Reverse Scaffolding: A Constructivist Design Architecture for Mathematics Learning With Educational Technology

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ABSTRACT
In the context of a design-based research effort to develop a technology-enabled constructivist algebra unit, a new activity architecture emerged that steps students through discovery levels. As they build a virtual model of a problem situation, students figure out technical principles for assuring the model's fidelity to the situation. These construction heuristics, we find, are precisely the conceptual foundations of algebra, such as tinkering with the model to assure that the variable quantity is of consistent size throughout the model. We articulated these principles as *situated intermediary learning objectives (SILOs)*. At each interaction level, the student discovers a SILO, and then the technology takes over by automatizing that SILO, thus freeing the student for further discovery. We call this architecture *reverse scaffolding*, because the cultural mediator thus relieves learners from performing what they *know* to do, not from what they *do not know* to do. In a quasi-experimental evaluation study (Grades 4 & 9; *n*=40), reverse-scaffolding students outperformed direct-scaffolding students, for whom the technical features were pre-automatized. We speculate on the architecture's generalizability.

Categories and Subject Descriptors
D.2.10 [Design]: Representations and Methodologies

General Terms
Design, Experimentation, Theory.

Keywords
Algebra, Learning, Pedagogy, Representation, Transparency.

1. INTRODUCTION
Educational design is a multifaceted practice. Apart from engineering and creating materials and activities, designers often make decisions regarding the activity flow—both the sequencing of tasks and assessment criteria for moving on from each task to the next along an optimized learning path. Determining an activity flow becomes imperative in technological media, wherein flow is encoded in procedures underlying human–computer interaction.

This implementation process of hard-coding activity facilitation can, at times, reveal to the designers implicit aspects of their own professional practice. In particular, when designers are required to delineate their instructional intuitions in the form of a regimented task flow, they may become cognizant of their own pedagogical beliefs that had been only implicit to their naturalistic interactions with students. Articulating one's pedagogical beliefs can prove to be a generative exercise, because the beliefs are thus revealed for scrutiny, perhaps for the first time. In turn, the designers can elaborate and evaluate these beliefs by instantiating them in additional contexts. As such, implicit pedagogical beliefs might emerge in the form of polished and portable design architecture. Such was the case of the design process reported herein.

The pedagogical architecture discussed in this paper, reverse scaffolding, was developed in the context of conducting a design-based research (DBR) project in the domain of mathematics. DBR projects are often motivated by a pedagogical conjecture that instantiates learning theory in the form of innovative artifacts (materials + activities). The conjecture is then evaluated by implementing the instructional design, collecting empirical records, analyzing these data, and reflecting on their implications for educational theory [7, 28]. The study cycle is typically iterated several times: from one study to the next, the investigators modify the research design so as both to improve the instructional intervention and focus on emerging phenomena of interest. DBR projects yield up to three types of deliverables: refined theoretical models of teaching and learning, new artifacts that may serve in instructional units, and reflexive insights onto the design process.

This paper reports on findings from a culminating study cycle in a DBR project that investigated the roots of algebraic cognition. Prior study iterations of this project [5] suggested the relevance of the cognitive theory of transparency [13, 18] to our pedagogical approach, instructional design, and analyses. In this particular study cycle we developed and evaluated a discovery-based learning environment, in which students were to bootstrap basic algebra notions via building virtual models of problem situations.

As the design process enfolded, we decided to parse the activity flow into an articulated sequence of several competency levels. Each level pertained to a particular construction heuristic—some proto-conceptual hands-on know-how—that we wished for students to figure out through tinkering with the modeling resources [23, 25, 33]. We had become aware that these modeling heuristics, which were specific for this algebra activity, in fact were technical embodiments of what we believe students ought to know about algebra. For example, students building a pictorial model of a word problem took care to ensure that two line segments were of the same spatial extent, a construction principle that anticipates the meaning of two equal expressions in an algebraic proposition (e.g., equal expressions in $3x + 14 = 5x + 6$).

We decided to take great care that all students discover these construction heuristics, and so we implemented into the technological design several interaction functions that would steer the students toward each of these principles. As we explain below, we found ourselves creating a type of activity flow that we had not encountered in any other work. We conceptualize an activity flow as a form of scaffolding, in the sense that we were simplifying a complex practice for pedagogical purposes, and yet at the same time it was an activity flow for discovery. We were scaffolding discovery and, as such, we never enacted for students what they could not yet perform themselves, as in classical scaffolding (e.g., think of attaching training wheels to a bicycle or helping a child to dress). Rather, we would only perform for the students what they had already discovered, what they *could* perform themselves. We therefore decided to call this activity flow "reverse scaffolding."

This paper elaborates on the notion of reverse scaffolding and reports on a quasi-experimental research study that sought to evaluate this activity architecture. In this study, we compared the impact of two comparable interventions on participating students' learning gains: a reverse-scaffolding activity, and another activity that consisted of the same materials and tasks only that the tasks flowed according to the normative "direct scaffolding." We conceptualized both interventions as implementing some form of scaffolding, because in both cases the interface was designed to co-enact the modeling task with the student. However, in the *direct scaffolding* condition, the software enacted actions that participants had not yet devised or even attempted to perform. We hypothesized that participants in the reverse-scaffolding study condition would develop greater understanding of the modeling system (and thus eventually of algebra), as compared to participants in the direct-scaffolding control condition.

The objective of this paper is to explain the rationale and operationalization of reverse scaffolding in our own design. We hope that our presentation will provide tools for other researchers to replicate and possibly extend and refine this activity flow both for algebra and perhaps beyond to other mathematics and further STEM concepts.

