses of students' conceptual trajectories suggest that these designs bear didactical potential. Aspect of these designs have been integrated into high-visibility units.

Common to this set of designs is that they each target an *a/b* concept, such as chance (favorable events / possible events), slope (rise / run), density (total object area / total area), and proportional equivalence in geometrical similitude ($a{:}b = c{:}d$). Further common to these designs is a general lesson plan by which to invite students first to articulate their naïve view with respect to a situation and only then engage in modeling, reflecting, and discourse by which to negotiate the formal view as complementary to, and empowering of their naïve view.

As such, activity sequences in this genre begin by presenting students not with mathematical definitions, notation, and worked examples—in fact, participants often do not know they are "doing math." Rather, the instructor presents students with a set of materials and asks them to cast a judgment with respect to some quantitative or logical property inherent in these materials. Importantly, the materials are crafted such that students' naïve inferences, though qualitative, ill-articulated, or tentative, nevertheless agree with mathematical analysis. That is, I do not attempt to cause cognitive conflict early on in the process by proving the students wrong; rather I attempt to embrace and affirm children's agency in making sense of the world in their natural, uninformed yet often sophisticated ways [29, 31, 91]. Thus students are expected initially to apply not analytical views, which they would not as yet share with the instructor, but—explicitly—their naïve views. In particular, students first experience the embedded magnitudes not analytically via *a/b* structuration but rather holistically, as a gestalt perceptual sensation ([71], pp. 46-49). Only then, in order to introduce the analytical view, the instructor provides students suitable media and guides them through the formal procedure of building a model of the situation. The emerging practical and theoretical question at the heart of research on this design genre has been whether and, if so, how and why students accept the mediated analytical view, proposed by the instructor, as complementary to their own naïve view on the source phenomenon.

Consider the probability subject matter of simple compound-event random generators, such as the rolling of two dice or the flipping of four coins. For this content, middle-school students should learn to perform combinatorial analysis procedures and make sense of resultant event spaces. For example, students are to determine the chance of getting an outcome with 2 heads (H) and 2 tails (T) from flipping four coins. This content has presented great challenges for students as for adults [49], and researchers implicate students' difficulty in appreciating that it is critical to attend to variations on combinations [23]. For example, students analyzing a four-coin flip do not discern among the equiprobable yet unique outcomes HTHT and THTH—they typically argue that the order is irrelevant to an analysis of chance (likewise, students view the dice-roll outcomes 3~5 and 5~3 as indexing literally the same event). Consequently, any useful intuitions and predictions that the students might have brought to bear on the problem become thwarted by the instructor's analysis. Students are thus expected to accept probability algorithms even if these conflict with their "normal thinking" [75, 80]. Granted, the algorithms enable the children to solve school assessment items, yet Wilensky [99] has warned of the "epistemological anxiety" ultimately bred by such reluctant acquiescence to ostensibly arbitrary routines (see also [24] on deutero learning, [94]).

We are thus searching for a situation that embodies the same mathematical problem as does the four-coin experiment yet in a form that is conducive to correct rather than incorrect intuitive prediction of actual experimental outcomes. Toward that design objective, we seek to create an opportunity for learners to express their predictions qualitatively, without any numerical indices. Only subsequent to these predictions will we guide the learners to coordinate meaningfully between their naturalistic view of the situation and the complementary mathematical view. This is the *gestalt-before-elements* principle of perception-based design [5].

In my design solution, the instructor presents the student with a small tub full of marbles—a mixture with equal amounts of green and blue marbles—accompanied by a utensil for drawing out exactly four marbles set in a 2-by-2 square configuration (see Figure 1). Students are asked to indicate the four-marbles event they believe is most likely to be drawn from the tub. The instructor then provides the students cards as well as a green and a blue crayon and guides them through combinatorial analysis of the stochastic experiment; this process results in the construction and assembly of the experiment's event space—a collection of sixteen iconic representations of all possible outcomes, organized in five stacks according to *k* (# of greens).[1]

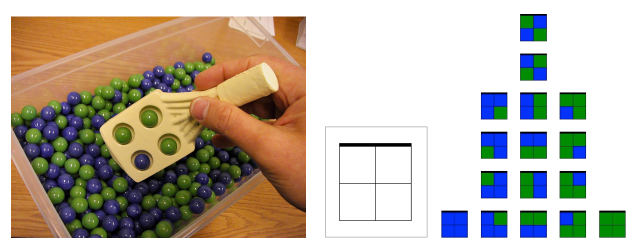

Figure 1: Selected materials from a design for the binomial. From left: an open urn full of green and blue marbles with a scooper for drawing out four marbles; a card for indicating possible outcomes using green and blue crayons; the event space made up of 16 such cards.

