Leveling Transparency via Situated Intermediary Learning Objectives (SILOs)

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Abstract: When designers set out to create a mathematics learning activity, they have a fair sense of its objectives: students will understand a concept and master relevant procedural skills. In reform-oriented activities, students first engage in concrete situations, wherein they achieve situated, intermediary learning objectives (SILOs), and only then they rearticulate their solutions formally. We define SILOs as heuristics learners devise to accommodate contingencies in an evolving problem space, e.g., monitoring and repairing manipulable structures so that they model with fidelity a source situation. Students achieve SILOs through problem-solving with media, instructors orient toward SILOs via discursive solicitation, and designers articulate SILOs via analyzing implementation data. We describe the emergence of three SILOs in developing the activity Giant Steps for Algebra. Whereas the notion of SILOs emerged spontaneously as a framework to organize a system of practice, i.e. our collaborative design, it aligns with phenomenological theory of knowledge as instrumented action.

When mathematics-education designers set out to create a new learning activity, they bear in mind the activity's *ultimate* pedagogical objective. Reform-oriented designers, however, bear in mind *intermediary* objectives, too, for students' immersive experiences in situated, multimodal, spatial–dynamical activities designed to foster grounded understanding of the ultimate target concepts. Broadly, reform-oriented activities unfold in two steps:

- In Step 1, learners interact with media—physical or virtual materials and ready-made objects—to solve problems that require manipulating, organizing, and/or transforming these media with attention to quantitative relations as well as emerging patterns or principles pertaining to these relations.
- In Step 2, learners are guided to reflect on, and rearticulate their insights using normative semiotic systems, including frames of reference, vocabulary, and symbolic notation and to reenact discovered processes as standard algorithms using the formal representations (Diénès, 1971; Freudenthal, 1983).

This paper focuses primarily on Step 1. Step 1 is of immense importance to the construction of knowledge (Kamii & DeClark, 1985; Piaget & Inhelder, 1969; Thompson, 2013). And yet, we find, educational designers have little, if any, conventional forms, nomenclature, or methodology for articulating Step 1 learning objectives prior to the design process. Perhaps, we submit, this disconcerting lacuna in the design toolkit is related to the ultimate futility of attempting to articulate Step 1 learning objectives prior to building and refining activities and observing people engage with them. Namely, Step 1 objectives emerge only through the design process. Yet this emergent nature of a design's Step 1 objectives, we further submit, should not deter us from eventually defining those objectives. This paper resulted from reflecting on an apparent omission in our own design process: Building a certain design, we kept referring nebulously to a set of latent, contextualized, mathematically oriented, informal ideas we wanted students to discover via engaging in its Step 1 activities. The objective of this paper is to name that unnamed class of ideas and define its role within the design process. We will name this class *situated, intermediary learning objectives* (SILOs) and demonstrate how this ontological innovation lends coherence to a comprehensive, complex, multi-stage process. We hope that, through this paper, fellow designers will join us in "learning and becoming in [design] practice" (the ICLS 2014 theme).

In the remainder of this paper we: overview relevant educational-research literature (Section 1); present Giant Steps for Algebra (Chase & Abrahamson, 2013) (Section 2); explain how three SILOs emerged via developing the design materials and analyzing pilot implementation data and how these SILOs inform our technological redesign (Section 3); and offer implications for theories of knowing and learning (Section 4).

Theoretical Background: Constructing Means for Constructing Meaning

When we design concrete activities for mathematics learning, what are our learning objectives for these activities? These are not quite mathematical learning objectives per se, because they may not be articulated in formal register and might not even involve numerical values. And yet we do eventually form clear ideas for what the students should be discovering about the target concepts through engaging in the concrete activities. In so doing, we implicitly exercise a theoretical view on the relation between the manual and the mental. One such view is ascribed to John Dewey, who characterized conceptual learning as the individual's process of formalizing their reflection on experience—their guided passage from implicit know-*how* through to articulated know-*that*. Such characterizations of grounded understanding are not only vital for building theories of learning but also bear direct implications for the practice of designing effective learning environments that seek to guide children from informal experience to formal concept. This schematic conceptualization of grounded mathematics learning as an experience-to-concept two-step process cuts across multiple theories and frameworks, including our own.