Section 2 of this article (Theoretical Frameworks) explains the construct of transparency, which is key to reverse scaffolding, as well as how we conceptualized a design architecture by which students are to develop deep understanding of content via literally constructing interactive models of problematic situations. We dwell on the construct of a SILO—a situated intermediary learning objective—that we developed as a means of articulating what it is that students learn when they build models of problem situations. In Section 3 (Design), we present the pedagogical challenge of algebra. We explain how the SILOs served us in parsing the educational interaction into a sequence of levels, where each level creates opportunities for the student to render transparent one (of three) critical aspects of the algebraic conceptual system. Section 4 (Methods) then details how we evaluated this proposed interaction architecture by comparing learning gains across two experimental conditions: one group engaged in the reverse-scaffolding activity flow, in which they had to discover the SILOs, and the other group engaged in direct-scaffolding activity flow and so did not have to discover the SILOs. Section 5 (Results) reports on the intervention's main effect and presents brief annotated vignettes from qualitative micro-genetic analyses of the tutor–student interactions. Section 6 (Conclusions) then offers a brief summary of the study, states its limitations, and suggests implications for research on the design of educational technology.

2. THEORETICAL FRAMEWORKS

In this section we first explain the construct of transparency. Next, we elaborate how we worked with this construct so as to engineer a constructivist design architecture for deep content learning.

2.1 Transparency

The theoretical construct of *transparency* is based on the premise that humans are thrust into a world that is rife with preexisting artifacts intrinsic to the social enactment of cultural practices. The theory is interested in each social agent's understanding of how features of these artifacts mediate their accomplishment of particular practices [18].

When people first encounter artifacts, they must learn how to use them. This is true even for simple cultural–historical tangible objects, such as a fork, but certainly for more complex objects such as an abacus. These artifacts are to serve as the instruments mediating and extending the humans' interaction with the environment's materials, whether these materials are pragmatic (e.g., a fork mediates interaction with food) or epistemic (e.g., an abacus mediates interaction with signified quantities). But for artifacts to serve us as bona fide instruments, we must adjust to the artifacts' interaction constraints [16, 17, 30]. Granted, we may become fluent users of some artifacts even without understanding how they mediate our intentionality, and so the artifacts will be effective yet remain opaque to us. However artifacts that are used to foster content learning should be transparent, because figuring out how they work—that is, exposing how their structure serves their function—is tantamount to understanding content (e.g., an abacus is better suited than a calculator for teaching the mathematical place-value system). Meira calls this process of learning-by-scrutinizing-an-artifact the subjective development of an artifact's transparency [18].

The theory of transparency is important for education. It focuses designers' choices in creating pedagogical artifacts, and specifically regarding which specific mathematical operations the artifacts should "blackbox" vs. "glassbox"—decisions that bear direct consequences for conceptual development. In the case of creating a modeling activity, for example, careful consideration must be given in deciding whether and how a digital learning tool should automate certain aspects of the activity, respond to user input, and provide feedback.

If learning is the subjective construction of artifact transparency, how do we design for this process? Moreover, if we implement this socio–cognitive approach in interactive technology, what pedagogical architecture might best inform our design decisions and organize our process?

This paper considers the above questions through presenting the case study of a design project for deep learning of introductory algebra content. Reflecting on empirical results from a quasi-experimental research design, we develop and evaluate a constructivist architecture for implementing educational activities based on the notion of subjective transparency. This architecture consists of engaging students in a sequence of activity levels, where each level corresponds with particular *figural* structure of the algebraic *conceptual* system. Importantly, students themselves determine these latent structural relations as their own pragmatic solutions for modeling word problems they attempt to solve. That is, the students figure out historical technique for rendering problem situations into diagrammatic structures.

We thus view algebra as a conceptual system emerging from modeling activity, and so our activity architecture was to ensure that key principles of this conceptual system are indeed deployed into the design in the form of discovery potentials. We conjectured that individual students would achieve subjective transparency of the conceptual system by tinkering with web-based construction resources in attempt to achieve functional fidelity with a given problem situation. Thus learning algebra would be tantamount to developing construction know-how, that is, a set of situated heuristics and technical criteria for evaluating whether the virtual model preserves information structures implicit to the problem situation.

This study, and in particular the emerging articulation of an activity architecture for educational technology, was initially motivated by intriguing communication problems that emerged during a collaboration between learning scientists and interaction designers [3]. We, the learning scientists, had been attempting to convert an educational activity for algebra from its earlier mechanical implementation into a digital environment [5]. In the mechanical version, a human tutor had supported the students by following a general protocol that informed decisions as to what to do and say and when and how to do so. In the computer-based implementation, however, much of this protocol would be encoded in software. To author this code, our collaborating interaction designers required from us to spell out aspects of our pedagogical content knowledge that were only implicit to our practice, and they needed a high-resolution proceduralization of our general instructional know-how. In a sense, our collaborators demanded that we render transparent for them our own tutorial tactics in the form of comprehensive structures that could then be deployed as the activity template, resources, and process. As a result, we created a new ontological entity, the set of *situated intermediary learning objectives* (SILOs) that spell out what students need to know about a particular content domain and how they develop this knowledge via the modeling activity [3].

2.2 Rethinking Scaffolding

Educational designers often begin a project by identifying a concept that is difficult to learn and teach, analyzing these difficulties, articulating a conjecture for improving instruction, and then creating tools in light of these analyses. The implementation of pedagogical design in the form of guided learning activities has been called "scaffolding." More broadly, scaffolding can be conceptualized as the asymmetrical social co-enactment of natural actions or cultural practice, wherein a more able agent implements, performs, or specifies, for a novice, elements of a challenging activity.