[1] Strictly speaking, this situation is a hypergeometric, not binomial experiment, because as each marble settles in the scooper, there is one less of its color in the urn. However, the large number of marbles in the urn makes this distinction practically negligible, hence I treat it as binomial.

In our studies, Grade 4 – 6 students, who had not formally studied probability, judged that the most likely four-marbles draw from the tub would have two green and two blue marbles. This is precisely what mathematicians would predict via probability theory, and yet the students did so based not on combinatorial analysis but, I submit, on hard-wired perceptual capacity to infer the representativeness of samples based on comparing color ratios in a sample and its source population [44, 102]. The students further judged that an all-green or all-blue draw would be the rarest type of draw, and so on. Importantly, these naïve inferences were couched in terms of the five possible combinations, with no reference to the variations on these combinations. Nevertheless, and critically, students were ultimately able to make sense of the event space as triangulating their naïve expectation, even though the event space does include those variations they had been ignoring. How do students achieve this coherence between tacit and cultural views on a stochastic situation, given that these views apparently carve the phenomenon at different joints [20]?

I have argued that students ground the analysis product as meaningful via a creative inferential process called abduction [70, 90]: students bootstrap the design's targeted mathematical content in the form of a *rule* they invent by which to construe the product (the mathematical model) as a *case* that explains as a *result* their unmediated perceptual judgment of the source phenomenon; students initiate this insight heuristically by aligning [58] and interpreting relations among elements of the mathematical model as analogous to relations among elements in their perceptual construction of the source phenomenon. For example, students notice that there are more possible outcomes of the two-green-and-two-blue color combination (6 outcomes) than outcomes of the three-green-and-one-blue combination (4 outcomes; see Figure 1, on the right); this inference evokes an "$A_{Outcomes} > B_{Outcomes}$" notion that coheres with a corresponding comparison of these same two events' intuited likelihood as inferred from the source phenomenon (Figure 1, on the left), that is, "$A_{Likelihood} > B_{Likelihood}$" (see [92]). This process is greatly supported by the instructor, who guides the child via multimodal discourse toward particular perceptual features, in the phenomenon and its model, whose highlighting, alignment, and coordination are crucial for achieving the abduction [4, 6].

Implementations of perception-based designs such as this have suggested an intriguing finding. Namely, under appropriate design conditions, students are able to make sense of the analysis *product*, that is, the material assembly that the educator views as a model of the situated phenomenon, before they appreciate the analytic *process* by which this model was built. For example, first students would succeed in accepting the event space as a meaningful representation of the anticipated outcome distribution, and only in retrospect would they accept the combinatorial analysis process by which this product was created. This is the *product-before-process* formalization sequence of perception-based design [5].

Perception-based design, while apparently holding potential for mathematics learning, also bears potential for *research* on learning, because it hones universal tension between students' informal resources for making sense of situations and instructors' formal reconstructions of these situations. I thus believe these empirical findings, theoretical developments, and investigative contexts are not only valid and useful within this design genre but, rather, might elucidate and even inform mathematics education more generally [5, 6, 7].

3.2 Action-Based Design

The action-based design genre emerged on the background of a growing body of theoretical and empirical research in the cognitive sciences implicating embodied activity as the source, substance, and process of human reasoning [21]. Within mathematics-education research literature, we witness an increasing support for a conceptualization of goal-oriented interaction—whether physically manifest or mentally simulated activity—as the epistemic grounding and intrinsic phenomenology of problem solving [55, 63, 79]. And yet, competent performance in the disciplines, specifically in mathematics, is instantiated within semiotic registers involving signs, forms, and procedures that bear little to no cues as to their spatio–temporal origin and meaning. We are faced with a continuity paradox: How does embodied action give rise to reflection, analysis, disciplinary forms, vocabulary, and inscription? How are these epistemically disparate resources linked through participation in learning activities? How does a teacher guide this process?