To begin with, our distinction between situated and general knowledge is a hallmark of Realistic Mathematics Education (RME). Freudenthal (1983), founder of RME, developed a pedagogical methodology based on the principle that children should create their own models of problematic realistic situations. Gravemeijer (1999) elaborates on the function of modeling activities in RME, emphasizing the imperative of letting students' models emerge: "The premise here is that students who work with these models will be encouraged to *(re)invent* the more formal mathematics" (p. 159, original italics). This progress from explorative actions to consistent rules that generalize these actions is theorized more explicitly in RME via the formulation of two related constructs, "model of" and "model for." A "model of" results from modeling a particular situation. A general "model for" eventually emerges from noticing homology across mathematically analogous "models of." Our SILOs (situated, intermediary learning objectives) can be viewed as a checklist detailing structural properties and relations inherent to a "model of." In turn, we deliberately articulate the SILOs in linguistic forms that would also capture general conceptual structures, just as in a "model for."

This ontological relation between actions, objects, and concepts has long fascinated theorists of human activity. For example, *distributed cognition* is a theory of human practice that elucidates relationships among participants to a collective human practice and the artifacts that mediate this collaboration (Clark, 2003). Broadly, the array of tools supporting our cognitive activities—pen and paper, calculator, computer, and so on—are *cognitive artifacts*, that is, artificial tools or devices that carry, elaborate, and report information during problem solving (Norman, 1991). As such, mathematical learning can be theorized as developing psychological structures for regulating the mental activity of distributing quantitative problems over available cognitive artifacts. This effect is dialectical: even as we learn to act and think in new ways as facilitated by these tools, they in turn bear the potential of reifying for our reflection what and how we act and think (Hutchins, 2010). It follows that different material instantiations of one and the same mathematical concept may bear different pedagogical affordances, because their uptake forges different cognitive routes, different neural residue. SILOs articulate this residue pragmatically in terms of the models' structural properties that students learn to monitor.

And yet, this emergence of cognitive structures from mediated actions with external media is not at all guaranteed. A novice might learn to problem-solve using a cognitive artifact that embodies a mathematical function yet without ever understanding this function or how the artifact embodies it. Is this cause for concern? We turn to discuss the psychological construct of *transparency*, which captures relations between, on the one hand, artifacts inherent to a cultural practice and, on the other hand, a social agent's understanding of how features of these artifacts mediate the accomplishment of their objectives. Thus when we say that an artifact is transparent, we refer to the *subjective* relation between a particular agent and the artifact (Meira, 2002).

For educational designers, the notion of transparency suggests a particular framing. Namely, the role of designers can be conceptualized as creating learning tools that learners can render subjectively transparent. In a word, the transparency perspective confers upon educators the role of enabling students to see and learn *how* mathematical artifacts do what they do. For example, in a study of physically distributed problem solving, Martin and Schwartz (2005) found that participants generated more salient and transferable conceptualizations of fractions when using "obdurate" square tiles as opposed to classical pie-shaped manipulatives. Why? From the theoretical lens of transparency, the pie pieces obscured the notion of "whole" precisely because the study participants did not need to assume agency in distributing onto those media their tacit sense of whole—the circle implicitly did that work for them. On the other hand, those students who worked with square tiles were obliged to construct the whole themselves, and that more challenging, agentive experience apparently endured.