Since its early theorization [33], the notion and methodology of scaffolding has become widely incorporated into all aspects of educational practice. Scaffolding is pervasively cited to motivate the design of informative and functional features of educational environments. These features distribute the enactment of processes as actions performed by the child in coordination with the interactive media. These media complement, emulate, and possibly enhance the variety of customized supports that human agents would provide in real time as they work with a child [18].

The didactical metaphor of scaffolding is now so ubiquitous in the rhetoric of education researchers and practitioners, that its meaning has become diffuse, its theoretical rationale unquestioned, and its pedagogical operationalization vague [24]. For example, Reiser describes how computer interfaces scaffold both the user's engagement in an activity ("structuring") and what the user thinks about the content ("problematizing"). With regards to structure, Reiser suggests, "If reasoning is difficult due to complexity or the open-ended nature of the task, then one way to help learners is to use the tool to reduce complexity and choice by providing additional structure to the task" [27, p. 283]. With regards to problematizing, he further suggests, "Rather than simplifying the task, the software leads students to encounter and grapple with important ideas or processes" [27, p. 287].

As we have discussed earlier, scaffolding originates from a socio-cultural theoretical perspective wherein learning is characterized as a process of co-production that eventually results in the learner's independence. However, mathematics-education researchers indicate that independent production per se may not always indicate or result in understanding [11, 21]. One might think of a student enacting a rote algorithmic procedure for dividing one fraction by another (or as the frivolous doggerel goes: "Ours is not to reason why—Just divide and multiply"). Some researchers suggest a constructivist approach, wherein meaningful learning is the process of subjectively reinventing cognitive structures for effectively enacting cultural practice [31]: "the constructivist model recognizes the benefits of students participating in tasks that enable the active construction of their own knowledge domain" [11, p. 28].

As such, pedagogical views inspired by constructivist philosophy support the argument for each child developing subjective transparency of the artifacts they use for performing their instructional tasks. At the same time, scaffolding is heralded by leading education scholars as a necessary pedagogical practice. We wondered, however, whether the practice of scaffolding—and in particular simplifying or co-enacting a task for the child—might rob that child of the critical opportunity to develop subjective transparency. When a child learns by imitation, we worried, she is liable to miss out on critical opportunities to reinvent those very operations and structures she was relieved of. If so, what role, if any, could the notion of scaffolding play in a conceptualization of discovery-oriented learning processes?

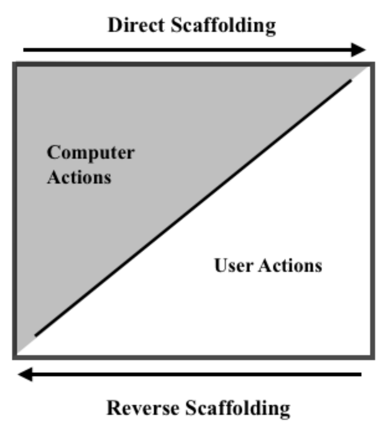


Figure 1. Fade out vs. fade in: In direct scaffolding, the technological environment enacts for learners what they *do not* know to do. In reverse scaffolding, the technology enacts what they *do* know to do.

We thus faced a dilemma between two apparently conflicting pedagogical principles, fostering subjective transparency and proffering scaffolding. Our resolution of this dilemma was in the way of modifying the classical notion of scaffolding. We coined the phrase *reverse scaffolding*. Traditional scaffolding captures the pedagogical essence of interactions designed to assist learners by enacting for them what they *do not* yet know to do. Reverse-scaffolding, on the other hand, captures the pedagogical essence of interactions designed to assist learners by enacting for them what they *know to do* (see Figure 1). We hypothesized that individual students would construct subjective transparency of mathematical concepts by constructing mathematical models leading to those concepts. Our pedagogical rationale differs from traditional conceptualizations of scaffolding, by which training wheels are progressively faded. Instead, we have students reinvent the wheel. Or, if you will, our scaffolds are faded *in* rather than out. And our tasks are designed accordingly to make this happen.

We are thus exploring a pedagogical architecture, reverse scaffolding, by which to foster students' subjective development of transparency for artifacts encapsulating mathematical knowledge. Algebra is our case study—our content context—for evaluating this general design rationale. In particular, we treat the case of Giant Steps for Algebra (GS4A), an experimental computer-based environment designed to foster student conceptual construction of rudimentary algebra concepts via hands-on construction of virtual models.

3. THE DESIGN

3.1 The Design Problem

Algebra has been described as a *praxis cogitans* [26]—a particular epistemic orientation toward situations. When we solve algebraic problem situations, we attend selectively to particular types of properties in the situation: the magnitudes of objects as well as quantitative relations among them, such as their numerosity, size, distance, or velocity. Algebraic solution methods are most powerful when the latent system of logical–quantitative relations inherent to a situation is modeled, that is, first converted into a diagrammatic structure and then perhaps further encoded as a set of propositions in the symbolic register. These symbolic propositions can be manipulated systematically—solving for x—toward determining new information about the situation [9, 20].

Algebra has been tagged as the "gatekeeper" high-school mathematics course, because students' failure to master algebra impedes their access to higher education [19]. Despite consistent attempts at reforming algebra curriculum and instruction, many students continue to struggle and fail [22]. Our choice of algebra as a case study for evaluating the pedagogical architecture of reverse scaffolding thus stands to inform curriculum and potentially to contribute toward addressing this gatekeeper barrier.