In approaching this traditional symbol-grounding problem, I agree with Harnad [45] that knowledge evolves "bottom up," and I characterize the "bottom" as deliberate embodied activity, yet I complement his position with sociocultural "top down" mediation via guided participation in social practice. In particular, I investigate the conjecture that individuals' mathematical understanding can emerge as they attempt to enhance, represent, or reflect on their own presymbolic situated action by utilizing cultural tools, that is, via enactive and discursive extension of embodied solution procedures [19, 25, 30, 43, 62]. In my design-based research work, I attempt to zoom in on this instrumentalization process of children adopting/adapting action-oriented artifacts available to them in the learning environment [17, 84, 98].

The rationale of action-based design coheres with empirical findings from the dynamic-systems perspective on motor development [93], cultural anthropology research on parentally promoted infant action routines [77], and cognitive anthropological research on vocational instruction of dexterous tool use within manual practices [26]. These disciplines all conceptualize skill development as the guided, repeated solution of similar motor problems via attuning to emerging affordances in the perceptuomotor field of interaction.

Finally, the rationale of action-based design resonates with, and draws inspiration from, Dourish's HCI (Human–Computer Interaction) notion of *embodied interaction*:

> [Embodied interaction] is an approach to the design and analysis of interaction that takes embodiment to be central to, even constitutive of, the whole phenomenon.… [E]embodied phenomena are those which by their very nature occur in real time and real space; embodiment is the property of engagement with the world that allows us to make it meaningful; Embodied Interaction is the creation, manipulation, and sharing of meaning through engaged interaction with artifacts. ([40], pp. 102-126)

To explore the potential of the action-based design genre for mathematics education, we built a technological device, the Mathematical Imagery Trainer (MIT). The MIT is an embodied-interaction system designed to foster the development of perceptuomotor schemas for grounded formalization of mathematical notions. Our first MIT was engineered specifically for proportion (MIT-P, [11, 46]).

Proportion is a pivotal curricular topic that has been presenting difficulty for many students from late-elementary school and through to college [56]. Research on students' incorrect solutions to rational numbers, more broadly, has implicated "additive reasoning" as underlying their numerical errors [27]. In particular, students attend to additive rather than multiplicative relations within and between number pairs. Looking at 6:10 = 9:x, for example, students attend to the difference of 4 between 6 and 10 rather than the factor of 10/6, or they attend to the difference of 3 between 6 and 9 rather than the factor of 9/6; consequently, they infer that 6:10 = 9:13 due to the equivalent differences of 4 within both number pairs, or due to the equivalent differences of 3 between corresponding elements (from 6 to 9 and from 10 to 13). In a sense, proportionality presents a novel situation involving the equivalence of number pairs bearing non-equivalent differences among corresponding elements. Somehow, students are to accept a new type of equivalence class in which different differences can be construed as "the same" [1]. The question is how they may develop this new equivalence class.

We maintain that students can and should ground proportional equivalence in additive reasoning, only that doing so requires appropriate cognitive structures, what [72] call dynamical imagistic schemas. And yet, we evaluate, everyday contexts do not occasion opportunities for people to develop these target schemas. That is, mundane activities do not afford the performance and practice of embodied coordinative routines that, with suitable guidance, could be signified quantitatively and symbolically as proportional. We thus wished to design a novel embodied-interaction activity by which children would develop a new pre-numerical schema bearing semiotic potential as a case of proportionality [23, 57]. The interaction would *initially* elicit the students' perceptuomotor schema presumed to underlie their additive reasoning yet *subsequently* treat this schema so that students would assimilate proportional relations. One might think of this intervention as "Feldenkrais somatic mathematical education" or just "*somathics.*" Once the new schema is established, we would steer students to signify this new "embodied artifact" as a mathematical artifact (cf. [41, 59]), so that the embodied artifact becomes a "conceptual performance" [95]. In short, action becomes concept.