Whereas mathematical models per se are often static, such as those fraction squares, they are created through active engagement. Indeed, scholars of embodiment pay close attention to perceptuomotor routines as these relate to conceptual knowledge. In particular, when students operate physically within concretized conceptual domains, design-based researchers attend to how the students carry out spatial–dynamical analogs of formal operations (Antle, 2013). An application of embodiment theory to mathematics education is *embodied design* (Abrahamson, 2009), "a pedagogical framework that seeks to promote grounded learning by creating situations in which students can be guided to negotiate tacit and cultural perspectives on phenomena under inquiry" (Abrahamson, 2013, p. 224). When students participate in embodied-design activities, they solve problems that initially do not bear symbolical notation, do not require calculation, and do not call for quantitative solutions; they call only for qualitative judgments, informal inference, or naïve physical actions.

Embodied designs clearly demarcate the two-step design framework that is thematic to this essay and, as such, underscore the informal nature of Step-1 situated, intermediary learning objectives (SILOs). That is, if we theorize perceptual judgment and motor action as bearing seeds of mathematical concepts, then we need language for bridging actions and concepts. SILOs articulate subtle elements of learners' informal inferential reasoning about perceptual judgments or motor-action solution strategies that they are to discover and refine.

With the introduction of embodied design, our literature survey shifts from evaluating implications of learning theory for pedagogical design to educational research work dealing directly with the development of design frameworks for grounded mathematical learning.

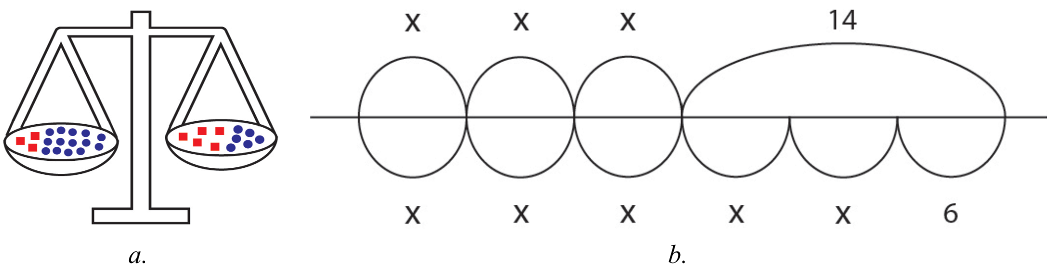
A profound contribution to the design of mathematics learning environments comes from Richard Noss and collaborators, whose learning theory and design frameworks co-emerged dialectically through empirical research studies (Noss, Healy, & Hoyles, 1997). Of particular relevance to our thesis is their set of design heuristics promoting students' *situated abstractions*, "in which abstraction is conceived, not so much as pulling away from context [i.e. the particular features of a situated learning activity], but as a process of constructing mathematical meanings by drawing context into abstraction, populating abstraction with objects and relationships of the setting" (Pratt & Noss, 2010, p. 94, citing Noss & Hoyles, 1996). Pratt and Noss (2010) implicate the epistemological root of mathematical concepts in children's purposeful construction of *utility* for new ideas that are instantiated into designed artifacts in the form of interaction potentialities. The SILOs framework differs from that of situated abstractions in terms of grain size, ontological and epistemological foci, and pedagogical underpinnings. In particular, SILOs articulate a set of initially unavailable interaction constraints that the learner determines, implicates, and wills as potentially conducive to more effective problem solving with a given cognitive artifact; in response, each of these willed constraints is then materialized into the artifact by the instructor who grants the learner's will by enabling into functionality a pre-programmed "hidden" constraint. SILOs are thus functional concretizations of the user's wish-list into working technological features of an interactive device. Yet SILOs are complementary to situated abstractions in the sense that SILOs can be conceptualized as articulating prerequisite structural conditions for enabling and appreciating utility.

In summary, although scholars may differ acutely in their epistemological positions on the constitution of mathematical knowledge, they generally agree that models—forms or structures that learners use in organized activities to promote problem-solving processes—can serve instrumental roles in conceptual development. Having both situated and singled out our proposed heuristic construct of SILO in a legacy of educational theory, philosophy of knowledge, and design frameworks, we now turn to demonstrate this construct's application in an actual case of design practice, namely Giant Steps for Algebra. The next section will explain the design problem that gave rise to this design, and then we explain the design itself.