3.2 Giant Steps for Algebra (GS4A)

The GS4A project seeks to investigate the potential of a new pedagogical approach to constructing algebraic transparency. Consider the algebraic proposition "$3x + 14 = 5x + 6$." The classical balance-scale metaphor for algebraic equations presents the proposition as the weighing of two expressions against each other (see Figure 2). Whereas the balance-scale metaphor supports the rationale of algebraic algorithms—adding (or subtracting) equivalent quantities on both sides of the equation so as to maintain balance—this structure may not directly model a variety of algebra problems and thus is not grounded intuitively in the situated context. GS4A draws instead on Dickinson and Eade [8], who proposed the double-number-line algebra model (Figure 3).

The number-line visualization of algebraic equivalence appears to facilitate an offloading of source information onto the diagram's inherent figural constraints. In this model, we can arrive at the solution, $x = 4$, by a sequence of visual deductions.

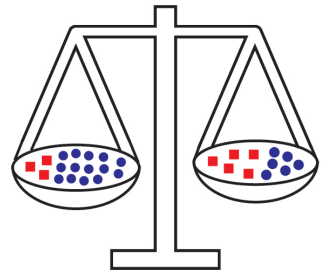


Figure 2. "3x + 14 = 5x + 6" on a balance scale

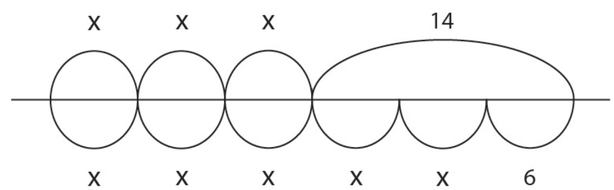


Figure 3. Number-line instantiation of "$3x + 14 = 5x + 6$"

We conjectured that the number-line model therefore bears greater potential, as compared to the balance-scale model, for students to develop subjective transparency of algebra situations. In particular, the spatial features of the number-line model render highly salient the logical relations between variable and integers, both within- and between expressions.

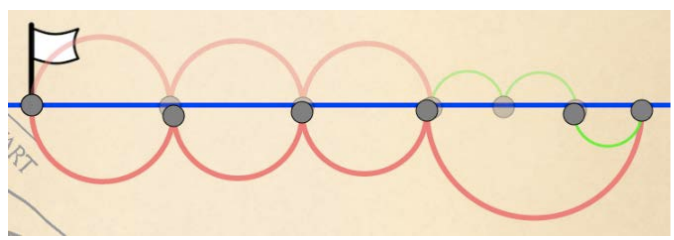


Figure 4. Model of a GS4A story. On both Day 1 and Day 2 the giant travels from the flag to the destination on the right. Red loops represent giant steps, green loops represent meters. Day 1 is marked above the line, Day 2 is marked below. This story corresponds with the proposition $3x + 2 = 4x - 1$.

We used the number-line model in designing our learning activity. GS4A is a situation-based model [32]. Per the embodied-design framework [1, 2] GS4A seeks to engage and leverage students' tacit knowledge about simple ambulatory motion and spatial relations. Figure 4 presents an excerpt from a screenshot showing a student's solution to the following problem:

> "A giant wanted to hide treasure. She walked 3 steps and then another 2 meters and buried the treasure. The next day, she wanted to bury more treasure in exactly the same place, but she was not sure where that was. She began from the same spot. She walked 4 steps and then, feeling she'd gone too far, walked back 1 meter. Yes! She found the treasure!"

Our pilot work used a range of mechanical devices in a variety of concrete media [5]. Qualitative analyses of student behaviors and tutor–student interactions suggested a relation between modeling, that is, building a structure that captures relevant properties of a problem situation, and conceptual learning. As students tinkered with the media to build models of the giant stories, they were implicitly determining a set of rules that we named SILOs (*situated intermediary learning objectives*). Yet these SILOs became evident in our analyses only as we turned to redeploy GS4A in a scalable technological format. Therein, the SILOs proved instrumental in formulating a blueprint for the activity architecture. In the technological version of GS4A, students transition from each interaction phase to the next when their electronic actions demonstrating mastery over one of the SILOs. The idea is thus to step learners through an activity while enabling them to build subjective transparency of their emerging model.

Borrowing the notion of "levels" from popular computer games—the gradual rewarding of manifest competency with increased power that is linked to increased skill—in GS4A we *level transparency* (see Table 1, two pages down, for the specific set of SILOs in relation to the interaction features). At each new level, the technology offloads the SILO from the user—it automatically generates and maintains the specific technical details and functional relations that the user had just discovered. We hypothesized that students participating in this activity would experience unique learning opportunities. First, they would develop subjective transparency of the content via constructing models of the situation. Second, at the discovery of each SILO they would be relieved of the tedium of painstakingly adjusting features on the screen and would thus have more resources for further problem solving. The Methods section, below, explains how we sought to isolate and measure these putative effects of reverse-scaffolding design.

We are not aware of similar discovery-based automated learning environments for algebra. The *MiGen* project [21] is our closest cousin, as it, too, is a microworld for discovering and generalizing proto-algebraic structures and processes. *MiGen* includes automated supports and constraints that steer students to formulate situated abstractions via testing their fledgling conjectures vis-à-vis the interface's output. The authors of *MiGen* view mathematical concepts as emerging from children's purposeful construction of utility for available interactive features of designed artifacts. The SILOs framework differs from that of situated abstractions in terms of grain size, ontological and epistemological foci, and pedagogical underpinnings. In particular, SILOs articulate a set of initially *un*available interaction constraints that the learner determines, implicates, and wills as potentially conducive to more effective problem solving with a given artifact; in response, each of these willed constraints is then materialized into the artifact by the instructor who enables into functionality a pre-programmed "hidden" constraint. SILOs are thus functional concretizations of the user's wish list into working technological features of an interactive device. Whereas situated abstractions sprout from examining what *is*, SILOs sprout from willing what *isn't*.