There have been numerous attempts to support the grounding of proportions, and these attempts vary, in part, in accord with the designer's conceptualization of multiplicative constructs. Some designs embark from an iteration rule for combining *a* and *b* discrete quantities into a succession of linked cumulative totals, for example: adding $2 and $3, respectively, into two separate piggybanks; tabulating the two linked running totals down the columns, such as 2-3, 4-6, 6-9, 8-12, etc.; and then highlighting multiplicative relations inherent to this tabulation as calculation shortcuts for moving between number pairs in the solution of proportion problems, such as scaling by a factor of 4 from 2-3 to 8-12 [8, 51]. Other designers launched the activities from non-additive situated multiplicative transformation, such as splitting a set of material elements into equally sized subsets [35].

Yet all those designs scarcely, if ever, considered what I view as the phenomenological core of proportional equivalence, namely the sensory experience of identity between two ratios ("sameness-relational" equivalence [47]). How might students experience 1:2 and 2:4 as sensuously identical? Ideally, I reflected, this sensuous identity should be instantiated in forms that are readily conducive to numerical quantification via measurement, so as to enable progressive formalization. Perhaps, I wondered, we could use technology to import into a learning environment the familiar "recipe" conceptualization of proportion by which, for example, 1-and-2 units of some substances "taste" the same as, respectively, 2-and-4 units of the same substances. In this design, the *a*, *b*, *c*, and *d* values of the *a:b* = *c:d* proportion would all be extensive quantities from the same measure space, and yet the physical enactment of the *a*-and-*b* pair and the *c*-and-*d* pair would somehow generate identical sensory effects, borrowing on the idea of *a/b* = *c/d* as intensive quantities.

Several inspirational prior designs satisfy some of my own design specifications [16, 20, 60, 64]. However either these designs introduce symbols too early, do not leverage NUI (Natural User Interfaces), or do not offer proportional equivalence as sensuous identity.

I thus sought to create an activity, in which learners could begin to construct proportionality initially by noticing that two physical postures—an *a*-and-*b* bimanual posture and a *c*-and-*d* bimanual posture—effect the same feedback; learners would then learn to move between the two postures, maintaining the target feedback. This is the *dynamical conservation* principle of action-based design: enacting continuous motion that varies positional/quantitative properties of topical elements yet sustains an overall target feedback. Students would discover and rehearse presymbolic action of proportional transformation as a new perceptuomotor form—a "proportion kata"—that maintains an invariant feedback across the different "ratio asanas." Only then would we introduce into the problem space mathematical tools, which students would recognize as bearing contextual utility. By appropriating these tools, students were implicitly to represent, reconfigure, and signify their embodied form in mathematical register. As such, the embodied artifact, initially performed as tight perceptuomotor coupling with an interactive technological device, would evolve into a standalone conceptual performance articulated in the discipline's semiotic system [10, 95].

The MIT-P remote-senses the heights of the user's hands above the datum line (see Figure 2a). When these heights (e.g., 2" and 4"; Figure 2b) relate in accord with the unknown ratio set on the interviewer's console (e.g., 1:2), the screen is green. If the user then raises her hands in front of the display maintaining a fixed distance between them (e.g., keeping the 2" interval, such as raising both hands farther by 6" each, resulting in 8" and 10"), the screen will turn red (Figure 2c), because the pre-set ratio has been violated. But if she raises her hands appropriate distances (e.g., raising her hands farther by 3" and 6", respectively, resulting in 5" and 10"), the screen will remain green (Figure 2d). Participants are tasked first to make the screen green and, once they have done so, to maintain a green screen while they move their hands.

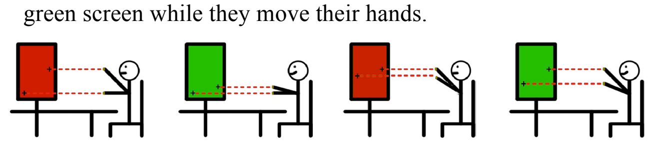

Figure 2. The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 ratio. The right hand needs to be twice as high along the monitor as the left hand.