Setting the Context: Designing Giant Steps for Algebra (GS4A)

The story of learning algebra in schools is often told as the challenge of progressing from arithmetic to algebra. A main character in this story is the "=" sign or, rather, students' evolving meanings for this sign (Herscovics & Linchevski, 1996). When students first encounter algebraic propositions, such as "$3x + 14 = 5x + 6$", their implicit framing of these symbols is *operational*, because the framing will have been fashioned by a history of solving arithmetic problems such as "3 + 14 = __", where you operate on the left-hand expression and then fill in your solution on the right (Carpenter, Franke, & Levi, 2003). Yet algebraic conceptualization of the "=" sign should be *relational*, as this sense contributes to correct treatment of algebraic equations (Knuth, Stephens, McNeil, & Alibali, 2006). Given that the arithmetic visualization of "=" apparently impedes students' transition to algebra, how might this visualization be countervailed? One way is to plant an alternative metaphor.

The *balance metaphor* is undoubtedly the most common visualization of algebraic propositions. This metaphor is typically introduced to students by invoking interactions with relevant cultural artifacts such as the twin-pan balance scale (see Figure 1a). The equivalence-as-balance conceptual metaphor enables a relational, rather than operational, view of algebraic equations. In particular, it grounds the rationale of algebraic algorithms, such as "Remove $3x$ from both sides of the equation," in interactions with a familiar artifact.



a. *b.*

Figure 1. (a) Balance scale and (b) number-line instantiations of "$3x + 14 = 5x + 6$"

Still, students' persistent difficulty in *transitioning* from arithmetic to algebra suggests that the balance metaphor may not be the ideal method for building a relational understanding of equations (Jones, Inglis, Gilmore, & Evans, 2013). Moreover, the historical substitution of twin-pan scales with electronic scales may have rendered the metaphor unfamiliar to many students. We thus wondered, "What alternative metaphor might facilitate students' passage from arithmetic to algebra?" Our search revealed that Dickinson and Eade (2004) tackled a similar problem. They used the number line as a diagrammatic form for modeling linear equations (see original work in Figure 1b.). Giant Steps for Algebra (GS4A) is based on this "double-measuring-stick" model.

Looking at the number-line diagram in Figure 1b, note the combination of above-the-line and under-the-line symbolic indices of one and the same line segment. This element offers two perceptually contrasting yet conceptually complementary visualizations of a single perceptual stimulus (Abrahamson & Wilensky, 2007). Further note how this number-line diagram "discloses" that $2x + 6 = 14$, so that $2x = 8$, and therefore $x = 4$.

In accord with distributed-cognition theory, this model of algebraic equivalence appears to facilitate the offloading of a rule onto a diagram's inherent logico-figural constraints, so that the problem solver can focus on critical inferences, all the while sustaining a sense of understanding for the solution steps. In the number-line model, but not in the twin-pan model, we are able to construct logical relations between variable and integers directly by attending at a single location to spatial properties such as adjacency and containment.

Finally, inspired by the RME principles, GS4A begins not directly with diagrammatic models of existing symbolic expressions but with an asymbolic situation that the student is required to model diagrammatically. This situation is presented in the form of a narrative about an agent who travels along a path, and the number-line emerges as a "model of" this journey. Per embodied design, we thus sought to engage and leverage students' tacit knowledge about simple ambulatory motion and spatial relations, and per constructivist pedagogy we draw on students' elementary arithmetic fluency.