4. METHODS

4.1 Participants

Forty Grade 4 and 9 students (18 male, 21 female) from an independent elementary and public high school voluntarily participated individually in task-based semi-structured clinical interviews [6, 12].

4.2 Experimental Design

The rationale of the experimental design was to measure and then compare learning under two conditions: a study-group worked *with* the leveling-transparency functionality (reverse-scaffolding group, RS) and the control-group worked *without* it (direct-scaffolding group, DS). In the DS group, users would not need to "earn" the automatic functionalities—they would receive *ab initio* a fully-fledged technological application. We hypothesized that the DS group students would thus experience reduced opportunity to develop subjective transparency of the conceptual system. See Table 1 for the distribution of participants by condition.

Table 1. Participant Distribution by Condition

Grade	Condition	
	RS	DS
4^{th}	11	9
9^{th}	10	10
Total	21	19

Upon completing the activity, all participants responded to two series of post-activity assessment problems. These problems were designed to evaluate participants' subjective transparency of pre-formal algebra concepts, as operationalized in the three SILOs. We used two sets of problems: (a) New-Context problems, in which we measured for the application of learned skills (transfer); and (b) In-Context problems that targeted the three SILOs directly within the familiar GS4A setting. The New-Context problems were narrative problems that had a similar structure to the Giant Steps problems yet dealt with different situations (the age of a turtle and the height of two buildings). The In-Context problems tasked participants with correcting hypothetical students' models that violated one or more of the SILOs.

The interviews took place in a quiet room on the school sites and were all videotaped for subsequent analysis. The interviewer (the first author) took field notes following each session and consulted on a daily basis with the design-research team so as to optimize the quality of collected data.

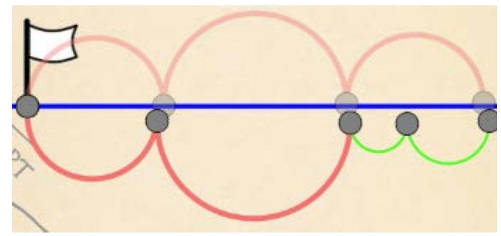
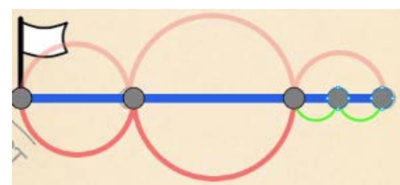
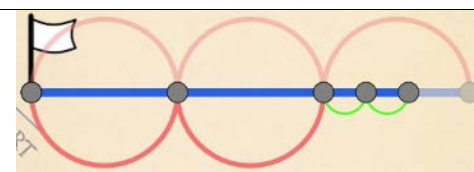
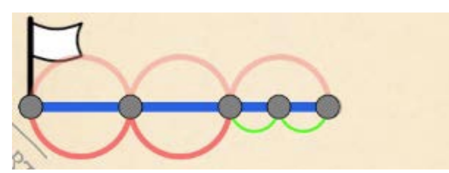
4.3 Analysis

The entire corpus of data was transcribed. Using micro-genetic analysis techniques [29], we developed a coding system for capturing the participants' development of subjective transparency of the algebra conceptual system. The coding structure was designed to identify and characterize moments when the participants articulated understanding of each SILO, that is, each structural feature of the model.

The post-activity assessment problems served as the primary source for empirical analysis. We scored participants' responses based on evidence that they achieved each SILO. After one analyst had coded all the post-activity assessment problems, a second analyst independently scored 21% of this data corpus. Results from an inter-rater reliability test were Kappa = 0.822 ($p < 0.001$), 95% CI (0.646, 0.998), almost perfect agreement.

For the purposes of this paper, we selected several annotated transcriptions. These vignettes include episodes both from the study and control group. To facilitate juxtaposition between students in the two groups, we selected events that all revolve around students' actions related to the same SILO. Namely, our analyses focused on students' attempts to spatially align the two journeys such that the relationship between the variable (giant steps) and integer (meters) could be identified and deciphered. In all cases, we also attempted to determine whether the tools obfuscated or illuminated the SILO.

Table 2. Leveling Transparency: Matched SILOs and Levels in the Giant Steps for Algebra Technological Design

SILO	Level	System Constraints, User Activity, and Behavior Criterion	Interface
1. Consistent Measures	1. Free Form	System offers no support in coordinating units or expressions.	
	Activity	User builds all parts of the model manually.	
	Criterion	User expresses frustration in equalizing units.	
2. Equivalent Expressions	2. Fixed Meters	System generates meter units in predetermined size and maintains uniform size automatically.	
	Activity	User builds variables manually.	
	Criterion	User expresses frustration with managing uniform variable units the lengths of Days 1 & 2.	
3. Shared Frame of Reference	3. Stretchy	System monitors for manual adjustment to the size of *any* of the variable units and accordingly adjusts the size of *all* variable units.	
	Activity	User adjusts the variable size to equalize the two propositions	

5. RESULTS

5.1 Main Effect

We expected to receive a positive difference that would indicate greater mean learning for the experimental condition as compared to the control condition. We therefore used a one-tailed independent-samples t-test to examine for differences between the mean scores of the two groups. Results from the New-Context tasks revealed that the RS experimental group ($M=5.17$, $SD=2.34$) scored significantly higher than the DS control group ($M=4.10$, $SD=2.76$); $t(38)=1.98$; $p=0.02$. Results from the In-Context tasks revealed that the RS experimental group ($M=5.88$, $SD=2.10$) scored significantly higher as compared to the DS group ($M=4.60$, $SD=1.90$); $t(38) = 2.00$; $p = 0.02$. Combining both results from these post-activity assessment problems revealed that the RS group ($M=11.59$, $SD=3.57$) scored significantly higher than the DS group ($M=8.71$, $SD=3.81$); $t(38) = 2.46$; $p < 0.01$.