The activity advances along a sequence of stages, each launched by the introduction of a new display overlay (see

Figure 3) immediately after the student has satisfied each of successive protocol criteria. For example, consider a student who is working with the crosshairs against a blank background (Figure 3b). Once he articulates a dynamical-conservation strategy for moving his hands while keeping the screen green, the activity facilitator introduces the grid (see Figure 3c).

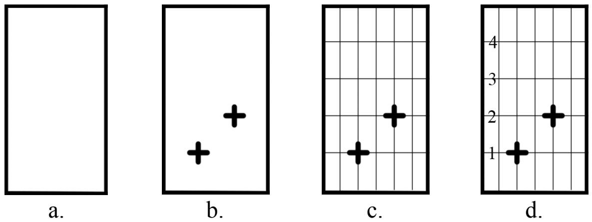

Figure 3. Sequence of MIT-P display overlays: (a) a blank screen: (b) crosshairs; (c) grid; and (d) numerals along the *y*-axis. Not featured here from the design is a ratio table.

We implemented the MIT-P design in the form of a tutorial task-based clinical interview with 22 Grade 4 – 6 students, who participated either individually or in pairs. Qualitative analyses of video data collected during those sessions suggest that the activities created opportunities for students to struggle productively with core conceptual challenges pertaining to the target content of proportions, at least per our embodied-cognition modeling of this mathematical topic. That is, the students discovered effective non-numerical strategies for utilizing instrumented gesture to enact dynamical conservation and then learned to re-describe these strategies numerically.

Initially, the students explored the space by waving their hands about until they chanced to turn the screen green, whereupon we asked them to find yet another green. All students moved both hands up (or both down), keeping a fixed distance between the hands. Thus, per our hypothesis, students' default schema for dynamical conservation is analogous to their typical numerical errors on rational-problem problems, such as 6:10 = 9:13 (albeit we cannot as yet support a claim for a *causal* relation). After further exploration, students articulated a strategy that relates between the hands' elevation and interval, for example, "The higher you go, the bigger the distance needs to be between them to make it green." They thus experienced different differences as "the same."

Next, students engaged the tools we overlaid onto the problem space, adapting them as enactive, semiotic, and epistemic means of enhancing their performance, discourse, and inquiry. In particular, students elaborated and generalized their qualitatively expressed, manipulation-based strategies into quantitatively expressed mathematical propositions. For example, they engaged the grid as a frame of reference that appeared better to enable an enactment of the "higher–bigger" strategy, yet in so doing they modulated into a new strategy: in the 1:2 setting they said: "For every 1 unit I go up on the left, I need to go up 2 units on the right" (*a*-per-*b* strategy, see [78])

Deeper analyses of students' conceptual microgenesis revealed that their discoveries of more sophisticated interaction strategies, such *a*-per-*b*, were neither premeditated by the students nor directly mediated by the instructors. Rather, these advanced strategies emerged as the students engaged the new mathematical tools to carry out an existing strategy for accomplishing the task [85, 88]. More specifically, in the micro-process of utilizing a new object to perform an existing strategy, the strategy's implicit perceptuomotor subgoals "hooked" the new object's embedded affordances, so that the strategy became redistributed and reconfigured. Consequently the strategy "shifted" and, in so doing, both its practical and mathematical power increased (the *hooks-and-shifts* principle of action-based research [12])

In the latter interviews of the study, we introduced a new protocol item: we asked the students to reason about any relations they discern *among* the different strategies they had devised, which—still using the 1:2 ratio as an example—also included moving the right hand double as fast as the left hand, placing the right hand double as high as the left, increasing the interval between the hands by 1 unit as they both rise, etc. In [9] we demonstrate cases of students coordinating between strategies, and we claim that they achieved this by inventing heuristic logico-mathematical causal mechanisms. One student said, for example, "[The right hand] is always going up by two, and [the left hand] is going up by one, which would mean that [the right hand] is always double [the left hand]." In the cognitive process of building this causal inference, the students coordinated multiplicative and additive conceptualizations of the dynamical conservation by re-visualizing additive elements multiplicatively. As such, the design achieved the objective of grounding proportionality in students' additive schemas.