The GS4A problem narrative depicts a quasi-realistic situation, in which the agent performs two consecutive journeys that begin at the same point of departure and end at the same destination yet differ in process. These two journeys correspond to two equivalent algebraic expressions. For example the algebraic proposition "$3x + 2 = 4x – 1$" is told as a Day-1 journey of "$3x + 2$" and a Day-2 journey of "$4x – 1$", as follows:

> Egbert the Giant has stolen the elves' treasure. He escaped their land and voyaged to a desert island. After docking, Egbert set off walking along a path. You are Eöl the Elf. You are positioned on this island to spy on Egbert and find out what he does with the treasure. Starting from the port and walking straight along the only path, Egbert walked 3 giant steps and then another 2 meters. He buried some of the treasure, covered it up really well, and then went back to the ship, covering up his tracks. On the next day, Egbert wanted to bury more treasure in exactly the same place, but he was not sure where that place was. Setting off along the same path, he walked 4 steps and then, feeling he'd gone too far, he walked back one meter. Yes! He'd found the treasure! He buried the rest of the treasure in exactly the same spot as the day before. Egbert then covered up the treasure as well as all his tracks, so that nobody will know where the treasure is. He returned to the ship and sailed off. Your job is to tell your fellow elves exactly where the treasure is: tell them how many meters they need to walk from the docks to the hidden treasure.

We thus designed GS4A as an environment wherein students develop a notion of variable as a specific quantity: a numerical value that is consistent within a local situation. The specific value of the variable would initially be unknown to the student but could eventually be determined by triangulating available information about the Day-1 and Day-2 journeys. Yet triangulating depictive information—as we learned by tinkering with the design ourselves and observing children attempt to solve the problem—carries certain implicit demands of structural precision and coordination. These "trivial" mechanical details surfaced as conceptually critical.

The Emergence of Situated, Intermediary, Learning Objectives in a Design Process

The GS4A SILOs emerged during our research team's meetings and coalesced over iterated cycles of analyzing empirical data collected in pilot implementations of the design. The SILOs enabled us to coordinate within a single linguistic nexus divergent aspects and objectives of our multi-disciplinary tasks: (1) the target concept (algebra); (2) elements of the design (GS4A); and (3) observations of student behavior (in videotaped studies).

During early trials of the design, we used a variety of different modeling media. This turned out to be fortuitous, in that it ultimately led to us identifying the SILOs. As we argue in Chase and Abrahamson (2013), when the students built a model from scratch, they understood its latent mathematical content better—it was more transparent to them—than in cases where the prefabricated media "did the work" for them (as in Martin & Schwartz, 2005). For example, students were more likely to understand the notion of a variable when they used paper and pencil to painstakingly scale up a drawing that depicted an unfolding sequence of giant steps, than when they were allowed to painlessly stretch an elastic ruler whose intervals scale up uniformly.

Qualitative data analyses suggested the following set of three SILOs for GS4A. (Note that although we articulate the SILOs here as rule-based propositions we do not wish to imply that participants used these forms.)

1. *Consistent measures*. All variable units (giant steps) and all fixed units (meters) are respectively uniform in size both within and between expressions (days);
2. *Equivalent expressions*. The two expressions (Day 1 and Day 2) are of identical magnitude—they share the "start" and the "end" points, so that they subtend precisely the same linear extent (even if the total distances traveled differ between days, e.g. when a giant oversteps and then goes back);
3. *Shared frame of reference*. The variable quantity (giant steps) can be described in terms of the unit quantity (meters).

Articulating the SILOs gradually increased the coherence and effectiveness of our work. In particular, it dawned on us that we should use these SILOs in planning a technological version of our mechanical design. In this technological redesign, the SILOs would form a blueprint for an activity architecture, wherein transitioning from each interaction phase to the next would be linked to demonstrating mastery over one of the SILOs. The idea was thus to step learners through an activity sequence, all the while enabling them to build and sustain subjective transparency of the emerging model. Each SILO would be implemented in this design in the form of some aspect of the model that the learner would be required to build manually (virtually) before that property was instantiated and monitored automatically. Borrowing the notion of "levels" from popular computer games—that is, the gradual rewarding of manifest competency with increased power that is linked to increased task demand—in GS4A we *level transparency.* That is, as the users master each SILO, they receive new control over the environment in the form of enhanced affordances that instantiate that specific SILO automatically.