5.2 Qualitative Analysis

We now present a selection of annotated vignettes from our participating students' guided interactions with the technological artifact. As we argue, based on these paradigmatic data excerpts, the two instructional conditions appear to have had differential effect on our participants' learning. These differences appear to be more nuanced than could have been measured by the post-intervention assessments alone. Building on this apparent finding of the study-group participants' advantage over the control-group participants, we will then use this finding as evidence supporting our claim that the difference lies in the different interventions and that, therefore, there is cognitive advantage in developing subjective transparency through leveled discovery. In turn, this apparent validation of the conjecture will lend support to the pedagogical architecture of reverse scaffolding.

The vignettes in this section are organized as matched pairs, with compatible reverse-scaffolding (RS) and direct-scaffolding (DS) study participants juxtaposed so as to bring out critical differences. We begin by featuring vignettes of two 4^{th}-grade participants, an RS participant and a DS participant, both rated by their teacher as having "high" mathematical abilities.

Susan (all names are pseudonyms) is working in the RS condition (study group). She is at Level #1, working on an informal narrative corresponding to the formal proposition "$4x = 3x + 2$." She has completed the Day 2 travel diagram (see in Figure 5 the four red loops above the horizontal line) and is now working on the Day 2 travel diagram below the line.

Res.: Ok. So she goes…
Susan: 3 giant steps and…..
Res.: …and then....
Susan: 2 meters. (Susan switches an interface feature to "meters" and draws below the line 2 equivalent meters that subtend the 4^{th} giant step immediately above the line.)
Res.: So she goes 2 meters and then she finds the right spot.
Susan: Yeah
Res.: So in your drawing did she find the right spot?
Susan: Hmmm well yeah.

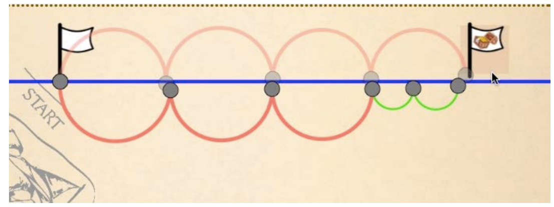


Figure 5. Susan's construction for a Giant Steps story corresponding to the algebra proposition $4x = 3x + 2$

Immediately, Susan has identified that the end point for both days is in the same screen location (see the treasure flag in Figure 5, on the right) and that, consequently, the 2 meters on Day 2 will subtend the same distance as the 4^{th} giant step on Day 1. Despite some imprecision in her modeling execution, for example the meters are not of precisely the same screen size, Susan has constructed the transparency of equivalent expressions (SILO 2).

We now turn to Karrie, a participant in the DS condition (control group), who is working on the same item (see Figure 6).

Karrie: It says she walks 2 steps further ahead and finds the treasure. But that doesn't make sense because it is more back than the other treasure. (Karrie has drawn a model in which the giant steps are too large, so that the respective ends of Day 1 and Day 2 are not co-located.)

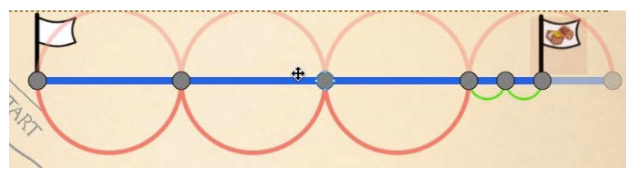


Figure 6. Karrie's construction for a Giant Steps story corresponding to the algebra proposition $4x = 3x + 2$

Concerned by this misalignment between the end points of Days 1 and 2, Karrie suggests inserting additional meters. The interviewer responds by stating that doing so would violate the information in the story. The conversation ensues as follows.

Karrie: I can change the size of the giant steps.

Pursuing on her new idea, Karrie attempts to stretch the Day 2 travel diagram toward the right so that it reach the treasure flag. Specifically, she stretches the giant steps in Day 2 (the red loops below the blue line, see Figure 6). Recall that in the direct-scaffolding condition the variable distances (all the red loops) are automatically interlinked, both within- and between days. Consequently, the variables in both Day 2 *and* Day 1 all stretched uniformly, and the two misaligned ends only became farther apart! Karrie then attempted the same maneuver by decreasing the step size in Day 1, but she stopped before the two ends met.

Karrie: It moves the whole thing?

Karrie was surprised to witness the automated-scaling feature that simultaneously adjusts corresponding variables in and across both days uniformly. Karrie had had meant to equalize the linear extents of the two days by first adjusting Day 2 and only then Day 1. Thus whereas Karrie was demonstrating SILO 3, equivalent expressions, she was doing so with disregard to SILO 1, consistent measures. Moreover, Karrie did not appear to appreciate the implication of uniform variable size for the fidelity of her story model. To Karrie, this feature is not transparent.

We now turn to our second comparison, two 9th-grade participants, both rated by their teacher as having "medium" mathematical abilities. Taylor is working in the RS condition. He is at Level #2, working on the narrative corresponding to the formal proposition "$2 + 2x + 3 = x + 1 + 2x + 1$." He has just begun reading the problem.