I have now introduced two design genres, perception-based and action-based design. In the reflective process of articulating all the above, I came to ask what these genres might have in common, given that both enable discovery-based learning. As I elaborate below, I believe that both genres create epistemic affordances for grounding conceptual knowledge yet they differ in the particular nature of these epistemic affordances. This proposed centrality of an epistemic factor in the learning process might clarify why I use the appellations "perception-based" and "action-based" to distinguish the genres even though clearly activities in both genres involve perceptuomotor activity! Namely, I am interested in implicating the epistemic root of sense-making—what the new mathematical concepts are grounded in—and differentiating this vital resource from pragmatic aspects of the activities.

4. JUMPING TO CONCLUSIONS: COMPARING TWO DESIGN GENRES

Both the perception-based and action-based design genres offer students a subjective sense of continuity from a relatively naïve, immediate form of effectively engaging a situation through to a scientific, analytical, mediated form of doing so. In both genres, the instructor embraces students' naïve forms of engaging the situation as valid and productive. In both genres, simple perceptuomotor engagement becomes restructured when students appropriate semiotic means of objectification available in the problem space. In both genre procedures, initial interactive embodiment and subsequent numerical signification are staggered rather than concurrent.

Still, by what criterion do learners judge that the naïve and mathematical schemas are commensurate such that the students experience continuity across these modes of engaging the activity? This question is important for the theory of education as for its practice, because the question touches on the old Socratic "learning paradox"—learners' universal capacity to build conceptual structures larger than the sum of their available parts [5, 20, 74].

Whether they participate in designs that accord with the perception- or action-based genre, children are led to instrumentalize available mathematical forms by evaluating the forms' ad hoc contextual utility vis-à-vis their own naive resources for engaging the situation. In perception-based design, the child compares two *inferences*: (a) the informal inference from looking directly at a phenomenon; and (b) the formal inference from studying its mathematical model. In action-based design, the child compares the *effects* of two strategies: (a) the naive strategy, in which the body moves in an acquired perceptuomotor kinesthetic routine that is well coupled with the environment; and (b) the reconfigured strategy that avails of enactive, discursive, and quantification affordances inherent in mathematical tools that are introduced into the interaction space.

Critically, both designs thus appear to afford learners a sense of meaning for the mathematical forms they first engage during the activity. In perception-based design, the sense of meaning emanates from achieving *inferential equivalence* between the immediate and mediated views on a source phenomenon. In action-based design, the sense of meaning emanates from *functional equivalence* across a naïve and an instrumented embodied-interaction strategy for effecting the targeted goal state of a technological system. Both inferential-equivalence in perception-based design and functional-equivalence in action-based design constitute for learners epistemic grounds for appropriating the mathematical signification of their embodied skill.

Finally, whereas—still prior to formalization—perception-based design avails of the child's pre-existing capacity, action-based design also constructs new perceptuomotor schemas.

The modest taxonomy of design genres offered in this paper is very much idiosyncratic to the work of one person. As such, the taxonomy might turn out to bear only little if any use to other mathematics-education designers and researchers, because it may be highlighting but a mere corner in what is otherwise a vast, multi-dimensional terrain of designs [69]. Nevertheless, to the extent that the rationale and methodology of this taxonomy agree with fellow design-based researchers, it may be worth their while to qualify or, hopefully, substantiate, complexify, and expand this taxonomy from the wealth of their own experiences. As such, this essay will have achieved its objective. For my part, I would be intrigued by follow-up studies that evaluated the generalizability and scope of this taxonomy by surveying prominent designs and asking, "In what epistemic resource are learners grounding the meaning of the core notion?"

In closing, I wish to reemphasize that rich learning activities may well include interaction elements availing of both the perception-based and action-based design genres. My objective was not to promote exclusivity of either design genres but rather to examine the epistemic source—the root of subjective meaning—that educational designers may offer mathematics learners. Ultimately, the ongoing research program is to continue my efforts in developing what I call *embodied design*. Embodied design is a pedagogical framework that seeks to promote grounded learning by creating situations in which students can be guided to negotiate tacit and cultural perspectives on phenomena under inquiry; tacit and cultural ways of perceiving and acting. To realize this vision, I have found, educational designers should keep the body in mind.

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