In GS4A, leveling transparency is engineered as follows. The user encounters a problem narrative and is encouraged to solve it on the screen. A continuous blue path extends horizontally across the screen (see Figure 2). On the left of this line there is a small flag (the "start" location). Below the line there is a standard drawing toolbox with buttons for either selecting a color (giant steps are red, meters are green), toggling between journey days (Days 1 or 2), or editing (removing or clearing model elements). A floating "treasure box" (see in Figure 2, in the top-right corner) can be placed at any location. If a user selects the "Giant Step" button and then clicks on the screen, a red arch will appear that connects the giant's last location along the path (a grey node) to the clicked location (a new grey node). Similarly, "Meters" are green arcs.

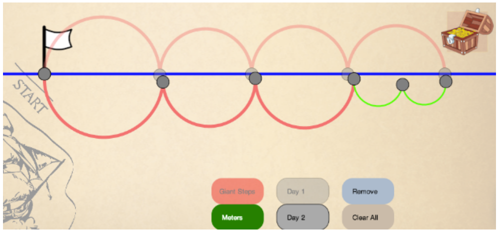


Figure 2. In Level 1, "Free Form," users create all parts of the model manually. Note that the giant steps (red arches) are not quite uniform in size; neither are the meters (green arches).

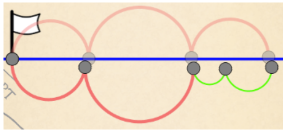
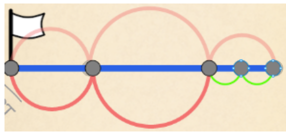
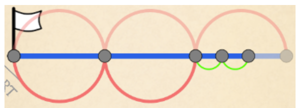
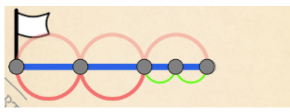
SILOs are psychological constructs—they are about what a child knows (or, at least, about the designer's best understanding of what the child knows). Levels, on the other hand, are technical constructs—they are about an activity's technological affordances, that is, what a pedagogical system performs for you. And yet SILOs and levels are closely related: each SILO articulates a knowledge criterion for entering a new level, and then each level, in turn, orients the child to achieve some next SILO, as outlined in Table 1 (see next page).

We shall now elaborate on this table, referring to its screenshot images. In Level 1, "Free Form," users construct all elements of their model in freehand, analogous to drawing with pencil and paper. Some production imprecision naturally ensues, such as steps that are not quite the same size. The importance of precision (SILO 1) will arise only once the learner attempts to coordinate measures across two journeys, marked above and below the path, and encounters misfits impeding the modeling process. Once users have articulated the imperative of consistency and labored over implementing this aspect in their models, they are evaluated as having graduated SILO 1, "Consistent Measures." As a first concession, the program enters Level 2, "Fixed Meter," in which the system relieves the learner of producing uniform meter units (see also Figure 3, next page).

At this new level, the system supplements manual interaction with optional symbolic interaction. Namely, the learner can now use a control (see in the bottom-right corner of Figure 3) to set a numerical value that determines how many meters will be generated; at a click of a button, the program creates these units as figural elements on the screen. Unburdened by the tedious task of maintaining uniform meters, the user now attempts to equalize the two journeys (Day 1 & Day 2) by adjusting the variable size. Note that one and the same variable, a giant step size, applies both within *each* journey day and across *both* days. As in the case of meters, it is difficult to manually coordinate both within-day and between-days equivalences of variables. Once the user articulates that the variable should be consistent across the entire model, the interface enters Level 3.