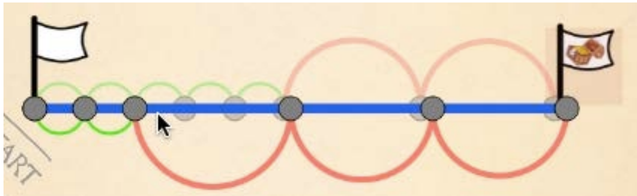


Figure 7. Taylor's completed model with the components reorganized for simplicity

Taylor: Ok. Two meters, (begins by drawing 5 meters, see Figure 7).
Res: Wait, what did you do?
Taylor: I put all the meters first. 'Cause, like, they're all going to go to the same place. (Taylor performs a sweeping hand gesture from left, the start, to right, the treasure location; see Figure 8). It doesn't really matter, the order.

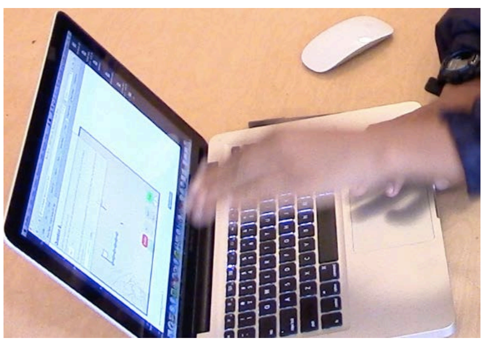


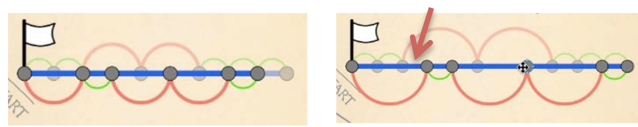
Figure 8. Taylor moves his hand from the start (left side of the screen) to the treasure (right side of the screen).

Taylor's insight captures an auxiliary objective for the design of this particular problem item. We intentionally created this item so as to foster an opportunity for participants to combine units (meters) and variables (giant steps). And yet the participants would need to be motivated to discover the utility of such combining. By stating that the order of distances traversed (the situated addends) does not change the final destination (the sum) Taylor is taking advantage of the commutative property of number, as instantiated in the form of a string of concatenated segments, to create a model that better utilizes the number-line solution form. Taylor clearly demonstrates that he has a flexible understanding of the model he is creating and has achieved all of the SILOs. In fact, Taylor is thoughtful in discussing how he can utilize this know-how so as to improve and interpret his model. His flexibility is reflected in his scores on the New-Context post-intervention assessment, where he received a score of 6 out of 8.

We now turn to Irene, a DS participant working on the same problem as Taylor. Recall that the DS condition automatically generates fixed meters and automatically rescales all of the giant steps for the participant (consult Table 1 for details).

Irene has just completed her model of the story narrative and realizes that the ends are not aligned (see Figure 9a).

Irene: Umm, So you need to make it bigger (she stretches the model so that the ends meet, see Figure 9b). There.
Res: Ok, so now they meet?
Irene: And I think each step is worth, ... not *worth* exactly (Irene appears to be groping for the word "equivalent") ...3 meters?
Res: hmmm
Irene: So….
Res: What makes you say that?
Irene: Like first on day 1 (switches the interface so that Day 1 is highlighted) it is 2 meters (scrolls over the first giant step on Day 2, which corresponds to 2 meters and a gap on Day 1). And I think if you had 1 more meter it would be 3 (scrolls over the gap, see Figure 9b – *After stretching*).



a. Before stretching ***b. After stretching***

Figure 9. Irene's model before and after stretching

Irene has achieved SILO 3, "shared frame of reference," as observed through her actions to align the ends of her Day 1 and Day 2 models in an attempt to determine how many meters make up one giant step. However, when it comes to determining a solution, Irene warrants her claims based on available visual information rather than the narrative information. She states, "I think if you had 1 more meter," yet she does not cross-check with the situation narrative. Furthermore, Irene's solution strategies do not exemplify the same level of sophistication and flexibility as her classmate Taylor. The results of this lack of flexibility are reflected in her scores on the New-Context post-activity question where she received a total score of only 3 out of 8.

Whereas there are moments in Irene's intervention that indicate her thoughtfulness, this thoughtfulness was not apparent later in the New-Context post-intervention assessment. The direct-scaffolding task-flow architecture of the intervention had enabled Irene to develop an effective yet inflexible and non-transferable modeling routine: (a) model each of the two Day narratives, respectively above and below the line; (b) stretch or shrink one or both day diagrams until the ends meet; and (c) calculate the meter value of a step. Importantly, the uniform stretching/shrinking of the variable quantity was a given automatic feature of the interaction. Irene never had to discover, challenge, or monitor this feature, and so this feature remained opaque—the feature did not appear to be grounded in any insight on the modeling system as relating to the narrative situation.

Note that we are not critiquing Irene. Her reasoning was logical, rational, and consistent. Rather, we underscore that Irene's reasoning was bound the particular contexts whence it developed.

Irene's hands-on problem-solving algorithm appears markedly different from the varied strategies Taylor employed. In the reverse-scaffolding task-flow architecture (gradual automatization) the user must modify the solution algorithm with the introduction of each new level. Doing so, we believe, offers the user opportunities to devise new and adaptive forms of manipulating the model's structural features as well as opportunities to interpret the emerging structural systems from multiple perspectives. Thus the user develops subjective transparency of the modeling system by exercising flexible visualization and manipulation. In particular, the user devises new operatory schemes that become articulated as the SILOs.