In Level 3, “Stretchy,” not only is the meter unit size maintained automatically, but the variable size changes uniformly. So when the user drags any of the nodes along the path line, all variable units change size accordingly (please also compare Figures 4a and 4b, two page down). This supplementary affordance enables the user more felicitously to match the end points of Day 1 and Day 2, as follows.

Table 1: Leveling Transparency: Matched SILOs and Levels in Giant Steps for Algebra Technological Design.

SILO	Level	System Constraints, User Activity, and Behavior Criterion	Interface
1. Consistent Measures	1. Free Form	System offers no support in coordinating units or expressions.	
	Activity	User builds all parts of the model manually; is perturbed by units’ unequal lengths within and between days; tries to equalize units via small adjustments but witnesses that increasing one unit decreases an adjacent unit sharing a node.	
	Criterion	User expresses frustration in equalizing units.	
2. Equivalent Expressions	2. Fixed Meters	System generates meter units in predetermined size and maintains uniform size automatically.	
	Activity	User builds variables manually; is perturbed by variable units’ unequal lengths within/between days; tries to equalize variable units but witnesses that increasing one unit decreases an adjacent unit sharing a node.	
	Criterion	User expresses frustration with managing uniform variable units particularly in an attempt to equalize the two propositions (the spatial extents of Days 1 & 2).	
3. Shared Frame of Reference	3. Stretchy	System monitors for manual adjustment to the size of *any* of the variable units and accordingly adjusts the size of *all* variable units.	
	Activity	User adjusts the variable size to equalize the two propositions	 
	Criterion	User reads off the value of a variable unit in terms of the number of known units (meters) it subtends, e.g., one giant step is 2 meters long.	

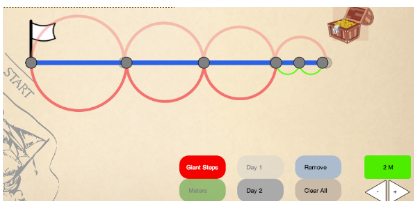
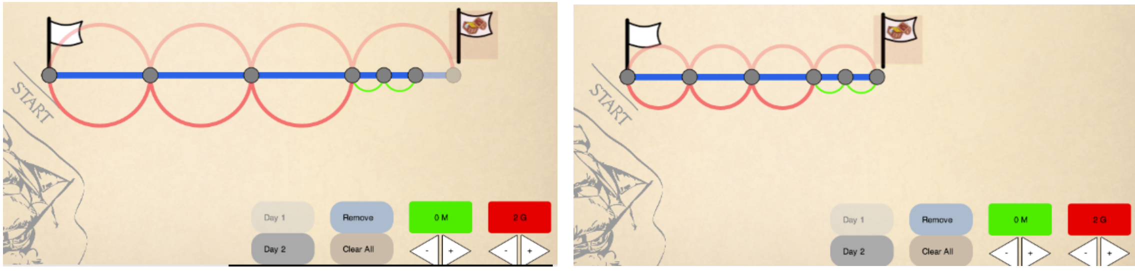


Figure 3. In Level 2, “Fixed Meter,” the (green) meters are automatically maintained as uniform in size (and therefore equal to each other), while the variable (red) giant steps are not automatically controlled thus. Users interact with a symbolic control (bottom-right corner) to generate meters.



a. *b.*

Figure 4. In Level 3, "Stretchy," green arches (meters) are invariable and thus equal to each other, while red arches (giant steps) are variable yet always equal to each other via uniform scaling. A new control (bottom-right corner) now enables the user to generate a specified number of giant steps, not only meters.

Note, in Figure 4a, that all the variable units are uniform, both above and below the line path, and yet the two journeys do not end at the same location, so that the only way of aligning the two trips would be by changing the uniform size of the variable (the red arcs). That is precisely what our hypothetical student did, so that the two trips ended in the same location thus determining the value of a single step as 2 meters (see Figure 4b).

A new hypothesis arises from the "leveling transparency" technological design architecture—a hypothesis informing our next study as well as a tentative theoretical insight. Namely, if users were introduced to the activity initially at Level 3, with its full slate of convenient interaction shortcuts, they could not appreciate these features as affordances, because they would not know *what* each feature accomplishes. As such, *learning as constructing transparency is the process of coming to visualize an artifact's features as affordances*.

Closing Words: SILOs Demarcate Structure-Oriented Mathematical Competency

We have introduced a mathematics-education construct we call SILOs—situated intermediary learning objectives. We explained that this construct emerged through our reflective engagement in the process of developing a design for algebra. Sensing the potential of these heuristics as something we might wish to understand, generalize, and share, we reified and refined these tacit elements of our practice in the form of the construct "SILO." SILOs are the structural and logical properties that a learner needs to figure out in order to utilize media made available in a particular learning environment so as to model a particular class of problem situations posed by the activity. Knowing a design's collective SILOs, we maintain, indexes conceptual ontogenesis of a student who is learning target content. Moreover, the creation of a set of SILOs indexes the progress of a designer who is learning about the student's learning process: by articulating the SILOs, the designer comes to know what the students should know who participate in activities enabled by the design. In a sense, SILOs are the educator's heuristics for engineering, orienting, and monitoring the learner's heuristics.

SILOs are not a to-do list of requisite *actions* required by an expert responding to a particular class of problems (e.g., production rules for solving picture-based pre-algebraic problems, Koedinger & Terao, 2002). Rather, SILOs are an artifact's set of necessary *properties*, any of whose violation would elicit from an expert adaptive action. We thus draw on the view of expertise not as the capacity of rote production but rather the skill of responsively recognizing and modifying perceived stimuli so that they embody target structures affording routine practice (Schoenfeld, 1998), such as inferring target information (e.g., the value of *x*). Of course this modeling skill must be developed. The practical function of SILOs is to organize and coordinate educators' efforts to create, moderate, and evaluate opportunities for learners to reinvent this expertise.

One might be tempted to describe GS4A as an exemplar of technological designs that *scaffold* algebra content. We hesitate to use that common term. In fact, our proposed design architecture for leveling transparency might be described as *reverse scaffolding*. Scaffolding is the asymmetrical social co-enactment of natural or cultural practice, wherein a more able agent performs for novices elements of a complex activity. The novices' participation is thus simplified, so that they experience the activity's purpose, meaning, and efficacy as well as a sense of competence. In GS4A, by way of contrast, the scaffolding is inherent to the design rationale but not the actual activity. That is, the design as a whole is *a fortiori* premeditated to enable and support guided reinvention of a mathematical concept. However, within the environment there is no co-enactment of any steps that students have not yet figured out themselves. The system co-constructs the model only once the student understands the necessity and functionality of each specific property of the model. Thus the pedagogical system relieves users of executing what they *know* to do rather than what they *do not know* to do.

SILOs are subjective achievements—they articulate learners' emergent, idealized system of target relations between reified elements in a problem space; they describe the "things" treated in the situation and imply how to treat them. As such, throughout this manuscript we have spoken of two emergent processes, each of which involves tinkering, discovery, and the objectification of implicit knowledge: (1) the child modeling a situation to infer quantitative information; and (2) the designer modeling the child's behavior to infer learning objectives. These two problem-solving processes are isomorphic, parallel, iterative, and reciprocal.

It is our hope that the idea of SILOs per se as well as the process by which they emerged will resonate with the experiences of fellow designers. A potentially productive focus of such a dialogue would be regarding the ontological status, or pedagogical role, of the external constructions children build as they work on a situated problem, whether concrete or virtual. Additionally, we are fascinated by the *designers'* early process of instantiating mathematical concepts. How does this process transpire? How do designers evaluate the quality, or epistemic fidelity, of these initial conceptual instantiations? We suspect that these two lines of inquiry—about design process and learning process, respectively—will turn out to be more similar than has been formerly suspected and, consequently, mutually informative.

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