Consider the case of Taylor. Recall that the task-flow change from Level 2 to Level 3 introduces the automatization of uniform Giant Steps. The moment this feature was enabled, Taylor recognized its utility, exclaiming, "Oh, I need that!" He immediately knew how this new control would function, understanding that it would generate and maintain consistent yet automatically scalable giant steps (SILO 1). In turn, the transparency of this Level 3 utility enabled Taylor to instantiate SILO 3, the shared frame of reference between variable and known quantities.

Now compare Taylor's case to that of Irene. During the post-intervention In-Context assessment, Irene is asked to interpret and possibly fix an incorrect model created by a hypothetical participant (see Figure 10).

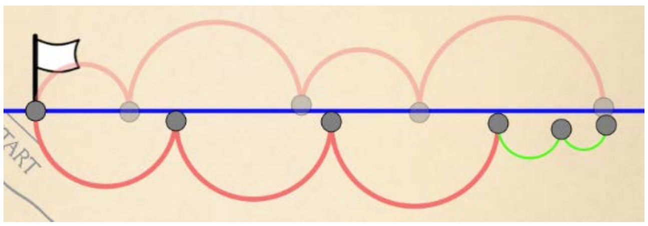


Figure 10. In-Context Question #2

Irene correctly identifies that the giant steps and the meters in this item are each modeled as non-uniform. In response, she wishes to enact her three-step solution strategy—model, stretch/shrink, calculate—so as to amend the apparent irregularity. Irene makes the following suggestions: (a) Pointing to the end nodes on the right that are not perfectly aligned, she says, "The giant steps on the bottom should be moved (toward the left) so that the ends (on the right) meet"; (b) "They should put big meters in big giant steps, and small meters in small giant steps…"; and finally (c) "…so that a giant step is 3 meters." Irene's suggestions for fixing the model intimate that she does not view the interaction affordances as instantiating critical features of an emerging conceptual system. At no point in her proposed solution does she directly address the non-uniform size of the giant steps. Her second solution step violates SILO 1, consistent meters. Her last solution step, while adhering to SILO 3, shared frame of reference, is incorrect.

Unlike Taylor, who expressly predicted the interface's affordances for the modeling the problem situation, Irene never wondered about the interface's action capabilities that were present in the construction of the model in Figure 10, namely that the interaction was manual. Irene is process oriented—she has developed an effective protocol for solving a particular class of problems under particular interaction conditions, and yet she never had to *will* those interactions and then acknowledge their arrival. She cannot appreciate how the model maintains or violates the SILOs, because the model's functions are opaque to her.

Based on these as well as other matching comparisons of annotated vignettes across the entire data corpus, we are inclined to assert that GS4A indeed fosters algebraic transparency. We further assert that the leveling-transparency activity-flow architecture was directly instrumental in mediating students' insights into the emerging algebraic system.

Furthermore, the generic notion of "scaffolding"—that is, that educators should facilitate learning via co-enacting for learners aspects of complex practice—does not foster algebra transparency too effectively. We therefore propose that an alternative instructional methodology should be considered. The leveling-transparency architecture offers just this: it can be conceptualized as reverse scaffolding—the technological system co-constructs the model with the student only after the student understands the necessity and functionality of each specific property of the model.

6. CONCLUSION

We reported on a design-based research project, in which we developed and evaluated a discovery-oriented activity for early algebra. The activity architecture is designed for users to gradually discover a set of situated features of the algebraic conceptual system. These latent features emerge and are articulated as "how-to" construction heuristics in the course of modeling assigned problem situations and enacting these models discursively. The activity is parsed into a sequence of levels. At each level users tinker with, negotiate, and reify what turn out to be critical features of the target content. The interface then automates these features. Thus an activity architecture designed for leveling discovery of modeling techniques supports the user's subjective construction of transparency for the conceptual system.

Emerging from the project is a new pedagogical approach called *reverse scaffolding*. In this approach, mathematics students discover properties of conceptual systems, such as algebra, by building virtual models of problem stories. As they do so, students figure out practical principles for building good models, that is, models that bear fidelity to the stories. These principles, as it turns out, are the situated embodiments of the conceptual system. When a student's actions demonstrate the discovery of one of these principles, the computer "takes over" by relieving the student from executing and monitoring that principle. Reverse scaffolding, much like traditional ("direct") scaffolding, captures teacher/student co-construction of content-relevant conceptual coordinations. Yet it is reversed, in that the interface scaffolds what students *know* to do rather than what they *do not know* to do.

Our results lend support to the thesis that the reverse-scaffolding is more effective in fostering conceptual understanding as compared to the traditional approach, direct scaffolding. In accord with our design conjecture, we demonstrated a statistically significant main effect, by which our study group outperformed the control group (total n=40) on measures of conceptual understanding. Results from qualitative analyses further suggest that the pedagogical design architecture of reverse scaffolding, implemented as a discovery-based leveled task flow, enables students to develop subjective transparency of the target content.

Students need not be spoon-fed ready-made solution strategies for mathematical problems they first encounter [4]. Indeed, struggle and inevitable failure in the process of inventing solution strategies are conceptually beneficial [14, 15]. Given appropriate design, students can discover key aspects of conceptual systems.

Reverse scaffolding should be seriously considered as pedagogical architecture for the design of learning environments, whether for algebra or other mathematical content and perhaps beyond for other STEM domains. Future DBR studies should thus focus on a variety of learning activities both within and outside of mathematics. Enduring design challenges to be addressed reside in automatizing the tutor's entire range of responses. If technological designs such as Giant Steps for Algebra are to go to scale, available for any child with access to a computer and internet, then projects such as this would benefit from collaborating with computer scientists with expertise in artificial intelligence, embedded assessment, and learning analytics.

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